18-0: **Language Class $\mathbf{P}$**

- A language L is polynomially decidable if there exists a polynomially bound deterministic Turing machine that decides it.
- A Turing Machine M is polynomially bound if:
  - There exists some polynomial function $p(n)$
  - For any input string w, M always halts within $p(|w|)$ steps
- The set of languages that are polynomially decidable is $\mathbf{P}$

18-1: **Language Class $\mathbf{NP}$**

- A language L is non-deterministically polynomially decidable if there exists a polynomially bound non-deterministic Turing machine that decides it.
- A Non-Deterministic Turing Machine M is polynomially bound if:
  - There exists some polynomial function $p(n)$
  - For any input string w, M always halts within $p(|w|)$ steps, for all computational paths
- The set of languages that are non-deterministically polynomially decidable is $\mathbf{NP}$

18-2: **Language Class $\mathbf{NP}$**

- If a Language L is in $\mathbf{NP}$:
  - There exists a non-deterministic Turing machine M
  - M halts within $p(|w|)$ steps for all inputs w, in all computational paths
  - If $w \in L$, then there is at least one computational path for w that accepts (and potentially several that reject)
  - If $w \notin L$, then all computational paths for w reject

18-3: **$\mathbf{NP}$ vs $\mathbf{P}$**

- A problem is in $\mathbf{P}$ if we can *generate* a solution quickly (that is, in polynomial time
- A problem is in $\mathbf{NP}$ if we can *check* to see if a potential solution is correct quickly
  - Non-deterministically create (guess) a potential solution
  - Check to see that the solution is correct

18-4: **$\mathbf{NP}$ vs $\mathbf{P}$**

- All problems in $\mathbf{P}$ are also in $\mathbf{NP}$
  - That is, $\mathbf{P} \subseteq \mathbf{NP}$
  - If you can generate correct solutions, you can check if a guessed solution is correct

18-5: **$\mathbf{NP}$ Problems**

- Finding Hamiltonian Cycles is $\mathbf{NP}$
  - Non-deterministically pick a permutation of the nodes of the graph

    - First, non-deterministically pick any node in the graph, and place it first in the permutation
    - Then, non-deterministically pick any unchosen node in the graph, and place it second in the permutation
    - ...
  - Check to see if that permutation forms a valid cycle

18-6: **NP Problems**

- Traveling Salesman decision problem is **NP**
  - Non-deterministically pick a permutation of the nodes of the graph
    - First, non-deterministically pick any node in the graph, and place it first in the permutation
    - Then, non-deterministically pick any unchosen node in the graph, and place it second in the permutation
    - ...
  - Check to see if the cost of that cycle is within the cost bound.

18-7: **Integer Partition**

- Integer Partition is **NP**
  - Non-deterministically pick a subset $P \subset S$
  - Check to see if:

$$\sum_{p \in P} p = \sum_{s \in S - P} s$$

18-8: **NP Problems**

- Satisfiability is **NP**
  - Count the number of variables in the formula
  - Non-deterministically write down True or False for each of the n variables in the formula
  - Check to see if that truth assignment satisfies the formula

18-9: **Reduction Redux**

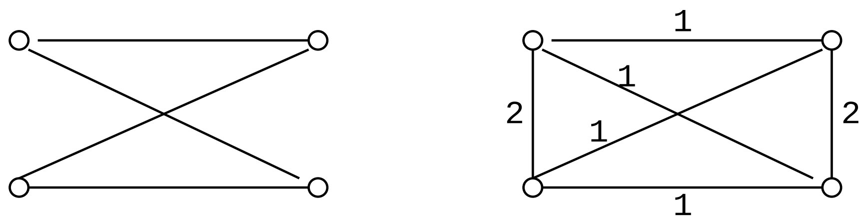
- Given a problem instance P, if we can
  - Create an instance of a different problem P', in polynomial time, such that the solution to P' is the same as the solution to P
  - Solve the instance P' in polynomial time
- Then we can solve P in polynomial time

18-10: **Reduction Example**

- If we could solve the Traveling Salesman decision problem in polynomial time, we could solve the Hamiltonian Cycle problem in polynomial time
  - Given any graph G, we can create a new graph G' and limit k, such that there is a Hamiltonian Circuit in G if and only if there is a Traveling Salesman tour in G' with cost less than k
  - Vertices in G' are the same as the vertices in G

