Automata Theory

CS411 & CS675 2015F-01

Set Theory & Proof Techniques

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01-0: **Syllabus**

- Office Hours
- Course Text
- Prerequisites
- Test Dates & Testing Policies
  - Check dates now!
- Grading Policies

01-1: How to Succeed

- Come to class. Pay attention. Ask questions.

01-2: How to Succeed

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  - A question as vague as “I don’t get it” is perfectly acceptable.
  - If you’re confused, *at least* 2 other people are, too.

01-3: How to Succeed

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  - I am *very* available to students.

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- Start the homework assignments early
  - Homework in this class requires “thinking time”

01-5: How to Succeed

- Come to class. Pay attention. Ask questions.
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- Come by my office
  - I am *very* available to students.
- Start the homework assignments early
  - Homework in this class requires "thinking time"
- Read the textbook.
  - Ask Questions! The textbook can be hard to follow –reading a dense, technical work is a "learning outcome" for this class

01-6: Class Goals

- Prove that there are some problems that *cannot be solved*
- Show that there are some problems that (are believed to) require an exponential amount of time to solve (NP-Complete)
  - Examine some strategies for dealing with these problems
- Along the way, learn how to model computation mathematically, and pick up some useful formalisms & techniques
  - DFA, regular expressions, CFGs, etc.

01-7: Review of the Basics

- Most (but perhaps not all) of the following material is review from discrete mathematics
- I will go fairly fast, assuming it is review
  - Ask me to slow down if you have any questions!

01-8: Sets – Definition

- A **set** is an unordered collection of objects
- $S = \{a, b, c\}$
  - a, b, c are **elements** or **members** of the set S

01-9: Sets – Definition

- A **set** is an unordered collection of objects
- $S = \{a, b, c\}$
  - a, b, c are **elements** or **members** of the set S
- Elements in a set need have no relation to each other
  - $S_1 = \{1, 2, 3\}$
  - $S_2 = \{$ red, farmhouse, π, -32 $\}$

01-10: Sets – Definition

- Sets can contain other sets as elements
  - $S_1 = \{3, \{3, 4\}, \{4, \{5, 6\}\}\}$
  - $S_2 = \{\{1, 2\}, \{\{4\}\}\}$
- Sets do not contain duplicates
  - NotASet = {4, 2, 4, 5}

01-11: Sets – Cardinality

- **Cardinality** of a set is the number of elements in the set
  - $|\{a, b, c\}| = 3$
  - $|\{\{a, b\}, c\}| =$?

01-12: Sets – Cardinality

- **Cardinality** of a set is the number of elements in the set
  - $|\{a, b, c\}| = 3$
  - $|\{\{a, b\}, c\}| = 2$ ($\{a, b\}$ and c)

01-13: Sets – Empty, Singleton

- Empty Set: $\{\}$ or $\emptyset$, $|\{\}| = |\emptyset| = 0$
- Singleton set – set with one element
  - {1}
  - {4}
  - {} ?
  - {{}} ?
  - {{3, 1, 2}} ?

01-14: Sets – Empty, Singleton

- **Empty Set:** $\{\}$ or $\emptyset$, $|\{\}| = |\emptyset| = 0$
- **Singleton** set – set with one element
  - {1} Singleton
  - {4} Singleton
  - {} Not a Singleton (empty)
  - {{}} Singleton
  - {{3, 1, 2}} Singleton

Sets – Membership

- Set membership: $x \in S$
  - $3 \in \{1, 3, 5\}$
  - $a \notin \{b, c, d\}$
  - $3 \in \{1, \{2, 3\}\}$?
  - $\{\} \in \{1, 2, 3\}$?
  - $\{\} \in \{1, \{\}, 4\}$?

