Automata Theory

CS411-2015F-11

Turing Machines

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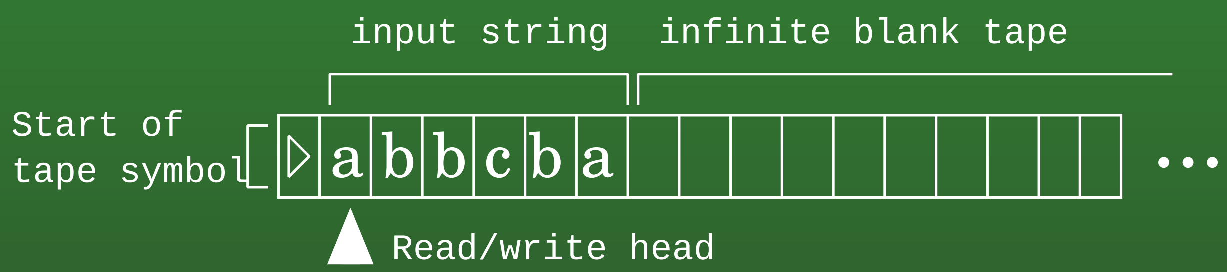
11-0: Turing Machines

- Machines so far (DFA, PDA) read input only once
- Next: Turing Machines
  - Can back up over the input
  - Can overwrite the input

11-1: Turing Machines

- Input string is written on a tape:



- At each step, machine reads a symbol, and then either
  - Writes a new symbol
  - Moves read/write head to right
  - Moves read/write head to left

11-2: Turing Machines

- A Turing Machine $M = (K, \Sigma, \delta, s, H)$
  - K is a set of states
  - Σ is the tape alphabet
  - $s \in K$ is the start state
  - $H \subset K$ are "Halting states" – y for accept, and n for reject
  - $\delta : (K - H) \times \Sigma \mapsto K \times (\Sigma \cup \{\leftarrow, \rightarrow\})$

11-3: Turing Machines

- TM for $L = \{0^n1^n : n > 0\}$
- Idea:
  - Mark off first 0, go to end of string and mark off last one (ensuring that the string starts with 0 and ends with 1
  - Repeat until all characters have been morked off
  - Make sure # of 0s match the # if 1s.

Turing Machines

TM for $L = \{0^n1^n : n > 0\}$

	0	1	$\sqcup$	X	Z
q_0	(q_1, X)	$(n, 1)$	$(n, \sqcup)$	(n, X)	(y, Z)
q_1	$(q_1, \rightarrow)$	$(q_1, \rightarrow)$	$(q_2, \leftarrow)$	$(q_1, \rightarrow)$	$(q_2, \leftarrow)$
q_2	$(n, 0)$	(q_3, Z)	$(n, \sqcup)$	(n, X)	(n, Z)
q_3	$(q_3, \leftarrow)$	$(q_3, \leftarrow)$	$(n, \sqcup)$	$(q_0, \rightarrow)$	$(q_3, \leftarrow)$

11-5: Turing Machines

TM for $L = \{0^n 1^n : n > 0\}$

	0	1	$\sqcup$	X	Z
q_0	(q_1, X)	no	no	no	yes
q_1	$(q_1, \rightarrow)$	$(q_1, \rightarrow)$	$(q_2, \leftarrow)$	$(q_1, \rightarrow)$	$(q_2, \leftarrow)$
q_2	no	(q_3, Z)	no	no	no
q_3	$(q_3, \leftarrow)$	$(q_3, \leftarrow)$	no	$(q_0, \rightarrow)$	$(q_3, \leftarrow)$

11-6: Turing Machines

TM for $L = \{0^n 1^n 2^n : n > 0\}$

11-7: Turing Machines

TM for $L = \{0^n 1^n 2^n : n > 0\}$

	0	1	2	$\sqcup$	X	Y	Z
q_0	(q_1, X)	no	no	no	no	$(q_4, \rightarrow)$	no
q_1	$(q_1, \rightarrow)$	(q_2, Y)	no	no	$(q_1, \rightarrow)$	$(q_1, \rightarrow)$	no
q_2	no	$(q_2, \rightarrow)$	(q_3, Z)	no	no	$(q_2, \rightarrow)$	$(q_2, \rightarrow)$
q_3	$(q_3, \leftarrow)$	$(q_3, \leftarrow)$	$(q_3, \leftarrow)$	no	$(q_0, \rightarrow)$	$(q_3, \leftarrow)$	$(q_3, \leftarrow)$
q_4	no	no	no	yes	$(q_4, \rightarrow)$	$(q_4, \rightarrow)$	$(q_4, \rightarrow)$

11-8: Turing Machines

TM for $L = \{ww^R : w \in \{a, b\}^*\}$

Turing Machines

TM for $L = \{ww^R : w \in \{a,b\}^*\}$

	a	b	$\sqcup$	X	Z
q_0	(q_1, X)	(q_4, X)	yes	no	yes
q_1	$(q_1, \rightarrow)$	$(q_1, \rightarrow)$	$(q_2, \leftarrow)$	$(q_1, \rightarrow)$	$(q_2, \leftarrow)$
q_2	(q_3, Z)	no	no	no	no
q_3	$(q_3, \leftarrow)$	$(q_3, \leftarrow)$	no	$(q_0, \rightarrow)$	$(q_3, \leftarrow)$
q_4	$(q_4, \rightarrow)$	$(q_4, \rightarrow)$	$(q_5, \leftarrow)$	$(q_4, \rightarrow)$	$(q_5, \leftarrow)$
q_2	no	(q_3, Z)	no	no	no

11-10: Turing Machine Diagrams

- “Table Notation” for Turing Machines can be difficult to read
- Define small machines, use them to build larger machines
    - *Not* changing the definition of a TM, just using a more convenient notation

Turing Machine Diagrams

- Writing machines: $M_a = (K, \Sigma, \delta, s, H)$
  - $K = \{s, h\}$
  - $s = s$
  - $H = \{h\}$
  - $\delta = \{((s, b), (h, a)) : \forall b \in \Sigma$
- Writes an 'a' on the tape, and then halts.

11-12: Turing Machine Diagrams

- Moving Machines: $M_{\leftarrow} = (K, \Sigma, \delta, s, H)$
  - $K = \{s, h\}$
  - $s = s$
  - $H = \{h\}$
  - $\delta = \{((s, b), (h, \leftarrow)) : \forall b \in \Sigma$
- Moves the head to the left, and then halts.