- For each pair of vertices x_i and x_j in G, if the edge (x_i, x_j) is in G, add the edge (x_i, x_j) to G' with the cost 1. Otherwise, add the edge (x_i, x_j) to G' with the cost 2.
- Set the limit $k =$ # of vertices in G

18-11: **Reduction Example**



Limit = 4

18-12: **Reduction Example**

- If we could solve TSP in polynomial time, we could solve Hamiltonian Cycle problem in polynomial time
  - Start with an instance of Hamiltonian Cycle
  - Create instance of TSP
  - Feed instance of TSP into TSP solver
  - Use result to find solution to Hamiltonian Cycle

18-13: **Reduction Example #2**

- Given any instance of the Hamiltonian Cycle Problem:
  - We can (in polynomial time) create an instance of Satisfiability
  - That is, given any graph G, we can create a boolean formula f, such that f is satisfiable if and only if there is a Hamiltonian Cycle in G
- If we could solve Satisfiability in Polynomial Time, we could solve the Hamiltonian Cycle problem in Polynomial Time

18-14: **Reduction Example #2**

- Given a graph G with n vertices, we will create a formula with n^2 variables:
  - $x_{11}, x_{12}, x_{13}, \ldots x_{1n}$
    $x_{21}, x_{22}, x_{23}, \ldots x_{2n}$
    $\ldots$
    $x_{n1}, x_{n2}, x_{n3}, \ldots x_{nn}$
- Design our formula such that x_{ij} will be true if and only if the ith element in a Hamiltonian Circuit of G is vertex # j

18-15: **Reduction Example #2**

- For our set of n^2 variables x_{ij}, we need to write a formula that ensures that:
  - For each i, there is exactly one j such that x_{ij} = true

- For each j, there is exactly one i such that x_{ij} = true
- If x_{ij} and $x_{(i+1)k}$ are both true, then there must be a link from v_j to v_k in the graph G

18-16: **Reduction Example #2**

- For each i, there is exactly one j such that x_{ij} = true
  - For each i in $1 \ldots n$, add the rules:
    - $(x_{i1} \vee x_{i2} \vee \ldots \vee x_{in})$
- This ensures that for each i, there is at least one j such that x_{ij} = true
- (This adds n clauses to the formula)

18-17: **Reduction Example #2**

- For each i, there is exactly one j such that x_{ij} = true

for each i in $1 \ldots n$
    for each j in $1 \ldots n$
        for each k in $1 \ldots n$ $j \neq k$
            Add rule $(\overline{x_{ij}} \vee \overline{x_{ik}})$

- This ensures that for each i, there is at most one j such that x_{ij} = true
- (this adds a total of n^3 clauses to the formula)

18-18: **Reduction Example #2**

- For each j, there is exactly one i such that x_{ij} = true
  - For each j in $1 \ldots n$, add the rules:
    - $(x_{1j} \vee x_{2j} \vee \ldots \vee x_{nj})$
- This ensures that for each j, there is at least one i such that x_{ij} = true
- (This adds n clauses to the formula)

18-19: **Reduction Example #2**

- For each j, there is exactly one i such that x_{ij} = true

for each j in $1 \ldots n$
    for each i in $1 \ldots n$
        for each k in $1 \ldots n$
            Add rule $(\overline{x_{ij}} \vee \overline{x_{kj}})$

- This ensures that for each j, there is at most one i such that x_{ij} = true
- (This adds a total of n^3 clauses to the formula)

18-20: **Reduction Example #2**

- If x_{ij} and $x_{(i+1)k}$ are both true, then there must be a link from v_i to v_k in the graph G

  for each i in $1 \ldots (n-1)$
      for each j in $1 \ldots n$
          for each k in $1 \ldots n$
              if edge (v_j, v_k) is *not* in the graph:
                  Add rule $(\overline{x_{ij}} \vee \overline{x_{(i+1)k}})$

- (This adds no more than n^3 clauses to the formula)

18-21: **Reduction Example #2**

- If x_{nj} and x_{0k} are both true, then there must be a link from v_j to v_k in the graph G (looping back to finish cycle)

  for each j in $1 \ldots n$
      for each k in $1 \ldots n$
          if edge (v_j, v_k) is *not* in the graph:
              Add rule $(\overline{x_{nj}} \vee \overline{x_{0k}})$

- (This adds no more than n^2 clauses to the formula)