01-16: Sets – Membership

- Set membership: $x \in S$
  - $3 \in \{1, 3, 5\}$
  - $a \notin \{b, c, d\}$
  - $3 \notin \{1, \{2, 3\}\}$
  - $\{\} \notin \{1, 2, 3\}$
  - $\{\} \in \{1, \{\}, 4\}$

01-17: Sets – Describing

- Referring to sets
  - List all members
    - {3, 4, 5}, {0, 1, 2, 3, . . .}
  - $S = \{x : x$ has a certain property$\}$
    $S = \{x \mid x$ has a certain property$\}$
    - $S = \{x : x \in \mathbf{N} \land x < 10\}$
      $\mathbf{N}$ is the set of natural numbers {0, 1, 2, . . .}
    - $S = \{x : x$ is prime $\}$
    - $A \cup B = \{x : x \in A \lor x \in B\}$
    - $A \cap B = \{x : x \in A \land x \in B\}$
    - $A - B = \{x : x \in A \land x \notin B\}$

01-18: Sets – $\cup, \cap$

- More Union & Intersection
  - A and B are **disjoint** if $A \cap B = \{\}$
  - S is a collection of sets (set of sets)

    $\bigcup S = \{x : x \in A$ for some $A \in S\}$
    - $\bigcup\{\{1, 2\}, \{2, 3\}\} = \{1, 2, 3\}$

    $\bigcap S = \{x : x \in A$ for all $A \in S\}$
    - $\bigcap\{\{1, 2\}, \{2, 3\}\} = \{2\}$

01-19: Sets – Subset

- Subsets & Supersets
  - A is a subset of B, $A \subseteq B$ if:
    - $\forall x, x \in A \implies x \in B$
    - $\forall (x \in A), x \in B$
  - A is a proper subset of B, $A \subset B$ if:
    - $A \subseteq B \wedge (\exists x, x \in B \wedge x \notin A)$
  - {} is a subset of any set (including itself)
  - {} is the only set that does not have a proper subset

01-20: Sets – Power Set

- Power set: Set of all subsets
- $2^S = \{x : x \subseteq S\}$
  - $2^{\{a,b\}} =$?
  - $2^{\{\}} =$?
- $|2^S| =$?

01-21: Sets – Power Set

- Power set: Set of all subsets
- $2^S = \{x : x \subseteq S\}$
  - $2^{\{a,b\}} = \{\{\}, \{a\}, \{b\}, \{a, b\}\}$
  - $2^{\{\}} = \{\{\}\}$
- $|2^S| = 2^{|S|}$

01-22: Sets – Partition

Π is a **partition** of S if:

- $\Pi \subset 2^S$
- $\{\} \notin \Pi$
- $\forall (X, Y \in \Pi), X \neq Y \implies X \cap Y = \{\}$
- $\bigcup \Pi = S$

{{a, c}, {b, d, e}, {f}} is a partition of {a,b,c,d,e,f}
{{a, b, c, d, e, f}} is a partition of {a,b,c,d,e,f}
{{a, b, c}, {d, e, f}} is a partition of {a,b,c,d,e,f}

01-23: Sets – Partition

In other words, a **partition** of a set S is just a division of the elements of S into 1 or more groups.

- All the partitions of the set {a, b, c}?

01-24: Sets – Partition

In other words, a **partition** of a set S is just a division of the elements of S into 1 or more groups.

- All the partitions of the set {a, b, c}?
  - {{a, b, c}}, {{a, b}, {c}}, {{a, c}, {b}}, {{a},{b, c}}, {{a}, {b}, {c}}

01-25: Ordered Pair

- (x, y) is an *ordered pair*
- Order matters – $(x, y) \neq (y, x)$ if $x \neq y$
  - hence *ordered*
- x and y are the **components** of the ordered pair (x, y)

01-26: Cartesian Product

$A \times B = \{(x, y) : x \in A \wedge y \in B\}$

- $\{1, 2\} \times \{3, 4\} = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$
- $\{1, 2\} \times \{1, 2\} =$?
- $2^{\{a\} \times \{b\}} =$?
- $2^{\{a\}} \times 2^{\{b\}} =$?