11-13: Turing Machine Diagrams

- Connecting Diagrams:

$$\begin{tikzcd} & M_2 \\ M_1 \arrow[ur, "a"] \arrow[dr, "b"'] & \\ & M_3 \end{tikzcd}$$

- Run M_1 until it halts, and then examine the symbol under the read/write head
  - If the symbol is an 'a', execute M_2 until it halts
  - If the symbol is a 'b', execute M_3 until it halts

Turing Machine Diagrams

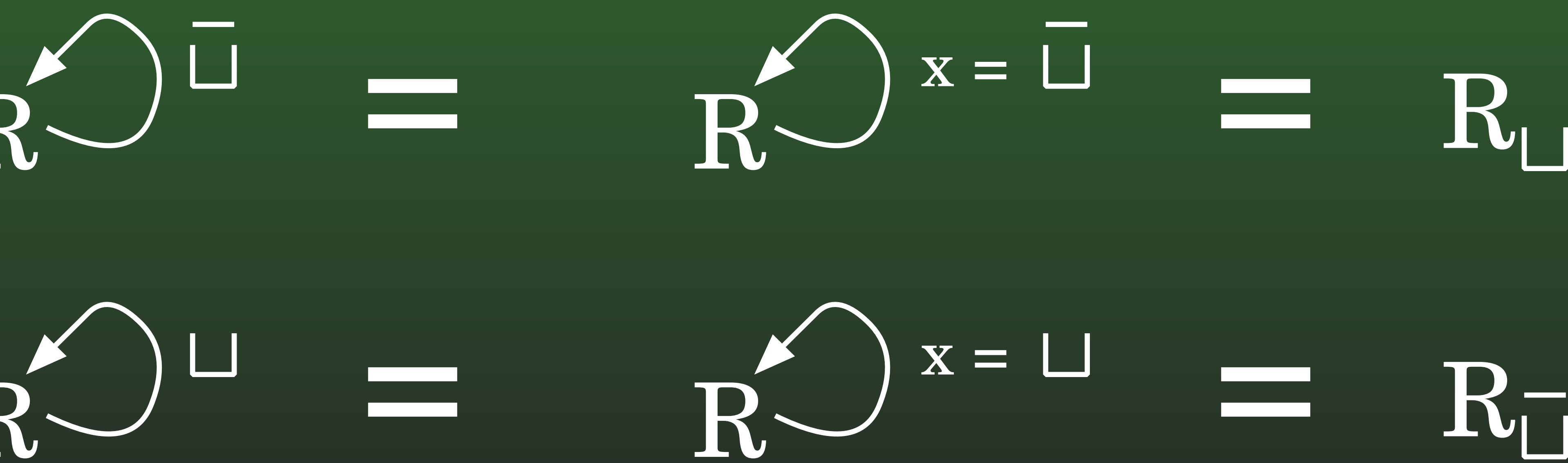
- Connecting Diagrams:

$$M_1 \xrightarrow{\quad a,b,c \quad} M_2$$

$$M_1 \longrightarrow M_2 \quad = \quad M_1 \xrightarrow{\quad \Sigma \quad} M_2$$

11-15: Turing Machine Diagrams

- Shorthand:
  - $M_a = a$
  - $M_\leftarrow = L$
  - $M_\rightarrow = R$
  - $R \rightarrow R \rightarrow a \rightarrow = RRa = R^2a$

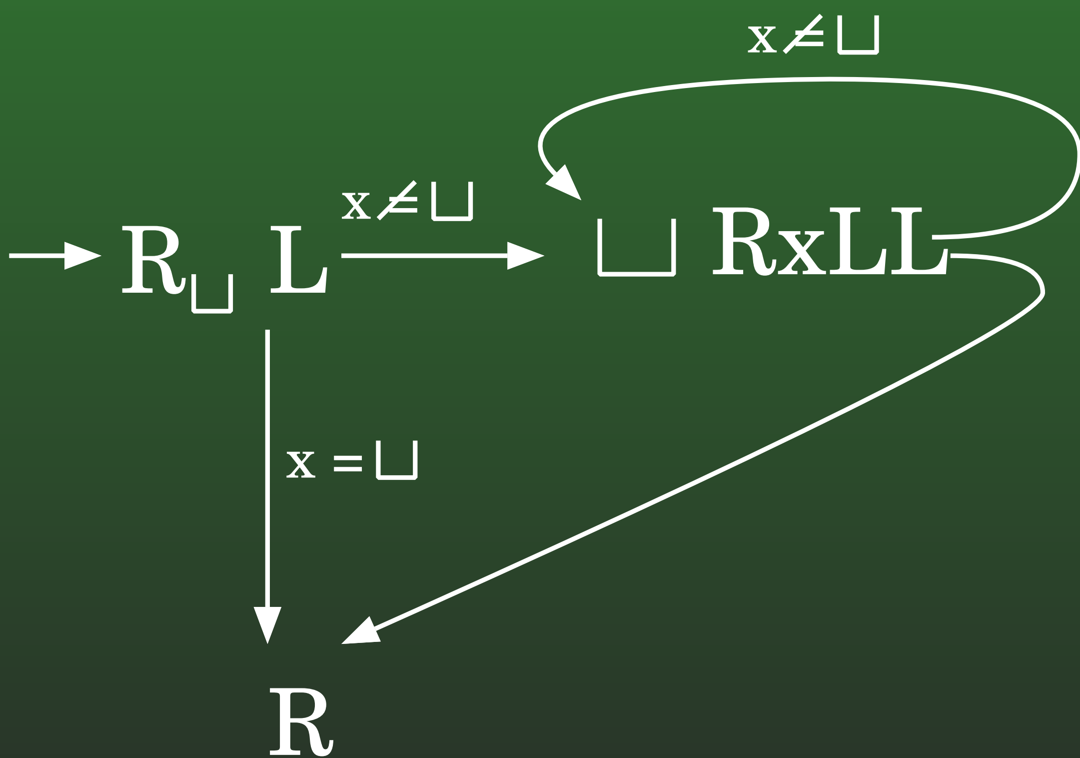


11-16: Turing Machine Diagrams

- Shift-right machine
  - Convert $\triangleright \underline{\sqcup} w$ to $\triangleright \sqcup \underline{\sqcup} w$

11-17: Turing Machine Diagrams

- Shift-right machine
  - Convert $\rhd\underline{\sqcup}w$ to $\rhd\sqcup\underline{\sqcup}w$