18-22: **Reduction Example #2**

- In order for this formula to be satisfied:
  - For each i, there is exactly one j such that x_{ij} is true
  - For each j, there is exactly one i such that x_{ji} is true
  - if x_{ij} is true, and $x_{(i+1)k}$ is true, then there is an arc from v_j to v_k in the graph G
- Thus, the formula can only be satisfied if there is a Hamiltonian Cycle of the graph

18-23: **$\mathbf{NP}$-Complete**

- A language L is $\mathbf{NP}$-Complete if:
  - L is in $\mathbf{NP}$
  - *If* we could decide L in polynomial time, then *all* $\mathbf{NP}$ languages could be decided in polynomial time
  - That is, we could reduce *any* $\mathbf{NP}$ problem to L in polynomial time

18-24: **$\mathbf{NP}$-Complete**

- How do you show a problem is $\mathbf{NP}$-Complete?
  - Given *any* polynomially-bound non-deterministic Turing machine M and string w:
    - Create an instance of the problem that has a solution if and only if M accepts w

18-25: **$\mathbf{NP}$-Complete**

- First $\mathbf{NP}$-Complete Problem: Satisfiability (SAT)
  - Given any (possibly non-deterministic) Turing Machine M, string w, and polynomial bound $p(n)$

- Create a boolean formula f, such that f is satisfiable if and only of M accepts w

18-26: **Cook's Theorem**

- Satisfiability is $\mathbf{NP}$-Complete
  - Given a Turing Machine M, string w, polynomial bound $p(n)$, we will create:
    - A set of variables
    - A set of clauses containing these variables
  - Such that the conjunction ($\wedge$) of the clauses is satisfiable if and only if M accepts w within $p(|w|)$ steps
- WARNING: This explaination is somewhat simplifed. Some subtleties have been eliminated for clarity.

18-27: **Cook's Theorem**

- Variables
  - $Q[i, k]$ at time i, machine is in state q_k
  - $H[i, j]$ at time i, the machine is scanning tape square j
  - $S[i, j, k]$ at time i, the contents of tape location j is the symbol k
- How many of each of these variables are there?

18-28: **Cook's Theorem**

- Variables
  - $Q[i, k]$ $|K| * p(|w|)$
  - $H[i, j]$ $p(|w|) * p(|w|)$
  - $S[i, j, k]$ $p(|w|) * p(|w|) * |\Sigma|$
- How many of each of these variables are there?

18-29: **Cook's Theorem**

G_1 At each time i, M is in exactly one state

G_2 At each time i, the read-write head is scanning one tape square

G_3 At each time i, each tape square contains exactly one symbol

G_4 At time 0, the computation is in the initial configuration for input w

G_5 By time $p(|w|)$, M has entered the final state and has hence accepted w

G_6 For each time i, the configuration of the M at $i + 1$ follows by a single application of δ

18-30: **Cook's Theorem**

G_1 At each time i, M is in exactly one state

$$(Q[i,0] \lor Q[i,1] \lor \ldots \lor Q[i,|K|])$$

for each $0 \leq i \leq p(|w|)$

$$(\overline{Q[i,j]} \lor \overline{Q[i,j']})$$

for each $0 \leq i \leq p(|w|), 0 \leq j < j' \leq |K|$ 18-31: **Cook's Theorem**

G_2 At each time i, the read-write head is scanning one tape square

$$(H[i,0] \lor H[i,1] \lor \ldots \lor H[i,p(|w|)])$$

for each $0 \leq i \leq p(|w|)$

$$(\overline{H[i,j]} \lor \overline{H[i,j']})$$

for each $0 \leq i \leq p(|w|), 0 \leq j < j' \leq p(|w|)$

18-32: **Cook's Theorem**

G_3 At each time i, each tape square contains exactly one symbol

$$(S[i,j,0] \lor S[i,j,1] \lor \ldots \lor S[i,j,|\Sigma|])$$

for each $0 \leq i \leq p(|w|), 0 \leq j \leq p(|w|)$

$$(\overline{S[i,j,k]} \lor \overline{S[i,j,k']})$$

for each $0 \leq i \leq p(|w|), 0 \leq j \leq p(|w|), 0 \leq k < k' \leq |\Sigma|$

18-33: **Cook's Theorem**

G_4 At time 0, the computation is in the initial configuration for input w

$Q[0,0]$
$H[0,1]$
$S[0,0,0]$
$S[0,1,w_1]$
$S[0,2,w_2]$
$\ldots$
$S[0,|w|,w_{|w|}]$
$S[0,|w|+1,0]$
$S[0,|w|+2,0]$
$\ldots$
$S[0,p(|w|),0]$

