Algorithmic Learning Theory

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The Halting Set

Definition. $\mathbb{K} = \{e \in \mathbb{N} \mid \varphi_e(e)\downarrow\}$.

The Halting Set is computably enumerable.

- It is the domain of a PCF.
- It is the range of a CF (such a function enumerates it).

Theorem. The Halting Set is NOT computable.

Indexed families of sets

An indexed family of sets $\{A_i \mid i \in \mathbb{N}\} \subseteq CE$ is *computably indexable* there is a computable function f that, given i as input, outputs a program number for set A_i.

So, $A_i = W_{f(i)}$.

Example. Let $B_n = \{n\}$.

Example. Let $A_i = \{n \in \mathbb{N} \mid n \neq i\}$.

Indexed families of sets

Exercise. Let $D = \{d_0, d_1, \ldots, d_k\}$ be a non-empty finite set. Let

$$i_D = \sum_{j=0}^{k} 2^{d_j}.$$

Let A_i be the finite set encoded by i as in the scheme above.

Find a computable function that, for each i, outputs an index for A_i.

Identification: Basic concepts

Definitions.

1. A *language* is a c.e. set.
2. A *text* is an infinite sequence of natural numbers.
3. The set of numbers appearing in text t is denoted $rng(t)$.
4. We say t *is a text for language* L if $rng(t) = L$.
5. The set of all possible texts is denoted $\mathcal{T}$.

Identification: Basic concepts

- $t = (0, 0, 2, 2, 4, 4, 6, 6, \ldots)$ is a text for $L = \{2n \mid n \in \mathbb{N}\}$.
- So is $s = (2, 0, 6, 4, 10, 8, \ldots)$.
- $t_4 = 2$, and $\overline{t_4} = (0, 0, 2, 2)$.
- $t_0 = (\ )$.

Notations: Let $t \in \mathcal{T}$ and $n \in \mathbb{N}$.

- $\overline{t_n}$ denotes the first n members of t.
- t_n is the nth member of t.

Identification: Basic concepts

Definitions.

1. SEQ is the set of all finite sequences.
2. $\mathcal{T}$ is the set of all infinite sequences.
3. For $\sigma \in SEQ$, we write $lh(\sigma)$ for the length of σ, and $rng(\sigma)$ for the *set* of members of σ.
4. For $\sigma \in SEQ$ and $t \in \mathcal{T}$, we say $\sigma \in t$ if and only if $\sigma = \overline{t_{lh(\sigma)}}$.
5. We can identify members of SEQ with numbers: $\langle\sigma\rangle = \langle lh(\sigma), \langle\sigma_1\langle\sigma_2, \ldots\rangle\rangle\rangle$

Identification: Basic concepts

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- So is $s = (2, 0, 6, 4, 10, 8, \ldots)$.
- $t_4 = 2$, and $\overline{t_4} = (0, 0, 2, 2)$.
- $t_0 = (\ )$.
- If $\sigma = (0, 0, 2, 2, 4)$, then $lh(\sigma) = 5$ and $rng(\sigma) = \{0, 2, 4\}$.
- If $\sigma = (2, 0)$, then $\langle \sigma \rangle = \langle 2, \langle 2, 0 \rangle \rangle = \frac{1}{2}(2 + \langle 2, 0 \rangle)(2 + \langle 2, 0 \rangle + 1) + \langle 2, 0 \rangle$.
  Now, $\langle 2, 0 \rangle = \frac{1}{2}(2 + 0)(2 + 0 + 1) + 0 = 3$, so if we plug in $\langle \sigma \rangle = 18$