01-27: Cartesian Product

$A \times B = \{(x, y) : x \in A \wedge y \in B\}$

- $\{1, 2\} \times \{3, 4\} = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$
- $\{1, 2\} \times \{1, 2\} = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$
- $2^{\{a\} \times \{b\}} = \{\{(a, b)\}, \{\}\}$
- $2^{\{a\}} \times 2^{\{b\}}$ =
  $\{(\{a\}, \{b\}), (\{a\}, \{\}), (\{\}, \{b\}), (\{\}, \{\})\}$

01-28: Cartesian Product

Which of the following is true:

- $\forall (A, B) \quad A \times B = B \times A$
- $\forall (A, B) \quad A \times B \neq B \times A$
- None of the above

01-29: Cartesian Product

Which of the following is true:

- $\forall(A, B) \quad A \times B = B \times A$
  - If and only if $A = B$
- $\forall(A, B) \quad A \times B \neq B \times A$
  - If and only if $A \neq B$

01-30: Cartesian Product

- Why “Cartesian”?
  - Think “Cartesian Coordinates” (standard coordinate system)
  - $\mathbf{R} \times \mathbf{R}$ is the real plane
    - Set of all points (x, y) where $x, y \in \mathbf{R}$
    - $\mathbf{R}$ is the set of real numbers (think “floats” if you’re CS)

01-31: Cartesian Product

- Can take the Cartesian product of > 2 sets.
- $A \times B \times C = \{(x, y, z) : x \in A, y \in B, z \in C\}$
- $\{a\} \times \{b, c\} \times \{d\}$ = $\{(a, b, d), (a, c, d)\}$
- (Techinally, $A \times B \times C = (A \times B) \times C$)
  - $\{a\} \times \{b, c\} \times \{d\} = \{((a, b), d), ((a, c), d)\}$
  - Often drop the extra parentheses for readability

01-32: Relations

- A **relation** R is a set of ordered pairs
- For example the relation $<$ over the Natural Numbers is the set:

$$
\{ \quad (0,1), (0,2), (0,3), \ldots \\
(1,2), (1,3), (1,4), \ldots \\
(2,3), (2,4), (2,5), \ldots \\
\ldots \quad \}
$$

01-33: Relations

- Often, relations are over the same set
  - that is, a subset of $A \times A$ for some set A
- Not all relations are over the same set, however
  - Relation describing prices of computer components
    {(Hard dive, \$55), (WAP, \$49), (2G DDR, \$44), $\ldots$}

Functions

- A **function** is a special kind of relation (all functions are relations, but not all relations are functions)
- A relation $R \subseteq A \times B$ is a function if:
  - For each $a \in A$, there is exactly one ordered pair in R with the first component a

01-35: Functions

- A function f that is a subset of $A \times B$ is written: $f : A \mapsto B$
  - $(a, b) \in f$ is written $f(a) = b$
  - A is the domain of the function
  - if $A' \subseteq A$, $f(A') = \{b : a \in A' \wedge f(a) = b\}$ is the **image** of A'
  - The **range** of a function is the image of its domain

01-36: Functions

A function $f : A \mapsto B$ is:

- **one-to-one** if no two elements in A match to the same element in B
- **onto** Each element in B is mapped to by at least one element in A
- a **bijection** if it is both one-to-one and onto

The **inverse** of a binary relation $R \subset A \times B$ is denoted R^{-1}, and defined to be $\{(b, a) : (a, b) \in R\}$

- A function only has an inverse if ...