18-34: **Cook's Theorem**

G_5 By time $p(|w|)$, M has entered the final state and has hence accepted w

$$Q[p(|w|), r]$$

Where q_r is the accept state

18-35: **Cook's Theorem**

G_6 For each time i, the configuration of the M at $i+1$ follows by a single application of δ

For each deterministic transtion $((q_k, \Sigma_a), (q_l, \rightarrow))$
For alll $0 \leq i \leq p(|w|), 0 \leq j \leq p(|w|)$
Add:
$Q[i,k] \wedge H[i,j] \wedge S[i,j,a] \Rightarrow H[i+1, j+1]$
$Q[i,k] \wedge H[i,j] \wedge S[i,j,a] \Rightarrow Q[i+1, l]$

18-36: **Cook's Theorem**

G_6 For each time i, the configuration of the M at $i+1$ follows by a single application of δ

For each deterministic transtion $((q_k, \Sigma_a), (q_l, \leftarrow))$
For alll $0 \leq i \leq p(|w|), 0 \leq j \leq p(|w|)$
Add:
$Q[i,k] \wedge H[i,j] \wedge S[i,j,a] \Rightarrow H[i+1, j-1]$
$Q[i,k] \wedge H[i,j] \wedge S[i,j,a] \Rightarrow Q[i+1, l]$

18-37: **Cook's Theorem**

G_6 For each time i, the configuration of the M at $i+1$ follows by a single application of δ

For each deterministic transtion $((q_k, \Sigma_a), (q_l, \Sigma_b))$
For alll $0 \leq i \leq p(|w|), 0 \leq j \leq p(|w|)$
Add:
$Q[i,k] \wedge H[i,j] \wedge S[i,j,a] \Rightarrow H[i+1, j]$
$Q[i,k] \wedge H[i,j] \wedge S[i,j,a] \Rightarrow Q[i+1, l]$
$Q[i,k] \wedge H[i,j] \wedge S[i,j,a] \Rightarrow S[i, j, b]$

18-38: **Cook's Theorem**

G_6 For each time i, the configuration of the M at $i+1$ follows by a single application of δ

For each non-deterministic transtion $((q_k, \Sigma_a), (q_l, \rightarrow))$ and $((q_k, \Sigma_a), (q_m, \rightarrow))$
For alll $0 \leq i \leq p(|w|), 0 \leq j \leq p(|w|)$
Add:
$Q[i,k] \wedge H[i,j] \wedge S[i,j,a] \Rightarrow H[i+1, j+1]$
$Q[i,k] \wedge H[i,j] \wedge S[i,j,a] \Rightarrow Q[i+1, l] \vee Q[i+1, m]$

18-39: **Cook's Theorem**

G_6 For each time i, the configuration of the M at $i+1$ follows by a single application of δ

- ... similar rules for other non-deterministic cases

18-40: **Cook's Theorem**

G_6 For each time i, the configuration of the M at $i+1$ follows by a single application of δ

$H[i, j] \wedge S[i, k, a] \Rightarrow S[i+1, k, a]$

for all values of k, j between 0 and $p(|w|)$ where $k \neq j$, and all values $0 \leq a < |\Sigma|$

18-41: **More NP-Complete Problems**

- So, if we could solve Satisfiability in Polynomial Time, we could solve *any* $\mathbf{NP}$ problem in polynomial time
  - Including factoring large numbers ...
- Satisfiability is $\mathbf{NP}$-Complete
- There are many $\mathbf{NP}$-Complete problems
  - Prove $\mathbf{NP}$-Completeness using a reduction

18-42: **More NP-Complete Problems**

- Exact Cover Problem
  - Set of elements A
  - $F \subset 2^A$, family of subsets
  - Is there a subset of F such that each element of A appears exactly once?