Identification: Basic concepts

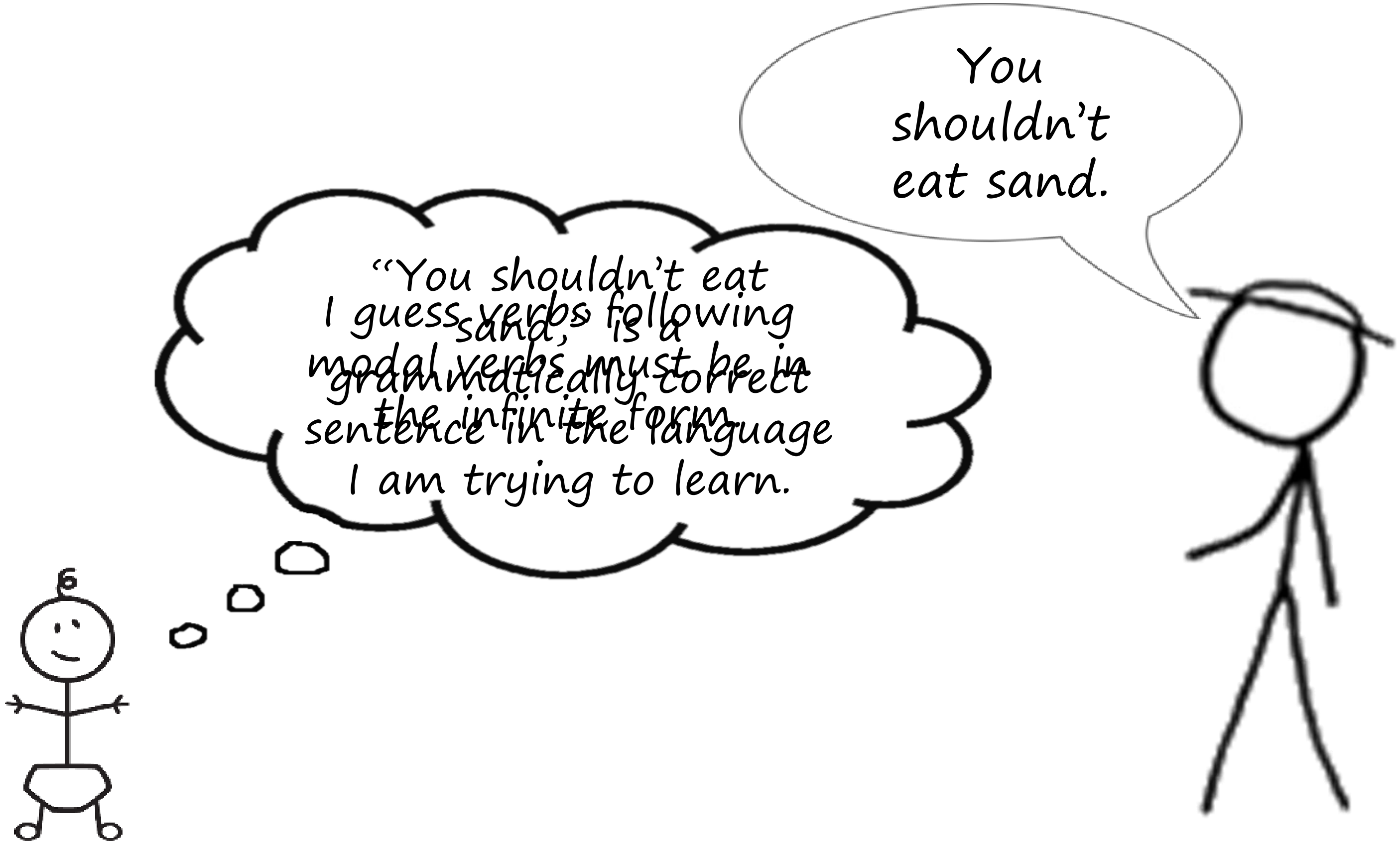
Definition. A *learning function* is an element $\mathcal{F}$, i.e., a function from $\mathbb{N}$ to $\mathbb{N}$, where the domain is typically thought of as SEQ. It may be partial or total, computable or not.

$$\langle(1,3,7,5,4,6)\rangle \longrightarrow \text{LEARNER} \longrightarrow e$$

Learning Languages

A child learning a language is playing the role of the learner in a more complex variant of this game.

- A language is a set of grammatically correct sentences.
- The child will be given clues in the form of sentences that are in that set.
- The child will try to figure out what is in that set (by guessing the grammatical rules of the language).