01-37: Functions

A function $f : A \mapsto B$ is:

- $\mathbf{one\text{-}to\text{-}one}$ if no two elements in A match to the same element in B
- $\mathbf{onto}$ Each element in B is mapped to by at least one element in A
- a $\mathbf{bijection}$ if it is both one-to-one and onto

The $\mathbf{inverse}$ of a binary relation $R \subset A \times B$ is denoted R^{-1}, and defined to be $\{(b, a) : (a, b) \in R\}$

- A function only has an inverse if it is a bijection

01-38: Functions

- What if we want to take the inverse of a function that is not a bijection – what can we do?
  - Want to preserve full information about the original function
  - Resulting inverse must be an actual function

Functions

- What if we want to take the inverse of a function that is not a bijection – what can we do?
  - Want to preserve full information about the original function
  - Resulting inverse must be an actual function
- How can we have an element map to 0, 1, or more elements, and still have a function? *HINT: If we modified the range ...*

01-40: Functions

- What if we want to take the inverse of a function that is not a bijection – what can we do?
  - Want to preserve full information about the original function
  - Resulting inverse must be an actual function
- $f : A \mapsto B$ $\qquad\qquad f^{-1} : B \mapsto 2^A$

(example on chalkboard)

01-41: Relations

- Q and R are two relations
- The composition of Q and R, $Q \circ R$ is:
  $\{(a, b) : (a, c) \in Q, (c, b) \in R$ for some $c\}$

$$Q = \{(a, c), (b, d), (c, a)\}$$
$$R = \{(a, c), (b, c), (c, a)\}$$

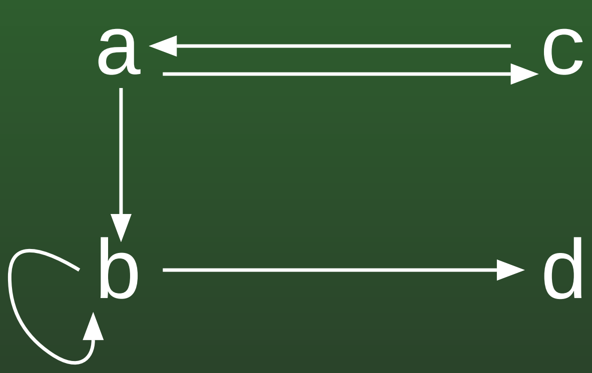
$$Q \circ R = \{(a, a), (c, c)\}$$

$Q \circ Q$? $(Q \circ R) \circ Q$?

01-42: Relation Graph

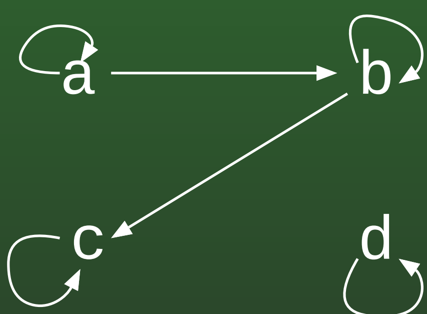
- Each element is a node in the graph
- if $(a, b) \in R$, then there is an edge from a to b in the graph

$$R = \{(a, b), (a, c), (c, a), (b, b), (b, d)\}$$

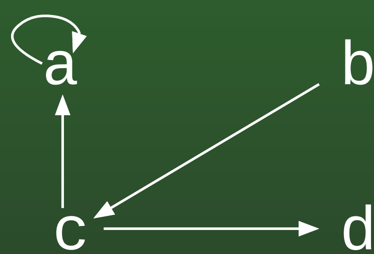


01-43: Relation Types

- A relation $R \subseteq A \times A$ is **reflexive** if
  - $(a, a) \in R$ for each $a \in A$
  - $\forall(a \in A), (a, a) \in R$
  - Each node has a self loop