18-43: **More NP-Complete Problems**

- Exact Cover Problem
  - $A = \{a, b, c, d, e, f, g\}$
  - $F = \{\{a, b, c\}, \{d, e, f\}, \{b, f, g\}, \{g\}\}$
  - Exact cover exists:
    $\{a, b, c\}, \{d, e, f\}, \{g\}$

18-44: **More NP-Complete Problems**

- Exact Cover Problem
  - $A = \{a, b, c, d, e, f, g\}$
  - $F = \{\{a, b, c\}, \{c, d, e, f\}, \{a, f, g\}, \{c\}\}$
  - No exact cover exists

18-45: **More NP-Complete Problems**

- Exact Cover is in $\mathbf{NP}$
  - Guess a cover
  - Check that each element appears exactly once
- Exact Cover is $\mathbf{NP}$-Complete

- Reduction from Satisfiability
- Given any instance of Satisfiability, create (in polynomial time) an instance of Exact Cover

18-46: **Exact Cover is NP-Complete**

- Given an instance of SAT:
  - $C_1 = (x_1, \vee \overline{x_2})$
  - $C_2 = (\overline{x_1} \vee x_2 \vee x_3)$
  - $C_3 = (x_2)$
  - $C_4 = (\overline{x_2}, \overline{x_3})$
- Formula: $C_1 \wedge C_2 \wedge C_3 \wedge C_4$
- Create an instance of Exact Cover
  - Define a set A and family of subsets F such that there is an exact cover of A in F if and only if the formula is satisfiable

18-47: **Exact Cover is NP-Complete**

$C_1 = (x_1 \vee \overline{x_2})$ $C_2 = (\overline{x_1} \vee x_2 \vee x_3)$ $C_3 = (x_2)$ $C_4 = (\overline{x_2} \vee \overline{x_3})$

$A = \{x_1, x_2, x_3, C_1, C_2, C_3, C_4, p_{11}, p_{12}, p_{21}, p_{22}, p_{23}, p_{31}, p_{41}, p_{42}\}$
$F = \{\{p_{11}\}, \{p_{12}\}, \{p_{21}\}, \{p_{22}\}, \{p_{23}\}, \{p_{31}\}, \{p_{41}\}, \{p_{42}\},$
$X_1, f = \{x_1, p_{11}\}$
$X_1, t = \{x_1, p_{21}\}$
$X_2, f = \{x_2, p_{22}, p_{31}\}$
$X_2, t = \{x_2, p_{12}, p_{41}\}$
$X_3, f = \{x_3, p_{23}\}$
$X_3, t = \{x_3, p_{42}\}$
$\{C_1, p_{11}\}, \{C_1, p_{12}\}, \{C_2, p_{21}\}, \{C_2, p_{22}\}, \{C_2, p_{23}\}, \{C_3, p_{31}\}, \{C_4, p_{41}\}, \{C_4, p_{422}\}\}$

18-48: **Knapsack**

- Given a set of integers S and a limit k:
  - Is there some subset of S that sums to k?
- {3, 5, 11, 15, 20, 25} Limit: 36
  - {5, 11, 20}
- {2, 5, 10, 12, 20, 27} Limit: 43
  - No solution
- Generalized version of Integer Partition problem

18-49: **Knapsack**

- Knapsack is NP-Complete
- By reduction from Exact Cover
  - Given any Exact Cover problem (set A, family of subsets F), we will create a Knapsack problem (set S, limit k), such that there is a subset of S that sums to k if and only if there is an exact cover of A in F

18-50: **Knapsack**

- Each set will be represented by a number – bit-vector representation of the set
  $A = \{a_1, a_2, a_3, a_4\}$

Set	Number
$F_1 = \{a_1, a_2, a_3\}$	1110
$F_2 = \{a_2, a_4\}$	0101
$F_3 = \{a_1, a_3\}$	1010
$F_4 = \{a_2, a_3, a_4\}$	0111

There is an exact cover if some subset of the numbers sum to ...

18-51: **Knapsack**

- Each set will be represented by a number – bit-vector representation of the set
  $A = \{a_1, a_2, a_3, a_4\}$

Set	Number
$F_1 = \{a_1, a_2, a_3\}$	1110
$F_2 = \{a_2, a_4\}$	0101
$F_3 = \{a_1, a_3\}$	1010
$F_4 = \{a_2, a_3, a_4\}$	0111

There is an exact cover if some subset of the numbers sum to 1111

18-52: **Knapsack**

- Bug in our reduction:
  $A = \{a_1, a_2, a_3, a_4\}$

Set	Number
$F_1 = \{a_2, a_3, a_4\}$	0111
$F_2 = \{a_2, a_4\}$	0101
$F_3 = \{a_3\}$	0010
$F_3 = \{a_4\}$	0001
$F_4 = \{a_1, a_3, a_4\}$	1011

- 0111 + 0101 + 0001 + 0010 = 1111
- What can we do?