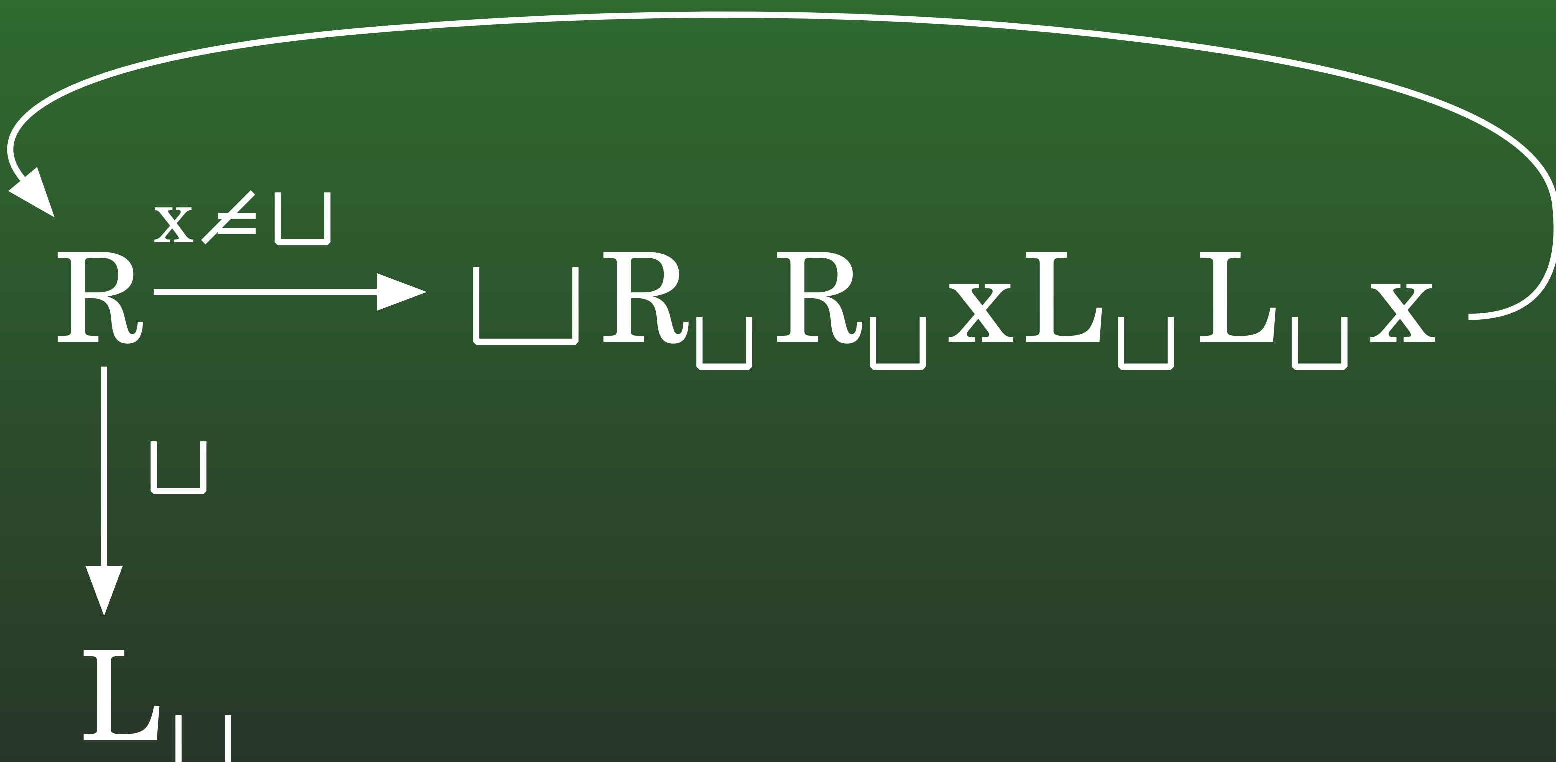


11-18: Turing Machine Diagrams

- Copy machine
  - Convert $\triangleright\underline{\sqcup}w$ to $\triangleright\underline{\sqcup}w \sqcup w$

11-19: Turing Machine Diagrams

- Copy machine
  - Convert $\triangleright\underline{\sqcup}w$ to $\triangleright\underline{\sqcup}w \sqcup w$



Turing Machine Diagrams

- Shift-left machine
  - Convert $\underline{\sqcup} w$ to $w\underline{\sqcup}$

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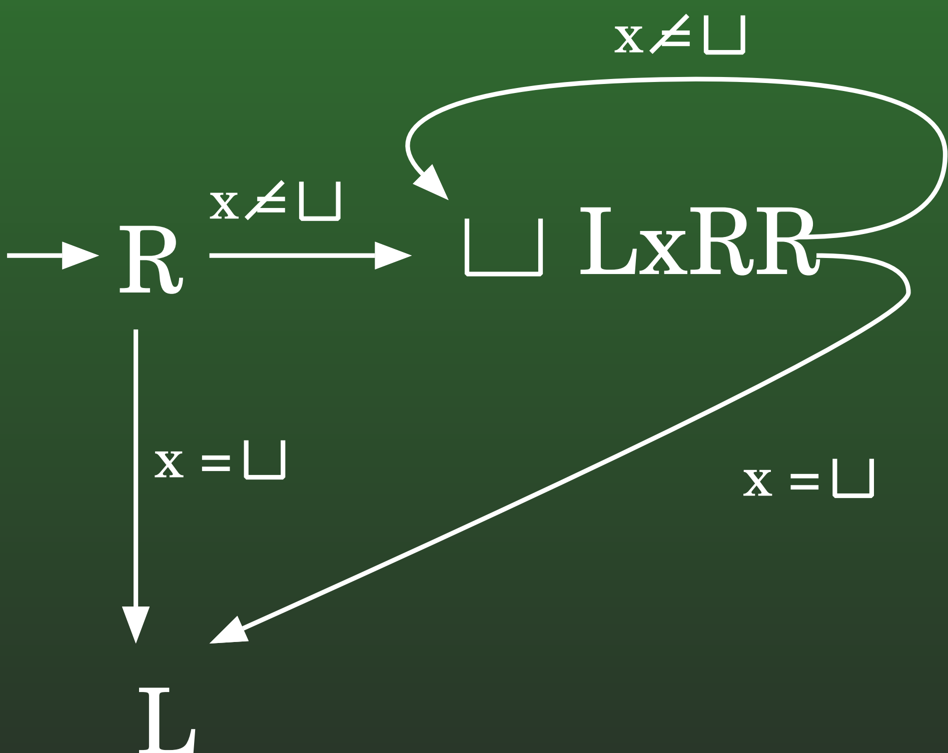


11-22: Turing Machine Diagrams

- Copy machine (part II)
  - Convert $\triangleright\underline{\sqcup}w$ to $\triangleright\underline{\sqcup}ww$
  - (Using other machines ....)