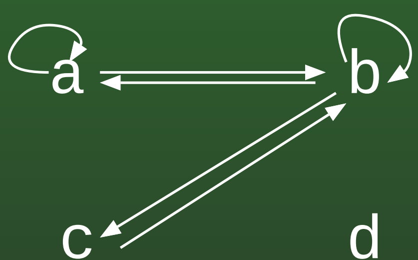
Reflexive



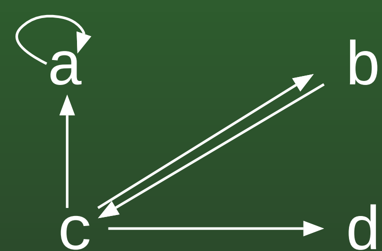
Not Reflexive

01-44: Relation Types

- A relation $R \subseteq A \times A$ is **symmetric** if
  - $(a, b) \in R$ whenever $(b, a) \in R$
  - $(a, b) \in R \implies (b, a) \in R$
  - Every edge goes “both ways”



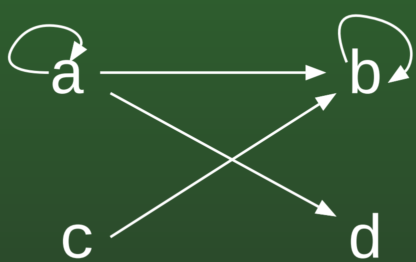
Symmetric



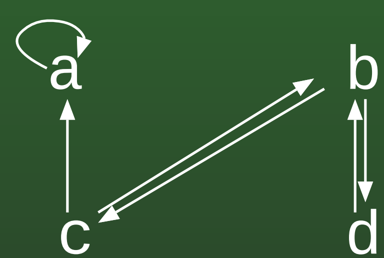
Not Symmeteric

01-45: Relation Types

- A relation $R \subseteq A \times A$ is **antisymmetric** if
  - whenever $(a, b) \in R$, a, b are distinct $(b, a) \notin R$
  - $(a, b) \in R \wedge a \neq b \implies (b, a) \notin R$
  - No edge goes “both ways”



Antisymmetric

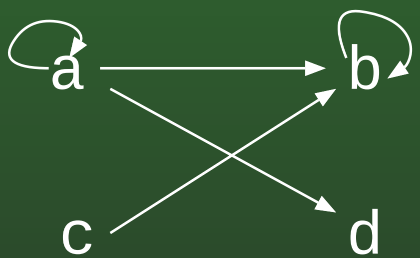


Not Antisymmeteric

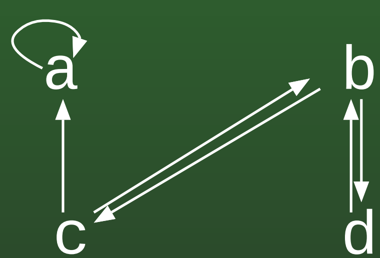
- Can a relation be neither symmetric nor antisymmetric?

01-46: Relation Types

- A relation $R \subseteq A \times A$ is **antisymmetric** if
  - whenever $(a, b) \in R$, a, b are distinct $(b, a) \notin R$
  - $(a, b) \in R \wedge a \neq b \implies (b, a) \notin R$
  - No edge goes "both ways"



Antisymmetric

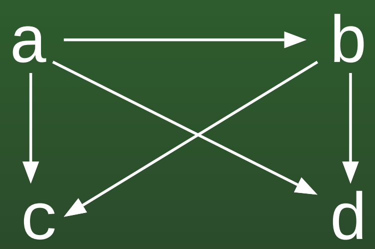


Not Antisymmeteric

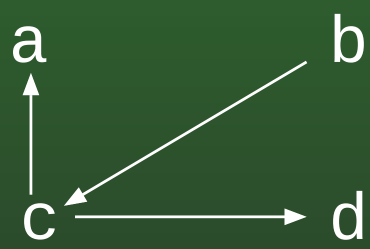
- Can a relation be both symmetric and antisymmetric?