18-53: **Knapsack**

- Construct the numbers just as before
- Do addition in base m, where m is the number of element in A. $A = \{a_1, a_2, a_3, a_4\}$

Set	Number
$F_1 = \{a_2, a_3, a_4\}$	0111
$F_2 = \{a_2, a_4\}$	0101
$F_3 = \{a_3\}$	0010
$F_3 = \{a_4\}$	0001
$F_4 = \{a_1, a_3, a_4\}$	1011

- 0111 + 0101 + 0001 + 0010 = 0223

- No subset of numbers sums to 1111

18-54: **Integer Partition**

- Integer Partition
  - Special Case of the Knapsack problem
  - “Half sum” H (sum of all elements in the set / 2) is an integer
  - Limit $k = H$
- Integer Partition is $\mathbf{NP}$-Complete
  - Reduce Knapsack to Integer Partition

18-55: **Integer Partition**

- Given any instance of the Knapsack problem
  - Set of integers $S = \{a_1, a_2, \ldots, a_n\}$ limit k
  - Is there a subset of S that sums to k?
- Create an instance of Integer Partition
  - Set of integers $S = \{a_1, a_2, \ldots, a_m\}$
  - Can we divde S into two subsets that have the same sum?
  - Equivalently, is there a subset if S that sums to $H = (\sum_{i=1}^{m} a_i)/2$

18-56: **Integer Partition**

- Given any instance of the Knapsack problem
  - Set of integers $S = \{a_1, a_2, \ldots, a_n\}$ limit k
- We create the following instance of Integer Partition:
  - $S' = S \cup \{2H + 2k, 4H\}$ (H is the half sum of S)

18-57: **Integer Partition**

- $S' = S \cup \{2H + 2k, 4H\}$ (H is the half sum of S)
  - If there is a partion for S', $2H + 2k$ and $4H$ must be in separate partitions (why)?

$$4H + \sum_{a_i \in P} a_i = 2H + 2k + \sum_{a_j \in S-P} a_j$$

18-58: **Integer Partition**

$$4H + \sum_{a_i \in P} a_i = 2H + 2k + \sum_{a_j \in S-P} a_j$$

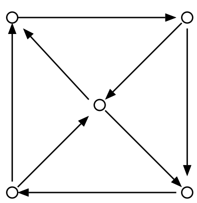
- Adding $\sum_{a_i \in P} a_i$ to both sides:

$$\begin{aligned} 4H + 2\sum_{a_i \in P} a_i &= 2H + 2k + \sum_{a_j \in S} a_j \\ 4H + 2\sum_{a_i \in P} a_i &= 4H + 2k \\ \sum_{a_i \in P} a_i &= k \end{aligned}$$

- Thus, if S' has a partition, then there must be some subset of S that sums to k

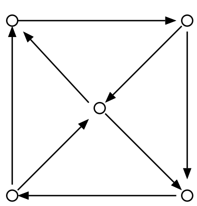
18-59: **Directed Hamiltonian Cycle**

- Given any directed graph G, determine if G has a a Hamiltonian Cycle
  - Cycle that includes every node in the graph exactly once, following the direction of the arrows



18-60: **Directed Hamiltonian Cycle**

- Given any directed graph G, determine if G has a a Hamiltonian Cycle
  - Cycle that includes every node in the graph exactly once, following the direction of the arrows



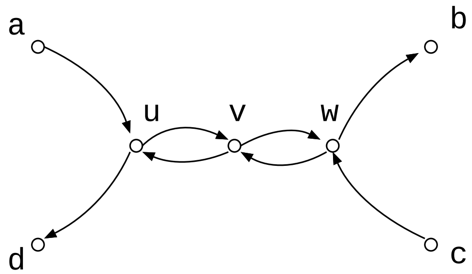
18-61: **Directed Hamiltonian Cycle**

- The Directed Hamiltonian Cycle problem is $\mathbf{NP}$-Complete
- Reduce Exact Cover to Directed Hamiltonian Cycle
  - Given any set A, and family of subsets F:
  - Create a graph G that has a hamiltonian cycle if and only if there is an exact cover of A in F

18-62: **Directed Hamiltonian Cycle**

- Widgets:

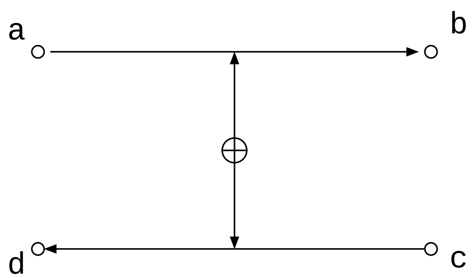
- Consider the following graph segment:



- If a graph containing this subgraph has a Hamiltonian cycle, then the cycle must contain either $a \to u \to v \to w \to b$ or $c \to w \to v \to u \to d$ – but not both (why)?

