08-0: **Context-Free Grammars**

- Set of Terminals (Σ)
- Set of Non-Terminals
- Set of Rules, each of the form:
  <Non-Terminal> → <Terminals & Non-Terminals>
- Special Non-Terminal – Initial Symbol

08-1: **Generating Strings with CFGs**

- Start with the initial symbol
- Repeat:
  - Pick any non-terminal in the string
  - Replace that non-terminal with the right-hand side of some rule that has that non-terminal as a left-hand side

  Until all elements in the string are terminals

08-2: **CFG Example**

$S \rightarrow aS$
$S \rightarrow Bb$
$B \rightarrow cB$
$B \rightarrow \epsilon$

Generating a string:

S	replace S with aS
aS	replace S wtih Bb
aBb	replace B wtih cB
$acBb$	replace B wtih ϵ
acb	Final String

08-3: **CFG Example**

$S \rightarrow aS$
$S \rightarrow Bb$
$B \rightarrow cB$
$B \rightarrow \epsilon$

Generating a string:

S	replace S with aS
aS	replace S wtih aS
aaS	replace S wtih Bb
$aaBb$	replace B wtih cB
$aacBb$	replace B wtih cB
$aaccBb$	replace B wtih ϵ
$aaccb$	Final String

08-4: **CFG Example**

$S \rightarrow aS$
$S \rightarrow Bb$
$B \rightarrow cB$
$B \rightarrow \epsilon$

Regular Expression equivalent to this CFG:

08-5: **CFG Example**

$S \to aS$
$S \to Bb$
$B \to cB$
$B \to \epsilon$

Regular Expression equivalent to this CFG:
a^*c^*b

08-6: **CFG Example**

CFG for $L = \{0^n1^n : n > 0\}$

08-7: **CFG Example**

CFG for $L = \{0^n1^n : n > 0\}$
$S \to 0S1$ or $S \to 0S1|01$
$S \to 01$

(note – can write:
$A \to \alpha$
$A \to \beta$
as
$A \to \alpha|\beta$)

(examples: 01, 0011, 000111) 08-8: **CFG Formal Definition**

$G = (V, \Sigma, R, S)$

- V = Set of symbols, both terminals & non-terminals
- $\Sigma \subset V$ set of terminals (alphabet for the language being described)
- $R \subset ((V - \Sigma) \times V^*)$ Finite set of rules
- $S \in (V - \Sigma)$ Start symbol

08-9: **CFG Formal Definition**

Example:
$S \to 0S1$
$S \to 01$

Set theory Definition:
$G = (V, \Sigma, R, S)$

- $V = \{S, 0, 1\}$
- $\Sigma \subset V = \{0, 1\}$
- $R \subset ((V - \Sigma) \times V^*) = \{(S, 0S0), (S, 01)\}$
- $S \in (V - \Sigma) = S$

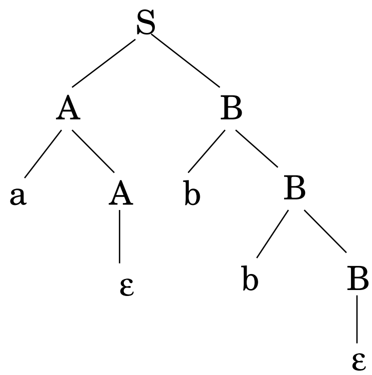
08-10: **Derivation**

A *Derivation* is a listing of how a string is generated – showing what the string looks like after every replacement.
$S \to AB$
$A \to aA|\epsilon$
$B \to bB|\epsilon$

$$\begin{aligned} S &\Rightarrow AB \\ &\Rightarrow aAB \\ &\Rightarrow aAbB \\ &\Rightarrow abB \\ &\Rightarrow abbB \\ &\Rightarrow abb \end{aligned}$$

08-11: **Parse Tree**

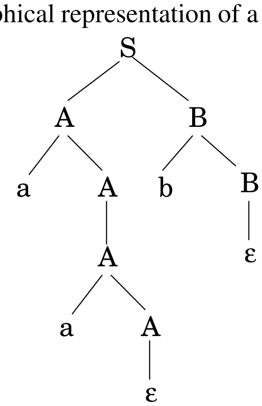
A *Parse Tree* is a graphical representation of a derivation.



$$\begin{aligned} S &\Rightarrow AB \\ &\Rightarrow aAB \\ &\Rightarrow aAbB \\ &\Rightarrow abB \\ &\Rightarrow abbB \\ &\Rightarrow abb \end{aligned}$$

08-12: **Parse Tree**

A *Parse Tree* is a graphical representation of a derivation.