Turing Machine Diagrams

- Copy machine (part II)
  - Convert $\triangleright \underline{\sqcup} w$ to $\triangleright \underline{\sqcup} ww$

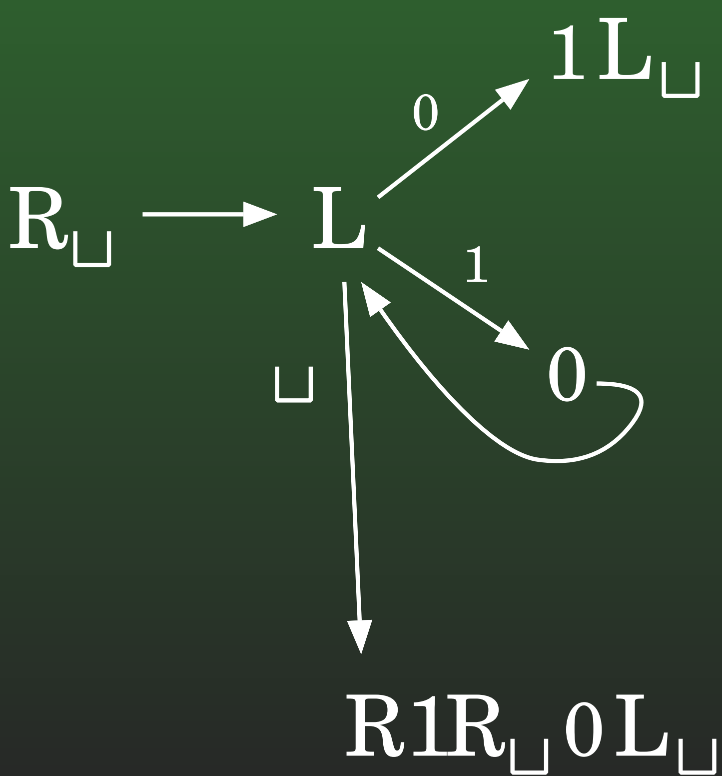
$$\mathrm{M}_{\mathrm{COPY}} \longrightarrow \mathrm{R}_{\sqcup} \longrightarrow \mathrm{M}_{\mathrm{LEFT\text{-}SHIFT}} \longrightarrow \mathrm{L}_{\sqcup}$$

11-24: Turing Machine Diagrams

- Successor
  - Convert $\triangleright\underline{\sqcup}w$ (where w is the binary representation of an integer) to $\triangleright\underline{\sqcup}v$ (where v is the binary representation of $(w+1)$
    $\triangleright\underline{\sqcup}11011 \Rightarrow \triangleright\underline{\sqcup}11100$
    $\triangleright\underline{\sqcup}1111 \Rightarrow \triangleright\underline{\sqcup}10000$

11-25: Turing Machine Diagrams

- Successor
  - Convert $\triangleright\underline{\sqcup}w$ (where w is the binary representation of an integer) to $\triangleright\underline{\sqcup}v$ (where v is the binary representation of $(w + 1)$



11-26: Turing Machine Diagrams

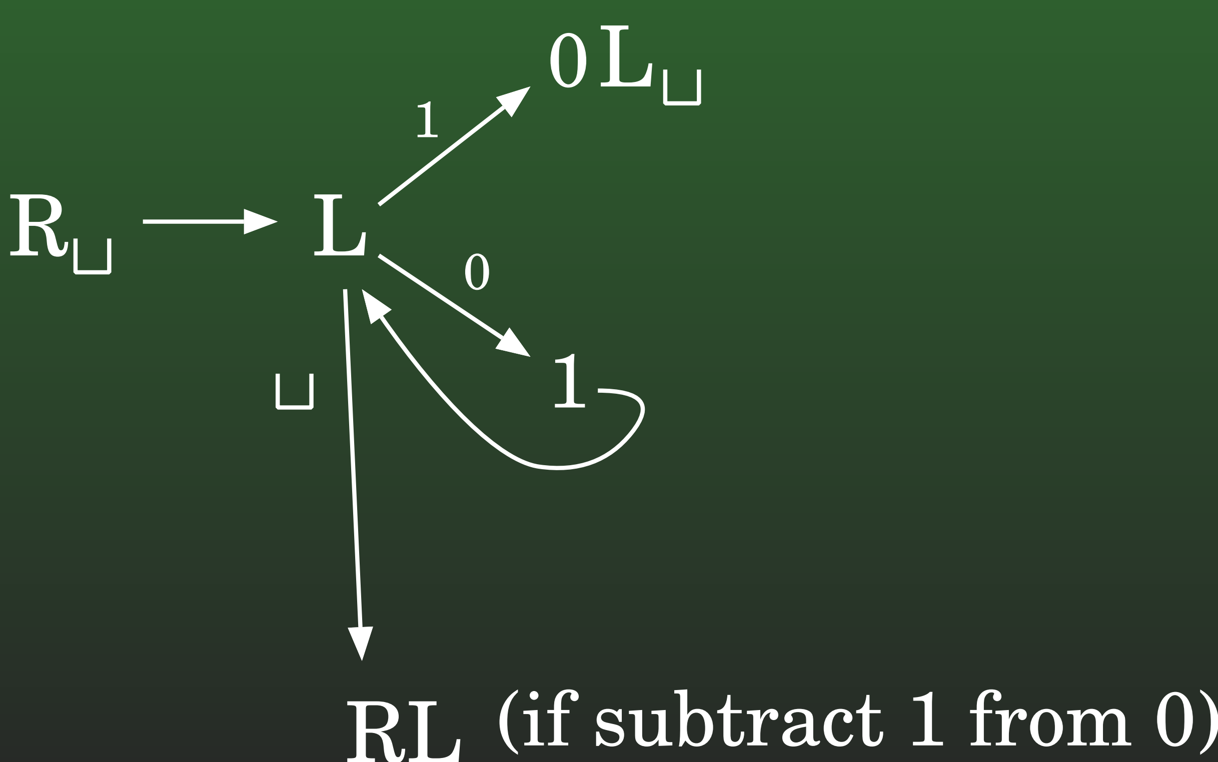
- Predecessor
  - Convert $\triangleright\underline{\sqcup}w$ (where w is the binary representation of an integer) to $\triangleright\underline{\sqcup}v$ (where v is the binary representation of $(w-1)$

    $\triangleright\underline{\sqcup}11100 \Rightarrow \triangleright\underline{\sqcup}11011$

    $\triangleright\underline{\sqcup}1111 \Rightarrow \triangleright\underline{\sqcup}10000$

11-27: Turing Machine Diagrams

- Predecessor
  - Convert $\triangleright \underline{\sqcup} w$ (where w is the binary representation of an integer) to $\triangleright \underline{\sqcup} v$ (where v is the binary representation of $(w-1)$