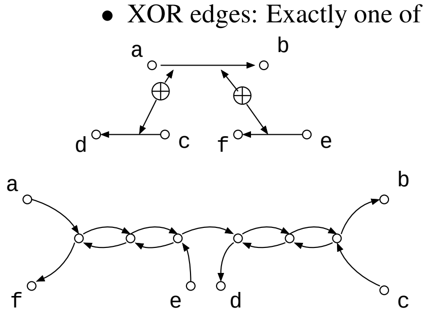
18-63: **Directed Hamiltonian Cycle**

- Widgets:
  - XOR edges: Exactly one of the edges must be used in a Hamiltonian Cycle



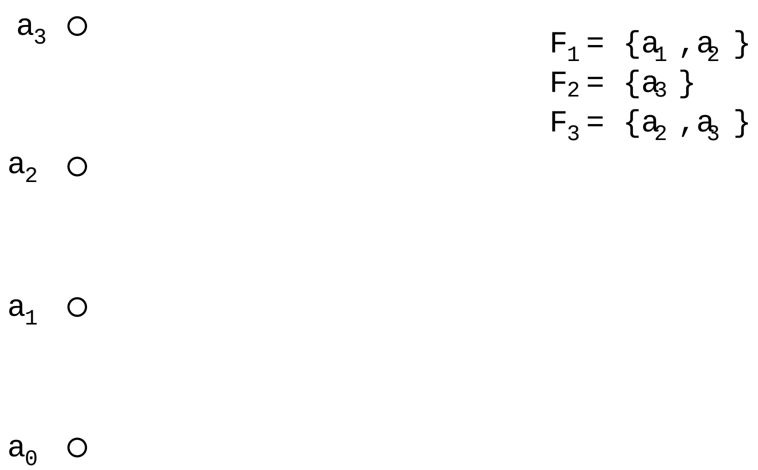
18-64: **Directed Hamiltonian Cycle**

- Widgets:
  - XOR edges: Exactly one of the edges must be used in a Hamiltonian Cycle



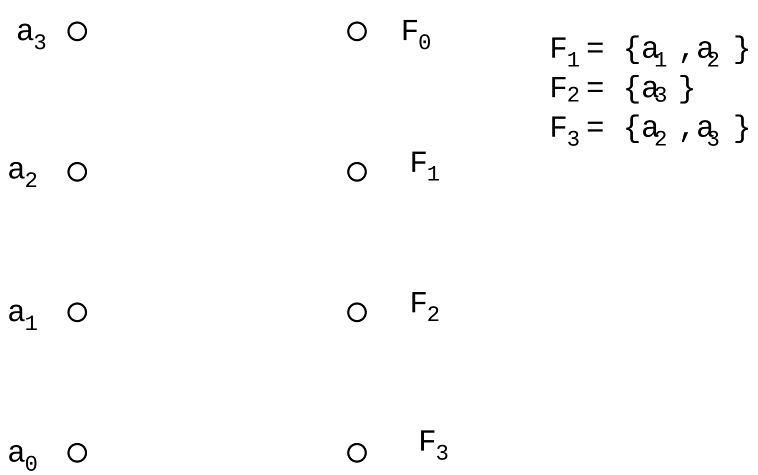
18-65: **Directed Hamiltonian Cycle**

- Add a vertex for every variable in A (+ 1 extra)