01-47: Relation Types

- A relation $R \subseteq A \times A$ is **transitive** if
  - whenever $(a, b) \in R$, and $(b, c) \in R$, $(a, c) \in R$
  - $(a, b) \in R \wedge (b, c) \in R \implies (a, c) \in R$
  - Every path of length 2 has a direct edge



Transitive



Not Transitive

01-48: Closure

- A set $A \subseteq B$ is closed under a relation $R \subseteq ((B \times B) \times B)$ if:
  - $a_1, a_2 \in A \wedge ((a_1, a_2), c) \in R \implies c \in A$
  - That is, if a_1 and a_2 are both in A, and $((a_1, a_2), c)$ is in the relation, then c is also in A
- $\mathbb{N}$ is closed under addtion
- $\mathbb{N}$ is not closed under subtraction or division

01-49: Closure

- Relations are also sets (of ordered pairs)
- We can talk about a relation R being closed over another relation R'
  - Each element of R' is an ordered triple of ordered pairs!

01-50: Closure

- Relations are also sets (of ordered pairs)
- We can talk about a relation R being closed over another relation R'
  - Each element of R' is an ordered triple of ordered pairs!
- Example:
  - $R \subseteq A \times A$
  - $R' = \{(((a, b), (b, c)), (a, c)) : a, b, c \in A\}$
  - If R is closed under R', then . . .

01-51: Closure

- Relations are also sets (of ordered pairs)
- We can talk about a relation R being closed over another relation R'
  - Each element of R' is an ordered triple of ordered pairs!
- Example:
  - $R \subseteq A \times A$
  - $R' = \{(((a,b),(b,c)),(a,c)) : a,b,c \in A\}$
  - If R is closed under R', then R is transitive!

01-52: Closure

- **Reflexive closure** of a relation $R \subseteq A \times A$ is the smallest possible superset of R which is reflexive
  - Add self-loop to every node in relation
  - Add (a,a) to R for every $a \in A$
- **Transitive Closure** of a relation $R \subseteq A \times A$ is the smallest possible superset of R which is transitive
  - Add direct link for every path of length 2.
  - $\forall(a, b, c \in A)$ if $(a, b) \in R \wedge (b, c) \in R$ add (a, c) to R.

(examples on board)

01-53: Relation Types

- Equivalence Relation
  - Symmetric, Transitive, Reflexive
- Examples:
  - Equality (=)
  - A is the set of English words, $(w_1, w_2) \in R$ if w_1 and w_2 start with the same letter

(example graphs)

01-54: Relation Types

- Equivalence Relation
  - Symmetric, Transitive, Reflexive
- Separates set into equivalence classes (all words that start with a, for example
- If $a \in A$, then $[a]$ represents equivalence class that contains a.

01-55: Relation Types

- Partial Order
  - Antisymmetric, Transitive, Reflexive
- Examples:
  - $\leq$ for integers
    A is the set of integers, $(a, b) \in R$ if $a \leq b$
  - Ancestor
    $R \subseteq A \times A = \{(x, y)$: x is an ancestor of y,
    or $x = y\}$

(example graphs)

01-56: Relation Types

- Total Order
  - $R \subseteq A \times A$ is a total order if:
    - R is a partial order
    - For all $a, b \in A$, either $(a, b) \in R$ or $(b, a) \in R$
  - Is $\leq$ a total order?
  - Is Ancestor a total order?

(example graphs)

01-57: Cardinality

- How can we tell if two sets A and B have the same cardinality?

01-58: Cardinality

- How can we tell if two sets A and B have the same cardinality?
  - Calculate $|A|$ and $|B|$, make sure numbers are the same
  - Match each element in A to an element in B
    - Create a bijection $f : A \mapsto B$ (or $f : B \mapsto A$)

Cardinality

- What about infinite sets? Are they all equinumerous (that is, have the same cardinality)?
- A set is **countable infinite** (or just **countable**) if it is equinumerous with $\mathbf{N}$.

01-60: Countable Sets

- A set is **countable infinite** (or just **countable**) if it is equinumerous with $\mathbf{N}$.
  - Even elements of $\mathbf{N}$?