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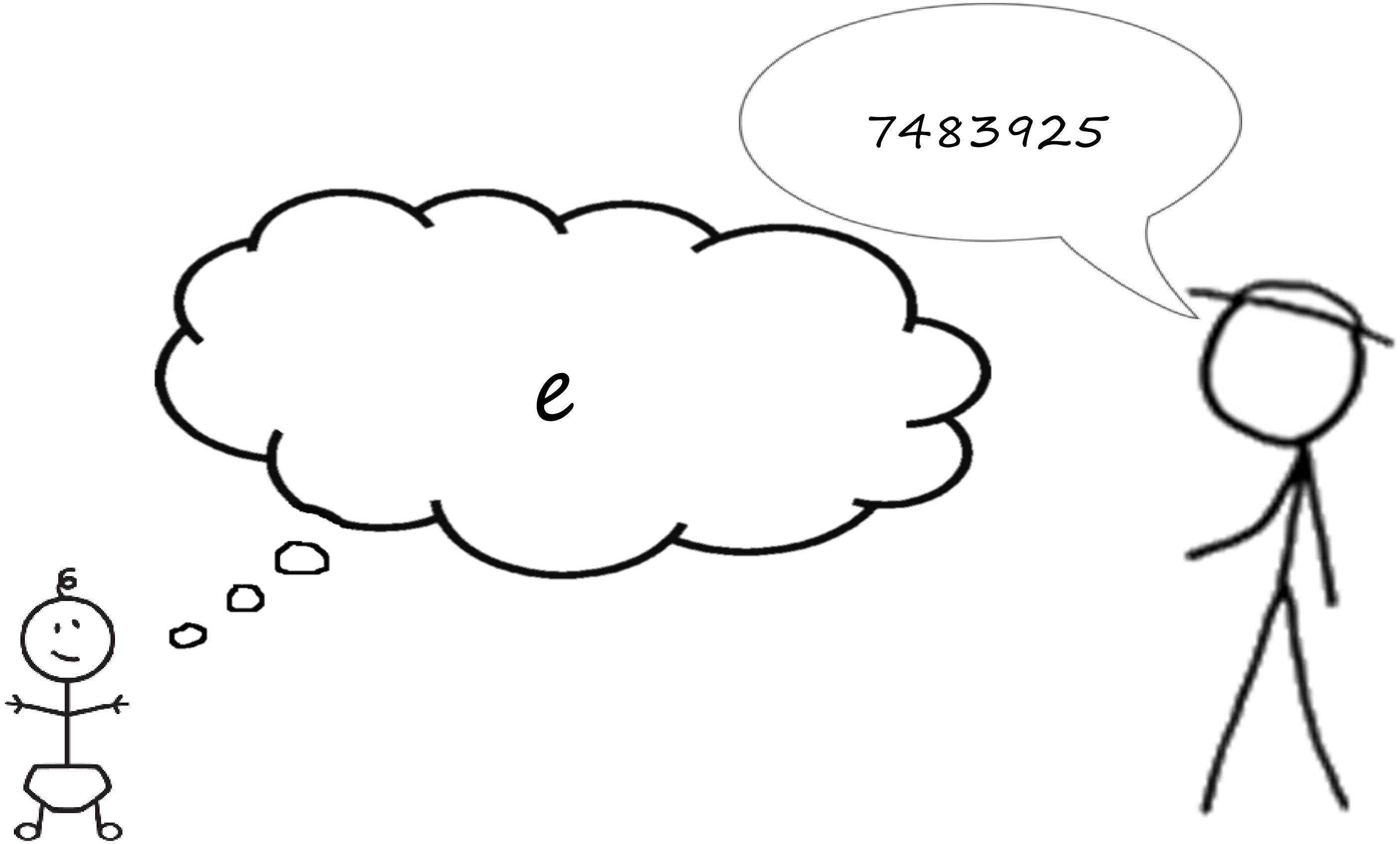

Identification of texts

Definitions. Let $f \in \mathcal{F}$ and $t \in \mathcal{T}$.

1. f is said to be *defined on* t just in case $f(\overline{t_n}) \downarrow$ for all $n \in \mathbb{N}$.
2. Let $e \in \mathbb{N}$. f is said to *converge on* t *to* e just in case f is defined on t and for all but finitely many $n \in \mathbb{N}$, $f(\overline{t_n}) = e$.
3. f is said to *identify* t just in case there is an $e \in \mathbb{N}$ such that f converges on t to e and $rng(t) = W_e$.

Identification of texts

Examples.

- Let $t = (2, 4, 6, 6, 6, \ldots)$
- Let $s = (0, 1, 2, 3, 4, 5, \ldots)$
- Let $r = (2, 2, 2, 3, 3, 3, 4, 4, 4, \ldots)$

- Let $f(\sigma) =$ the least index for $rng(\sigma)$.
- Let $g(\sigma) = 5$.
- Let $h(\sigma) =$ the smallest i so that $rng(\sigma) \subseteq W_i$.

Identification of languages

Definitions. Let $f \in \mathcal{F}$ and $L \in RE$.

f is said to *identify* L just in case f identifies every text for L.

1. Let $L = \{2, 4, 6\}$. Which learning functions identify L?
2. What does g identify?
3. Let n_0 be an index for $L = \{0, 1\}$. Does h identify L?

- Let $f(\sigma) =$ the least index for $rng(\sigma)$.
- Let $g(\sigma) = 5$.
- Let

$$h(\sigma) = \begin{cases} n_0 & \text{if } \sigma \text{ does not end in a } 1 \\ lh(\sigma) & \text{otherwise} \end{cases}$$

Identification of collections of languages

Definitions. Let $f \in \mathcal{F}$, and $\mathcal{L} \subseteq RE$ be a collection of languages.

f is said to *idenfity* $\mathcal{L}$ just in case f identifies every $L \in \mathcal{L}$.

$\mathcal{L}$ is said to be *identifiable* just in case there is some learning function f that identifies it.

- The empty collection of languages is identifiable.
- Every singleton collection of languages $\{L\}$ is identifiable.

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"…questions about the identifiabity of collections of more than one language are often nontrivial, for many such questions receive negative answers… Such is the consequence of requiring a single learning function to determine which of several languages is inscribed in a given text."

Systems that Learn, OSB

Identification of collections of languages

Examples.

Proposition. $RE_{FIN} = \{A \subset \mathbb{N} \mid A \text{ is finite.}\}$ is identifiable.

Proposition. Let $\mathcal{L} = \{\mathbb{N} - \{x\} \mid x \in \mathbb{N}\}$. Then $\mathcal{L}$ is identifiable.

Proposition. Every finite collection of languages is identifiable.