$$\begin{aligned} S &\Rightarrow AB \\ &\Rightarrow AbB \\ &\Rightarrow aAbB \\ &\Rightarrow aaAbB \\ &\Rightarrow aaAb \\ &\Rightarrow aab \end{aligned}$$

08-13: **Fun with CFGs**

- Create a Context-Free Grammar for all strings over {a,b} which contain the substring "aba"

08-14: **Fun with CFGs**

- Create a Context-Free Grammar for all strings over {a,b} which contain the substring "aba"

$S \rightarrow A\text{aba}A$
$A \rightarrow \text{a}A$
$A \rightarrow \text{b}A$
$A \rightarrow \epsilon$

- Give a parse tree for the string: bbabaa

08-15: **Fun with CFGs**

- Create a Context-Free Grammar for all strings over {a,b} that begin or end with the substring bba (inclusive or)

08-16: **Fun with CFGs**

- Create a Context-Free Grammar for all strings over {a,b} that begin or end with the substring bba (inclusive or)

$S \rightarrow$ bba$A$
$S \rightarrow A$bba
$A \rightarrow$bA
$A \rightarrow$aA
$A \rightarrow \epsilon$

08-17: L_{CFG}

The Context-Free Languages, L_{CFG}, is the set of all languages that can be described by some CFG:

- $L_{CFG} = \{L : \exists$ CFG $G \wedge L[G] = L\}$

We already know $L_{CFG} \not\subseteq L_{REG}$ (why)?

- $L_{REG} \subset L_{CFG}$?

08-18: $L_{REG} \subseteq L_{CFG}$

We will prove $L_{REG} \subseteq L_{CFG}$ in two different ways:

- Prove by induction that, given any regular expression r, we create a CFG G such that $L[G] = L[r]$
- Given any NFA M, we create a CFG G such that $L[G] = L[M]$

08-19: $L_{REG} \subseteq L_{CFG}$

- To Prove: Given any regular expression r, we can create a CFG G such that $L[G] = L[r]$
- By induction on the structure of r

08-20: $L_{REG} \subseteq L_{CFG}$

Base Cases:

- r = a, a $\in \Sigma$

08-21: $L_{REG} \subseteq L_{CFG}$

Base Cases:

- r = a, a $\in \Sigma$

$S \rightarrow$ a

08-22: $L_{REG} \subseteq L_{CFG}$

Base Cases:

- $r = \epsilon$

08-23: $L_{REG} \subseteq L_{CFG}$

Base Cases:

- $r = \epsilon$

$S \rightarrow \epsilon$

08-24: $L_{REG} \subseteq L_{CFG}$

Base Cases:

- $r = \emptyset$

08-25: $L_{REG} \subseteq L_{CFG}$
Base Cases:

- $r = \emptyset$

$S \rightarrow SS$
08-26: $L_{REG} \subseteq L_{CFG}$
Recursive Cases:

- $r = (r_1 r_2)$

$L[G_1] = L[r_1]$, Start symbol of $G_1 = S_1$
$L[G_2] = L[r_2]$, Start symbol of $G_2 = S_2$
08-27: $L_{REG} \subseteq L_{CFG}$
Recursive Cases:

- $r = (r_1 r_2)$

$L[G_1] = L[r_1]$, Start symbol of $G_1 = S_1$
$L[G_2] = L[r_2]$, Start symbol of $G_2 = S_2$

G = all rules from G_1 and G_2, plus plus new non-terminal S, and new rule:

$S \rightarrow S_1 S_2$

New start symbol S
08-28: $L_{REG} \subseteq L_{CFG}$
Recursive Cases:

- $r = (r_1 + r_2)$

$L[G_1] = L[r_1]$, Start symbol of $G_1 = S_1$
$L[G_2] = L[r_2]$, Start symbol of $G_2 = S_2$
08-29: $L_{REG} \subseteq L_{CFG}$
Recursive Cases:

- $r = (r_1 + r_2)$

$L[G_1] = L[r_1]$, Start symbol of $G_1 = S_1$
$L[G_2] = L[r_2]$, Start symbol of $G_2 = S_2$

G = all rules from G_1 and G_2, plus new non-terminal S, and new rules:

$S \rightarrow S_1$
$S \rightarrow S_2$

Start symbol = S
08-30: $L_{REG} \subseteq L_{CFG}$
Recursive Cases:

- $r = (r_1^*)$

$L[G_1] = L[r_1]$, Start symbol of $G_1 = S_1$

08-31: $L_{REG} \subseteq L_{CFG}$

Recursive Cases:

- $r = (r_1^*)$

$L[G_1] = L[r_1]$, Start symbol of $G_1 = S_1$

G = all rules from G_1, plus new non-terminal S, and new rules:

$S \rightarrow S_1 S$
$S \rightarrow \epsilon$

Start symbol = S

(Example) 08-32: $L_{REG} \subseteq L_{CFG}$ **II**

- Given any NFA
  - $M = (K, \Sigma, \Delta, s, F)$
- Create a grammar
  - $G = (V, \Sigma, R, S)$ such that $L[G] = L[M]$
- Idea: Derivations like "backward NFA configurations", showing past instead of future
  - Example for all strings over {a, b} that contain aa, not bb

08-33: $L_{REG} \subseteq L_{CFG}$ **II**

- $M = (K, \Sigma, \Delta, s, F)$
- $G = (V, \Sigma', R, S)$
  - V
  - Σ'
  - R
  - S

08-34: $L_{REG} \subseteq L_{CFG}$ **II**

- $M = (K, \Sigma, \Delta, s, F)$
- $G = (V, \Sigma', R, S)$
  - $V = K \bigcup \Sigma$
  - $\Sigma' = \Sigma$
  - $R = \{(q_1 \rightarrow a q_2) : q_1, q_2 \in K$ (and V), $a \in \Sigma, ((q_1, a), q_2) \in \Delta\} \cup \{(q \rightarrow \epsilon) : q \in F\}$
  - $S = s$

(Example)

08-35: **CFG – Ambiguity**

- A CFG is *ambiguous* if there exists at least one string generated by the grammar that has ¿ 1 different parse tree
- Previous CFG is ambiguous (examples)
  $S \rightarrow A\text{aba}A$
  $A \rightarrow \text{a}A$
  $A \rightarrow \text{b}A$
  $A \rightarrow \epsilon$

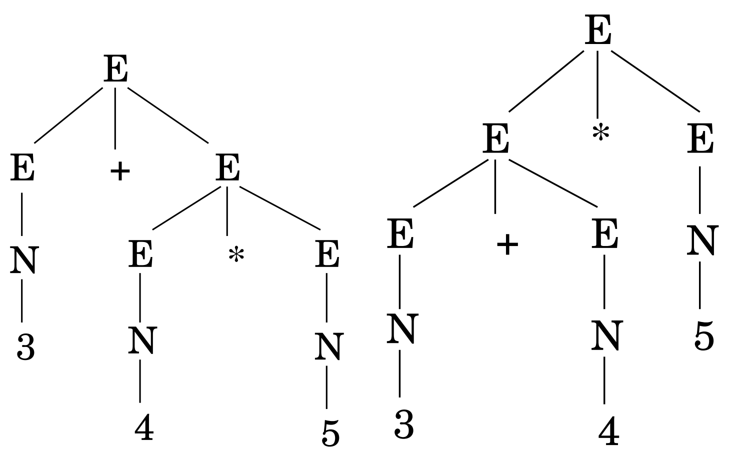
08-36: **CFG – Ambiguity**

- Consider the following CFG:

$E \rightarrow E + E | E - E | E * E | N$
$N \rightarrow 0|1|2|3|4|5|6|7|8|9$

- Is this CFG ambiguous?
- Why is this a problem?

08-37: **CFG – Ambiguity**
$E \rightarrow E + E | E - E | E * E | N$
$N \rightarrow 0|1|2|3|4|5|6|7|8|9$



08-38: **CFG – Ambiguity**
$E \rightarrow E + E | E - E | E * E | N$
$N \rightarrow 0|1|2|3|4|5|6|7|8|9$

- If all we care about is removing ambiguity, there is a (relatively) easy way to make this unambiguous (make all operators right-associative)

08-39: **CFG – Ambiguity**
$E \rightarrow E + E | E - E | E * E | N$
$N \rightarrow 0|1|2|3|4|5|6|7|8|9$
Non-ambiguous:
$E \rightarrow N | N + E | N - E | N * E$
$N \rightarrow 0|1|2|3|4|5|6|7|8|9$

- If we were writing a compiler, would this be a good CFG?
- How can we get correct associativity

08-40: **CFG – Ambiguity**

- Ambiguous:

  $E \to E + E|E - E|E * E|N$
  $N \to 0|1|2|3|4|5|6|7|8|9$

- Unambiguous:

  $E \to E + T|E - T|T$
  $T \to T * N|N$
  $N \to 0|1|2|3|4|5|6|7|8|9$

Can add parentheses, other operators, etc. (More in Compilers)

08-41: **Fun with CFGs**

- Create a CFG for all strings over $\{(,)\}$ that form balanced parenthesis
  - ()
  - ()()
  - (()())((()()()))
  - (((())))

08-42: **Fun with CFGs**

- Create a CFG for all strings over $\{(,)\}$ that form balanced parenthesis

  $S \to (S)$
  $S \to SS$
  $S \to \epsilon$

- Is this grammar ambiguous?