01-61: Countable Sets

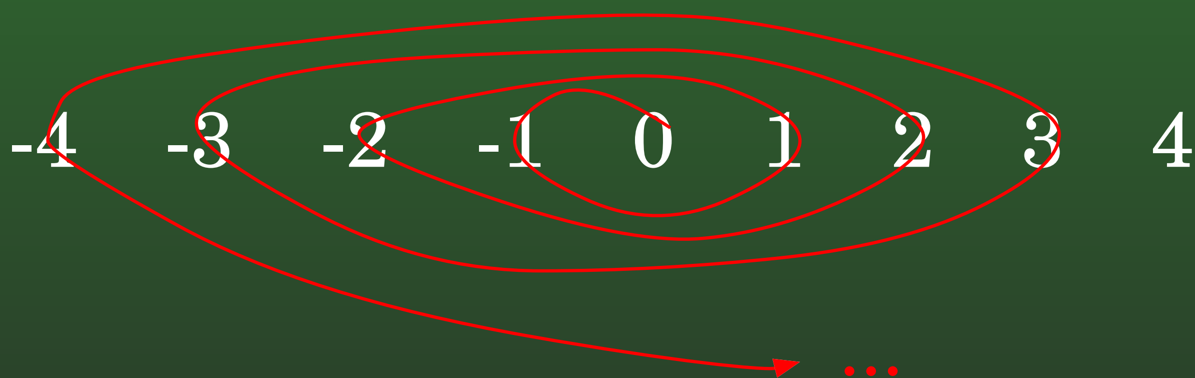
- A set is **countable infinite** (or just **countable**) if it is equinumerous with $\mathbf{N}$.
  - Even elements of $\mathbf{N}$?
  - $f(x) = 2x$

01-62: Countable Sets

- A set is **countable infinite** (or just **countable**) if it is equinumerous with $\mathbf{N}$.
  - Integers ($\mathbf{Z}$)?

01-63: Countable Sets

- A set is **countable infinite** (or just **countable**) if it is equinumerous with $\mathbf{N}$.
  - Integers ($\mathbf{Z}$)?
  - $f(x) = \lceil \frac{x}{2} \rceil * (-1)^x$



01-64: Countable Sets

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  - Union of 3 (disjoint) countable sets A, B, C?

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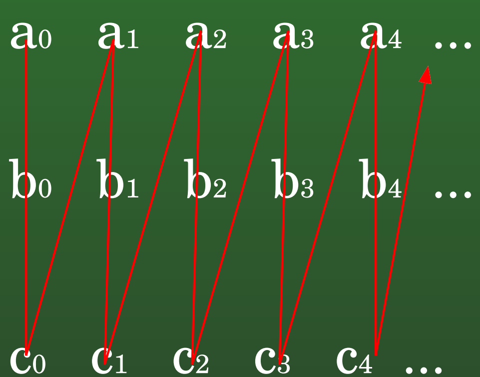
  - Union of 3 (disjoint) countable sets A, B, C?



- $f(x) = \begin{cases} a_{\frac{x}{3}} & \text{if x mod 3 = 0} \\ b_{\frac{x-1}{3}} & \text{if x mod 3 = 1} \\ c_{\frac{x-2}{3}} & \text{if x mod 3 = 2} \end{cases}$

01-66: Countable Sets

- A set is **countable infinite** (or just **countable**) if it is equinumerous with $\mathbf{N}$.
  - $\mathbf{N} \times \mathbf{N}$?

$$\begin{array}{cccccc}
(0,0) & (0,1) & (0,2) & (0,3) & (0,4) & \ldots \\
(1,0) & (1,1) & (1,2) & (1,3) & (1,4) & \ldots \\
(2,0) & (2,1) & (2,2) & (2,3) & (2,4) & \ldots \\
(3,0) & (3,1) & (3,2) & (3,3) & (3,4) & \ldots \\
(4,0) & (4,1) & (4,2) & (4,3) & (4,4) & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

01-67: Countable Sets

- A set is **countable infinite** (or just **countable**) if it is equinumerous with $\mathbf{N}$.
  - $\mathbf{N} \times \mathbf{N}$?