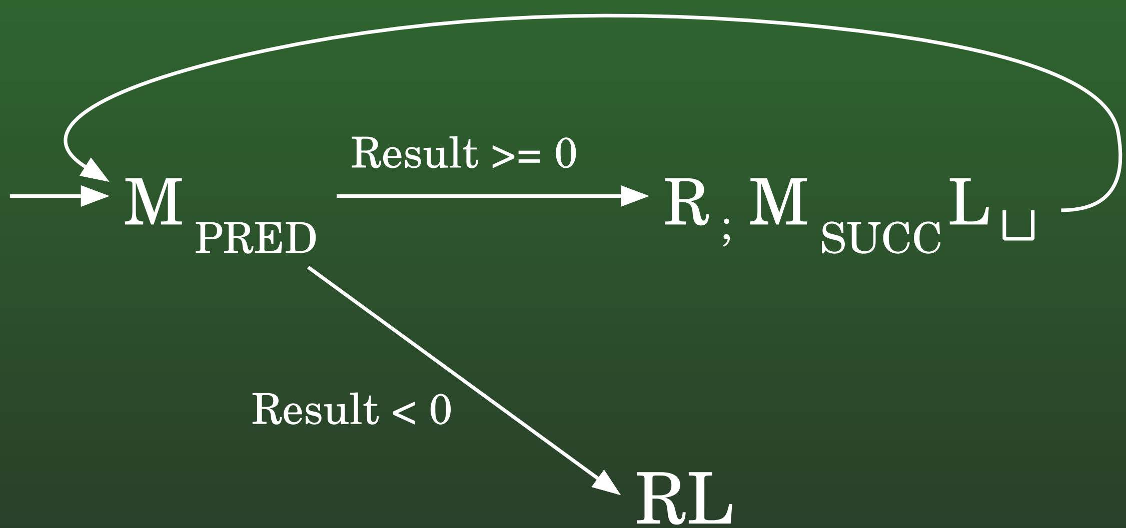


11-28: Turing Machine Diagrams

- Add
  - Convert $\triangleright\underline{\sqcup}w;v$ to $\triangleright\underline{\sqcup}w+v$
  - (first, convert to $\triangleright\underline{\sqcup}0\ldots0;w+v$)

Turing Machine Diagrams

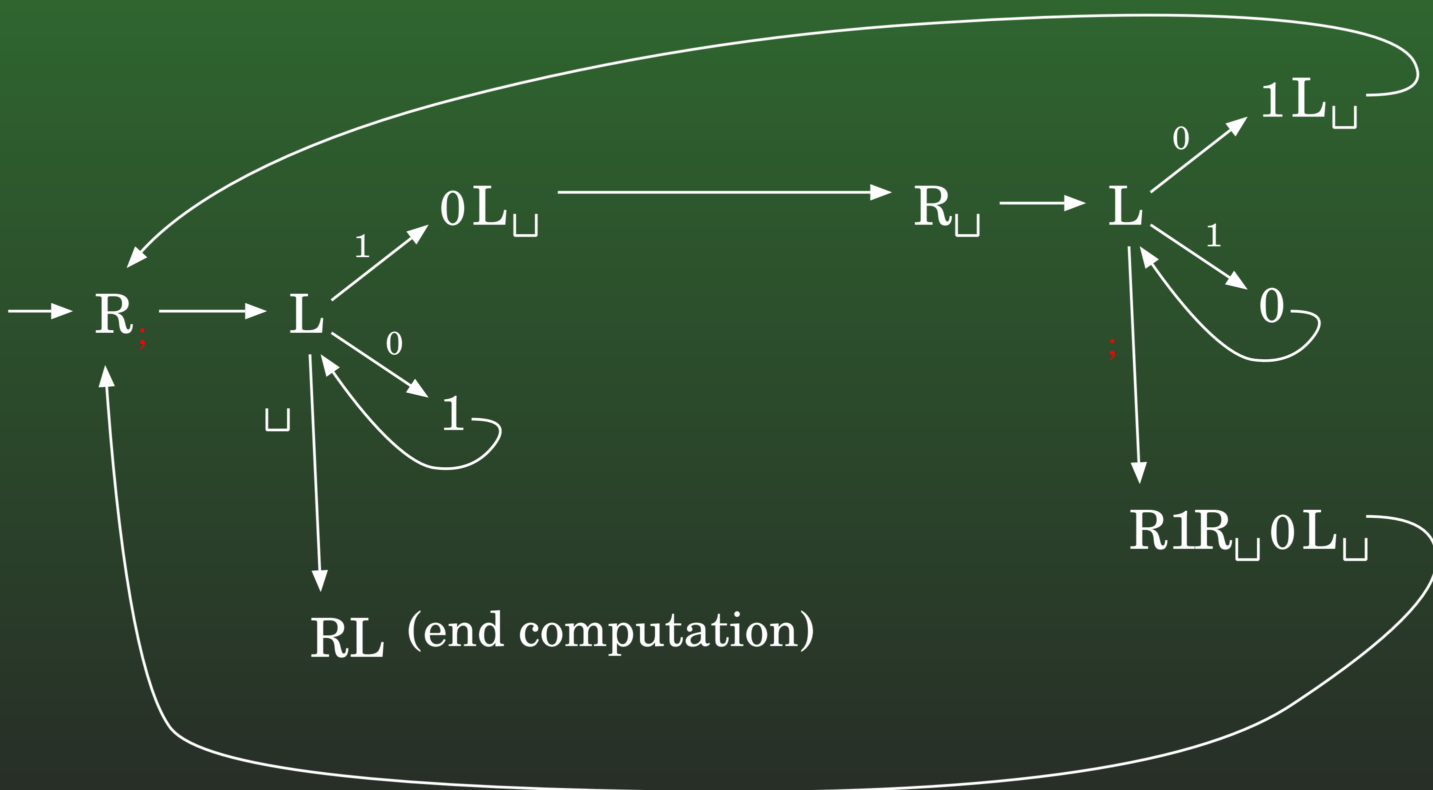
- Add
  - Convert $\triangleright\underline{\sqcup}w; v$ to $\triangleright\underline{\sqcup}w + v$
  - (first, convert to $\triangleright\underline{\sqcup}0 \ldots 0; w + v$)



(note – need to change M_{SUCC} so that it expects a ; instead of a $\sqcup$ at the beginning of the string)

11-30: Turing Machine Diagrams

- Add
  - Convert $\triangleright\underline{\sqcup}w;v$ to $\triangleright\underline{\sqcup}w+v$
  - (first, convert to $\triangleright\underline{\sqcup}0\ldots0;w+v$)