08-43: **Fun with CFGs**

- Create a CFG for all strings over $\{(,)\}$ that form balanced parenthesis

  $S \to (S)$
  $S \to SS$
  $S \to \epsilon$

- Is this grammar ambiguous?
  - YES! (examples)

08-44: **Fun with CFGs**

- Create an *unambiguous* CFG for all strings over $\{(,)\}$ that form balanced parenthesis

08-45: **Fun with CFGs**

- Create an *unambiguous* CFG for all strings over $\{(,)\}$ that form balanced parenthesis

$S \to AS$
$S \to \epsilon$
$A \to (S)$

08-46: **Ambiguous Languages**

- A language L is ambiguous if all CFGs G that generate it are ambiguous
- Example:
  - $L_1 = \{a^i b^i c^j d^j | i, j > 0\}$
  - $L_2 = \{a^i b^j c^j d^i | i, j > 0\}$
  - $L_3 = L_1 \cup L_2$
- L_3 is inherently ambiguous

(Create a CFG for L_3) 08-47: **Ambiguous Languages**

- $L_1 = \{a^i b^i c^j d^j | i, j > 0\}$
- $L_2 = \{a^i b^j c^j d^i | i, j > 0\}$
- $L_3 = L_1 \cup L_2$

$S \rightarrow S_1 | S_2$
$S_1 \rightarrow AB$
$A \rightarrow aAb | ab$
$B \rightarrow cBd | cd$
$S_2 \rightarrow aS_2d | aCd$
$C \rightarrow bCc | bc$

What happens when $i = j$? 08-48: **(More) Fun with CFGs**

- Create an CFG for all strings over {a, b} that have the same number of a's as b's (can be ambiguous)

08-49: **(More) Fun with CFGs**

- Create an CFG for all strings over {a, b} that have the same number of a's as b's (can be ambiguous)

$S \rightarrow aSb$
$S \rightarrow bSa$
$S \rightarrow SS$
$S \rightarrow \epsilon$

08-50: **(More) Fun with CFGs**

- Create an CFG for $L = \{ww^R : w \in (a + b)^*\}$

08-51: **(More) Fun with CFGs**

- Create an CFG for $L = \{ww^R : w \in (a + b)^*\}$

$S \rightarrow aSa$
$S \rightarrow bSb$ 08-52: **(More) Fun with CFGs**
$S \rightarrow \epsilon$

- Create an CFG for all palindromes over $\{a, b\}$. That is, create a CFG for:
  - $L = \{w : w \in (a + b)^*, w = w^R\}$

08-53: **(More) Fun with CFGs**

- Create an CFG for all palindromes over $\{a, b\}$. That is, create a CFG for:

- $L = \{w : w \in (a+b)^*, w = w^R\}$

$S \rightarrow aSa$
$S \rightarrow bSb$
$S \rightarrow \epsilon$
$S \rightarrow a$
$S \rightarrow b$

08-54: **(More) Fun with CFGs**

- Create an CFG for $L = \{a^i b^j c^k : j > i + k\}$

08-55: **(More) Fun with CFGs**

- Create an CFG for $L = \{a^i b^j c^k : j > i + k\}$

HINT: We may wish to break this down into 3 different langauges ...

08-56: **(More) Fun with CFGs**

- Create an CFG for $L = \{a^i b^j c^k : j > i + k\}$

$S \rightarrow ABC$
$A \rightarrow aAb$
$A \rightarrow \epsilon$
$B \rightarrow bB$
$B \rightarrow b$
$C \rightarrow bCc|\epsilon$

08-57: **(More) Fun with CFGs**

- Create an CFG for all strings over $\{0, 1\}$ that have the an even number of 0's and an odd number of 1's.
  - *HINT:* It may be easier to come up with 4 CFGs – even 0's, even 1's, odd 0's odd 1's, even 0's odd 1's, odd 1's, even 0's – and combine them ...

08-58: **(More) Fun with CFGs**

- Create an CFG for all strings over $\{0, 1\}$ that have the an even number of 0's and an odd number of 1's.

S_1 = Even 0's Even 1's
S_2 = Even 0's Odd 1's
S_3 = Odd 0's Even 1's
S_4 = Odd 0's Odd 1's

$S_1 \rightarrow 0S_3|1S_2$
$S_2 \rightarrow 0S_4|1S_1$
$S_3 \rightarrow 0S_1|1S_4$
$S_4 \rightarrow 0S_2|1S_3$