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	...
(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	...
(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	...
(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	...
(4,0)	(4,1)	(4,2)	(4,3)	(4,4)	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

  - $f((x, y)) = \frac{(x+y)*(x+y+1)}{2} + x$

01-68: Countable Sets

- A set is **countable infinite** (or just **countable**) if it is equinumerous with $\mathbf{N}$.
  - Real numbers between 0 and 1 (exclusive)?

01-69: Uncountable R

- Proof by contradiction
  - Assume that R between 0 and 1 (exclusive) is countable
    - (that is, assume that there is some bijection from $\mathbf{N}$ to $\mathbf{R}$ between 0 and 1)
  - Show that this leads to a contradiction
    - Find some element of $\mathbf{R}$ between 0 and 1 that is not mapped to by any element in $\mathbf{N}$

01-70: Uncountable R

- Assume that there is some bijection from $\mathbf{N}$ to $\mathbf{R}$ between 0 and 1

0	0.3412315569...
1	0.0123506541...
2	0.1143216751...
3	0.2839143215...
4	0.2311459412...
5	0.8381441234...
6	0.7415296413...
$\vdots$	$\vdots$

01-71: Uncountable R

- Assume that there is some bijection from $\mathbf{N}$ to $\mathbf{R}$ between 0 and 1

0	0.3412315569...
1	0.0123506541...
2	0.1143216751...
3	0.2839143215...
4	0.2311459412...
5	0.8381441234...
6	0.7415296413...
$\vdots$	$\vdots$ $\ddots$

Consider: 0.425055...

01-72: Proof Techniques

- Three basic proof techniques used in this class
  - Induction
  - Diagonalization
  - Pigeonhole Principle

Induction

Can create exact postage for any amount $\geq$ \$0.08 using only 3 cent and 5 cent stamps

Induction

Can create exact postage for any amount $\geq$ \$0.08 using only 3 cent and 5 cent stamps

- Base case

Can create postage for 0.08 using one 5-cent and one 3-cent stamp

Induction

Can create exact postage for any amount $\geq$ \$0.08 using only 3 cent and 5 cent stamps

- Inductive case
  - To show: if we can create exact postage for \$x using only 3-cent and 5-cent stamps, we can create exact postage for \$x + \$0.01 using 3-cent and 5-cent stamps
  - Two cases:
    - Exact postage for \$x uses at least one 5-cent stamp
    - Exact postage for \$x uses no 5-cent stamps

Induction

- To show: if we can create exact postage for \$x using only 3-cent and 5-cent stamps, we can create exact postage for \$x + \$0.01 using 3-cent and 5-cent stamps
  - Exact postage for \$x uses at least one 5-cent stamp
    - Replace a 5-cent stamp with two 3-cent stamps to get \$x + \$0.01
  - Exact postage for \$x uses no 5-cent stamps
    - Replace three 3-cent stamps with two 5-cent stamps to get \$ + \$0.01

Pigeonhole Principle

- A, B are finite sets, with $|A| > |B|$, then there is no one-to-one function from A to B
- If you have n pigeonholes, and $> n$ pigeons, and every pigeon is in a pigeonhole, there must be at least one hole with > 1 pigeon.

Pigeonhole Principle

- Show that in a relation R over a set A, if there is a path from a_i to a_j in R, then there is a path from a to b whose length is at most $|A|$.

01-79: Pigeonhole Principle

Proof by Contradiction

- Assume that there exists some shortest path from a_i to a_j of length $> |A|$.
- By pigeonhole principle, some element must repeat:
  - $\{a_i, \ldots, a_k, \ldots, a_k \ldots a_J\}$
- We can create a shorter path by removing elements between a_ks.
- We've just found a shorter path from a_i to a_j – a contradiction