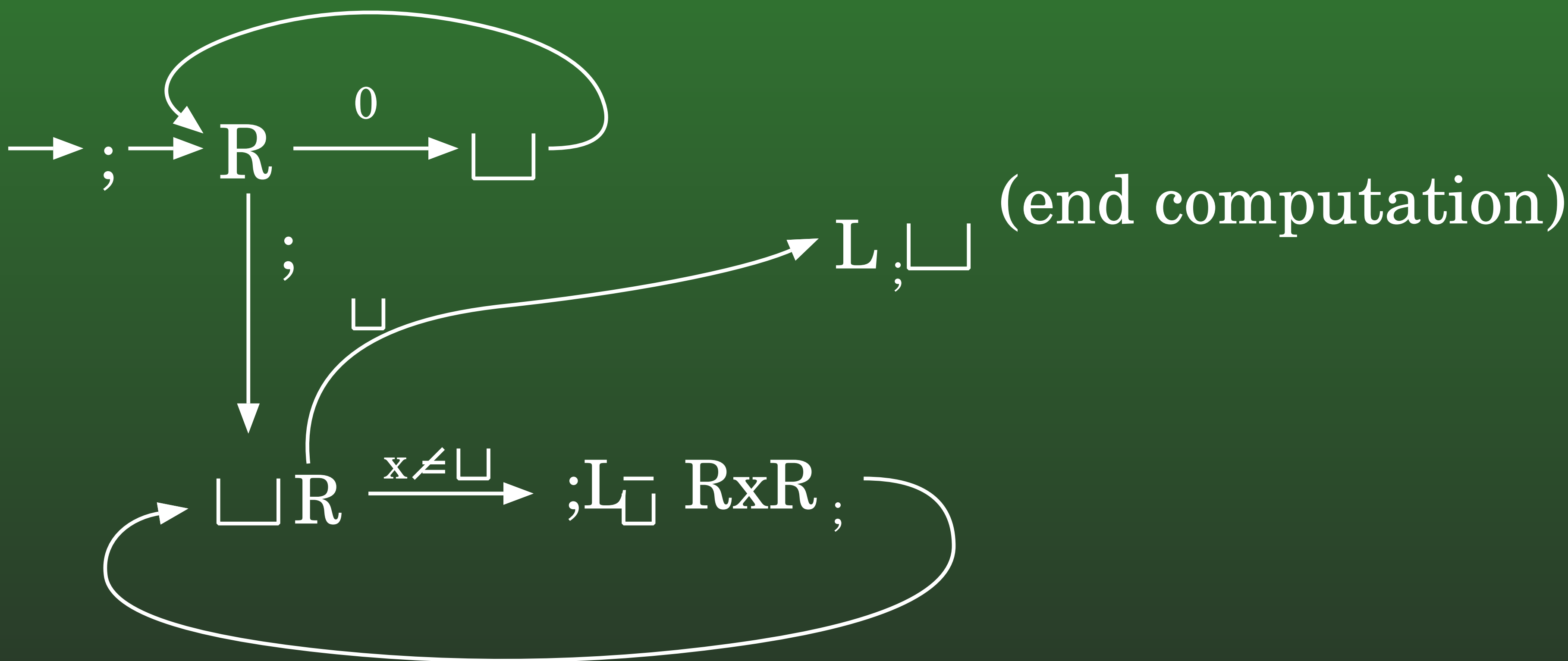


11-31: Turing Machine Diagrams

- Covert $\triangleright\underline{\sqcup}0\ldots0;w$ to $\triangleright\underline{\sqcup}w$, where $w \in \{0,1\}$

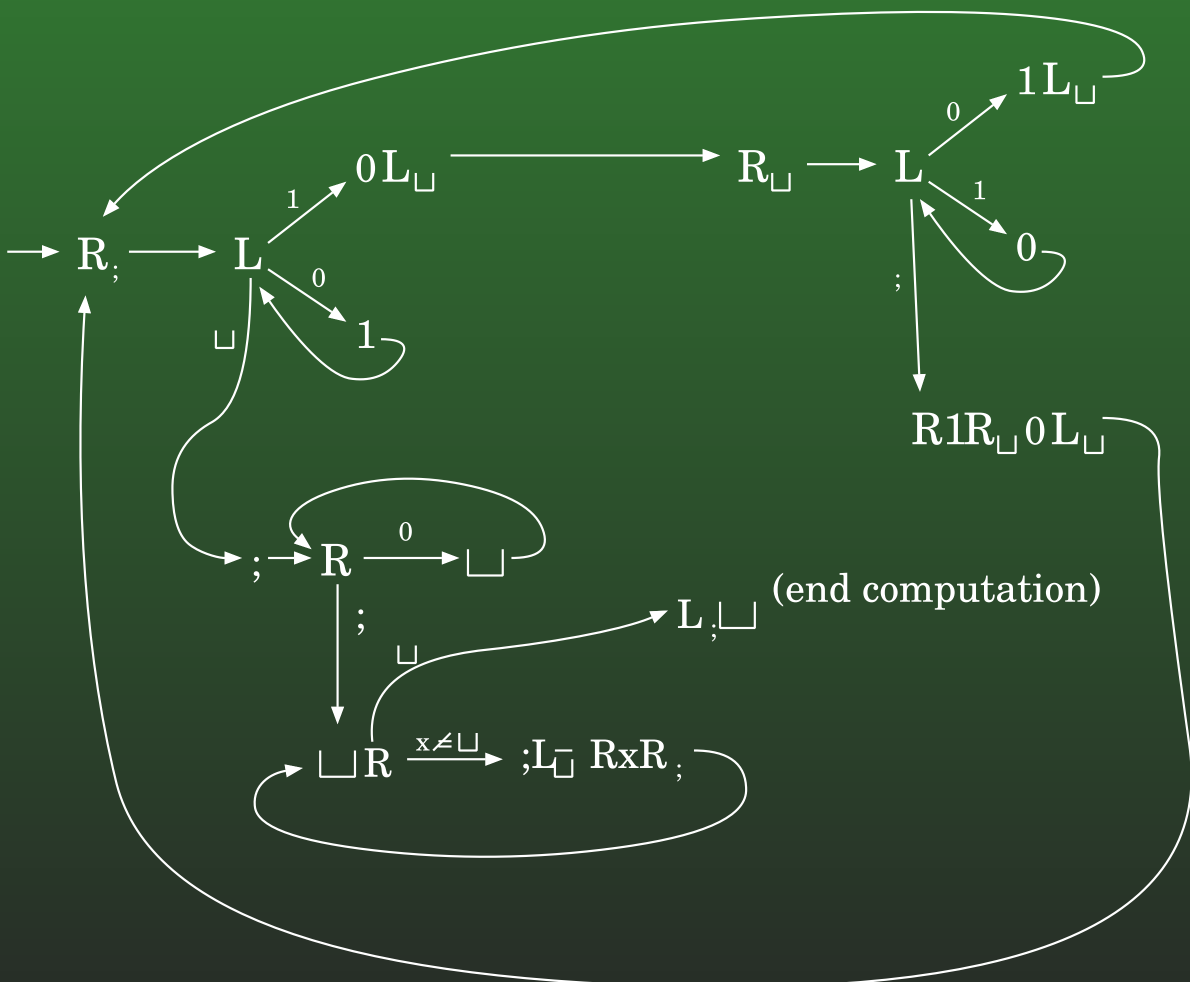
11-32: Turing Machine Diagrams

- Covert $\triangleright\underline{\sqcup}0\ldots0;w$ to $\triangleright\underline{\sqcup}w$, where $w \in \{0,1\}$



11-33: Turing Machine Diagrams

- Add: Convert $\triangleright\underline{\sqcup}w;v$ to $\triangleright\underline{\sqcup}w+v$

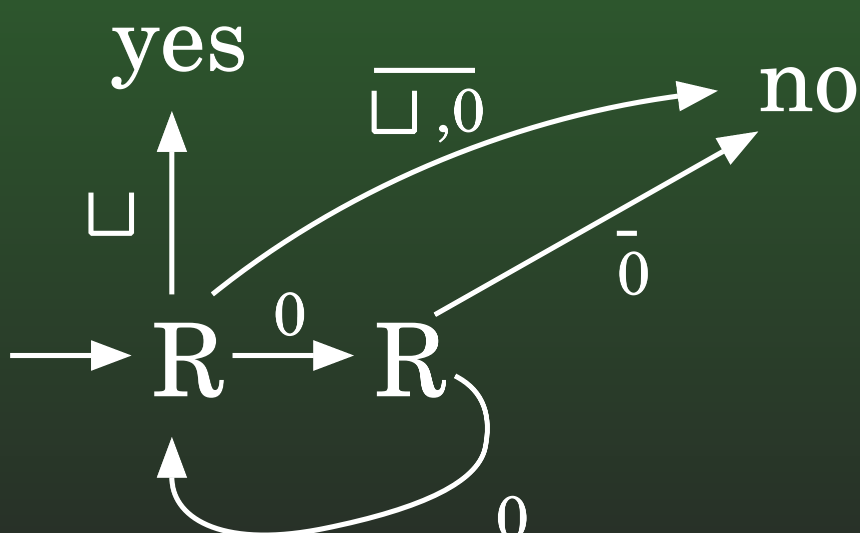


11-34: Turing Machine Diagrams

- We can add “yes” and “no” machines to our diagrams for machines that accept and reject strings
- Diagram for a Turing Machine that accepts the language $L = \{0^{2n} : n \geq 0\}$

Turing Machine Diagrams

- We can add "yes" and "no" machines to our diagrams for machines that accept and reject strings
- Diagram for a Turing Machine that accepts the language $L = \{0^{2n} : n \geq 0\}$



Turing Machine Diagrams

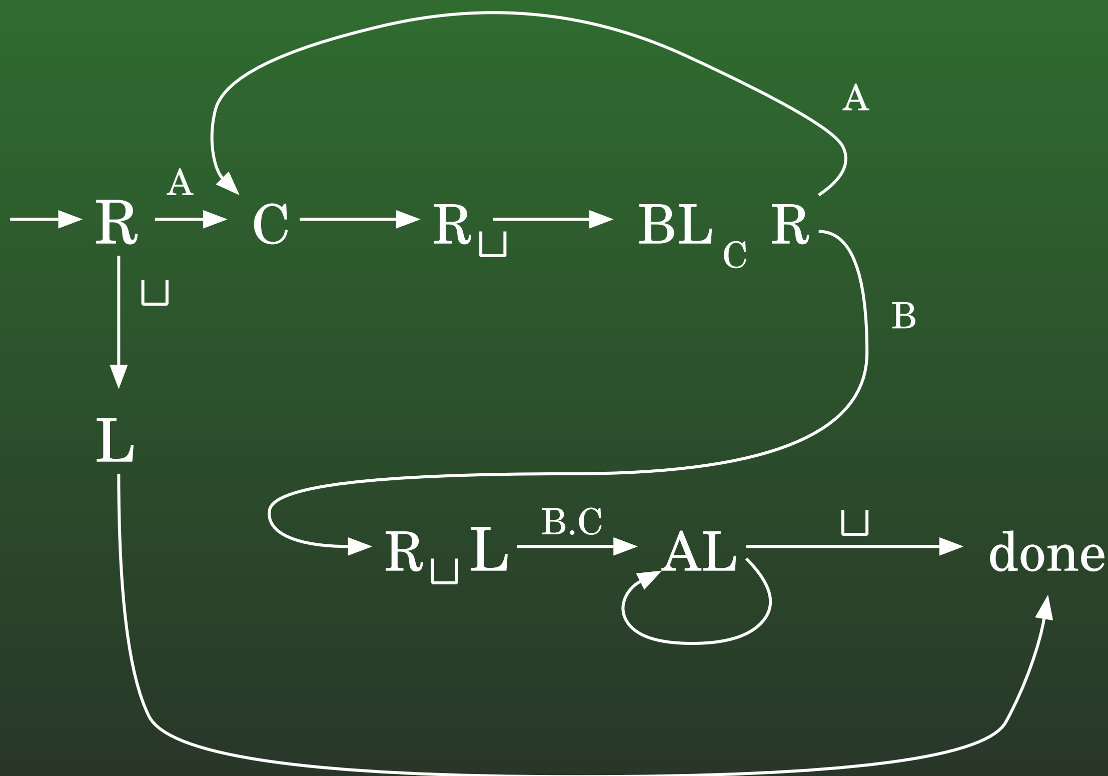
- Diagram for a Turing Machine that accepts the language $L = \{a^{2^n} : n \geq 0\}$

11-37: Turing Machine Diagrams

- Diagram for a Turing Machine that accepts the language $L = \{a^{2^n} : n \geq 0\}$
- First: Write a TM that doubles a string of A's (converts $w = A^n$ to $w' = A^{2n}$
  - (Can use other tape symbols if desired)

11-38: Turing Machine Diagrams

- Write a TM that doubles a string of A's (converts $w = A^n$ to $W = A^{2n}$



11-39: Turing Machine Diagrams

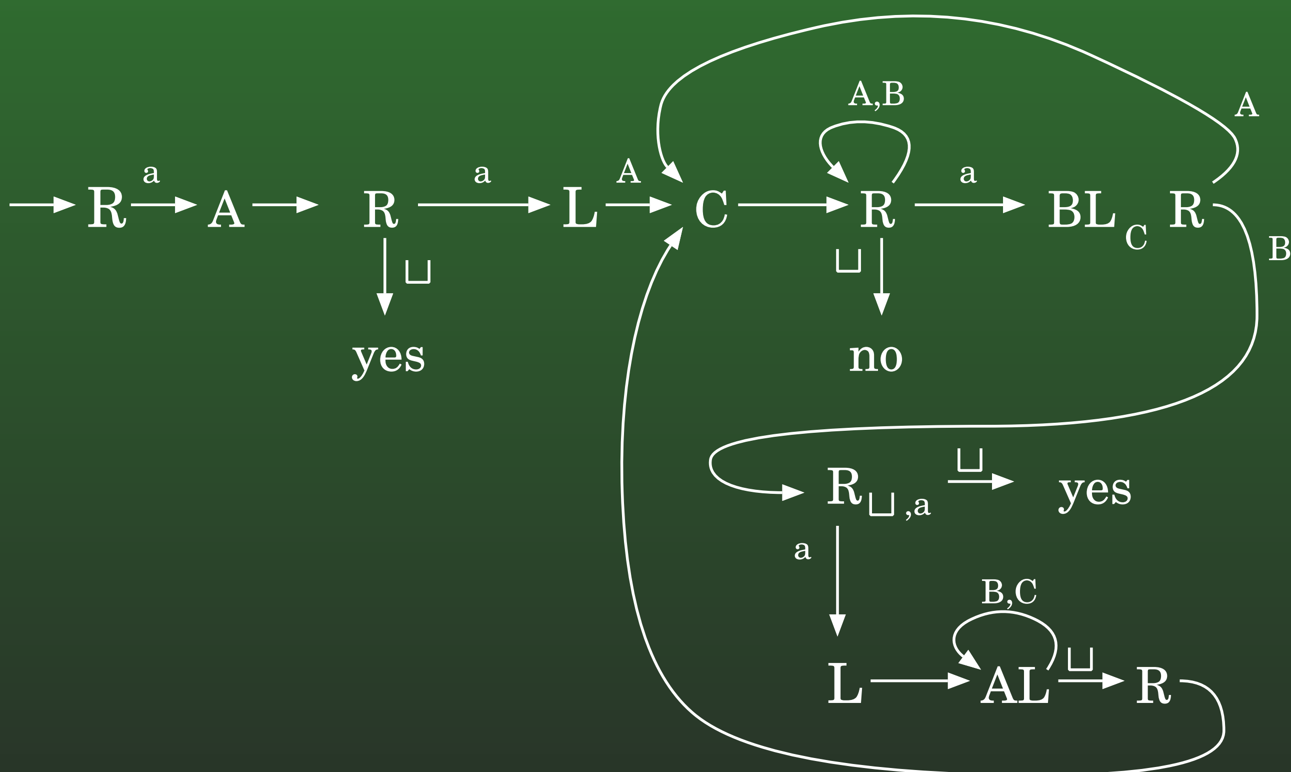
- Given a machine that converts $w = A^n$ to $w' = A^{2n}$, how can we accept the language $L = \{a^{2^n} : n \geq 0\}$?

11-40: Turing Machine Diagrams

- Given a machine that converts $w = A^n$ to $w' = A^{2n}$, how can we accept the language $L = \{a^{2^n} : n \geq 0\}$?
  - Check if the string is a. If so, accept.
  - Otherwise, overwrite the first a with an A.
  - Repeat:
    - Doulble the # of A's – if a $\sqcup$ is overwritten in this process, halt and reject
    - After doubling, check to see if the next symbol is a $\sqcup$. If so, halt and accept

11-41: Turing Machine Diagrams

- Diagram for a Turing Machine that accepts the language $L = \{a^{2^n} : n \geq 0\}$



Non-Halting TMs

- It is possible to create a Turing Machine that does not halt

$$\rightarrow R$$

- It is possible to create a Turing Machine that only halts on some inputs

$$\rightarrow R_a \rightarrow \text{yes}$$

Deciding TM

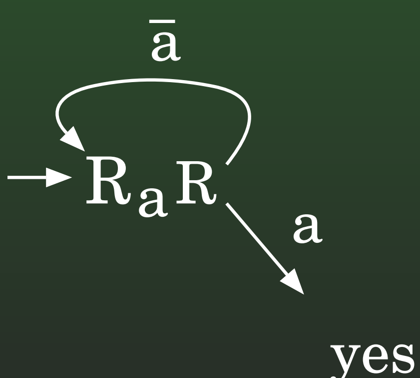
- A Turing Machine M *Decides* a language L if:
  - M halts on all inputs
  - M accepts all strings $w \in L$
  - M rejects all strings $w \notin L$
- We've already seen an example of Turing machines that decide languages (a^{2n}, a^{2^n})

11-44: Semi-Deciding TM

- A Turing Machine M *Semi-Decides* a language L if:
  - M halts on all strings $w \in L$
  - M accepts all strings $w \in L$
  - M *runs forever* on all strings $w \notin L$
- TM that semi-decides L = all strings over {a, b} that contain the substring aa

11-45: Semi-Deciding TM

- A Turing Machine M *Semi-Decides* a language L if:
  - M halts on all strings $w \in L$
  - M accepts all strings $w \in L$
  - M *runs forever* on all strings $w \notin L$
- TM that semi-decides L = all strings over {a, b} that contain the substring aa



11-46: Recursive Languages

- The *Recursive Languages* is the set of all languages that are decided by some Turing Machine
    - $L_{REC} = \{L : \exists$ Turing machine M, M decides $L\}$
- Is L_{REC} closed under complementation?
    - That is, if $L \in L_{REC}$, must $\overline{L} \in L_{REC}$?

11-47: Recursive Languages

- The *Recursive Languages* is the set of all languages that are decided by some Turing Machine
  - $L_{REC} = \{L : \exists$ Turing machine M, M decides $L\}$
- L_{REC} is closed under complementation?
  - $L \in L_{REC} \implies \overline{L} \in L_{REC}$ (flip yes/no states)

11-48: r.e. Languages

- The *Recursively Enumerable (r.e.) Languages* is the set of all languages that are semi-decided by some Turing Machine
  - $L_{re} = \{L : \exists$ Turing machine M, M semi-decides $L\}$
- Is $L_{REC} \subseteq L_{re}$?

11-49: r.e. Languages

- The *Recursively Enumerable (r.e.) Languages* is the set of all languages that are semi-decided by some Turing Machine
  - $L_{re} = \{L : \exists$ Turing machine M, M semi-decides $L\}$
- Is $L_{REC} \subseteq L_{re}$
  - Replace “no” states with non-halting machine

11-50: r.e. Languages

- The *Recursively Enumerable (r.e.) Languages* is the set of all languages that are semi-decided by some Turing Machine
  - $L_{re} = \{L : \exists$ Turing machine M, M semi-decides $L\}$
- Is $L_{re} \not\subseteq L_{REC}$
  - More on this later ...