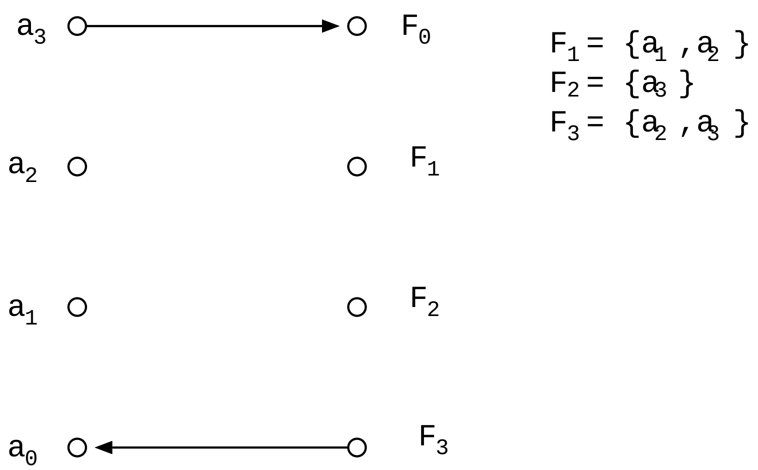
18-66: **Directed Hamiltonian Cycle**

- Add a vertex for every subset F (+ 1 extra)



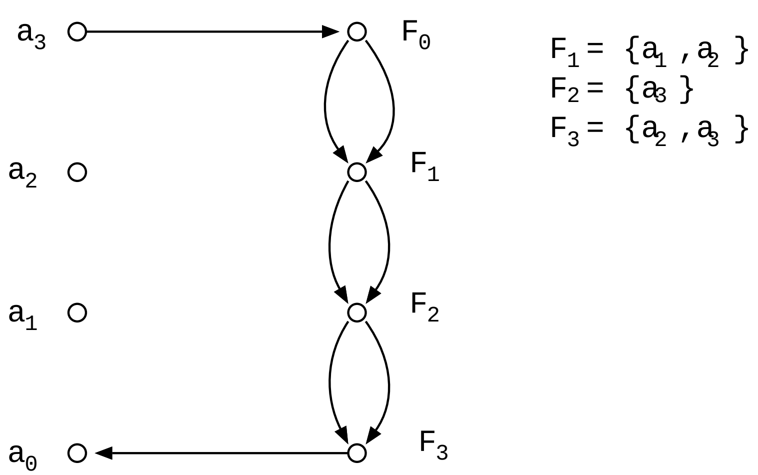
18-67: **Directed Hamiltonian Cycle**

- Add an edge from the last variable to the 0th subset, and from the last subset to the 0th variable



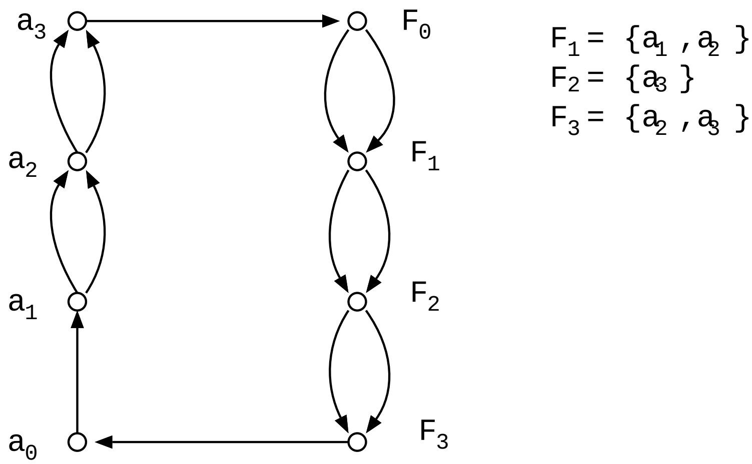
18-68: **Directed Hamiltonian Cycle**

- Add **2** edges from F_i to F_{i+1}. One edge will be a "short edge", and one will be a "long edge".



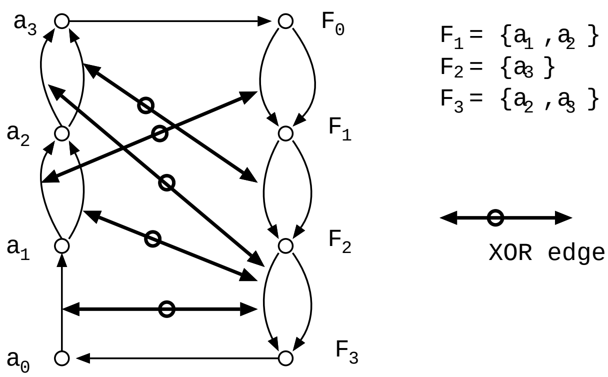
18-69: **Directed Hamiltonian Cycle**

- Add an edge from a_{i-1} to a_i for **each** subset a_i appears in.

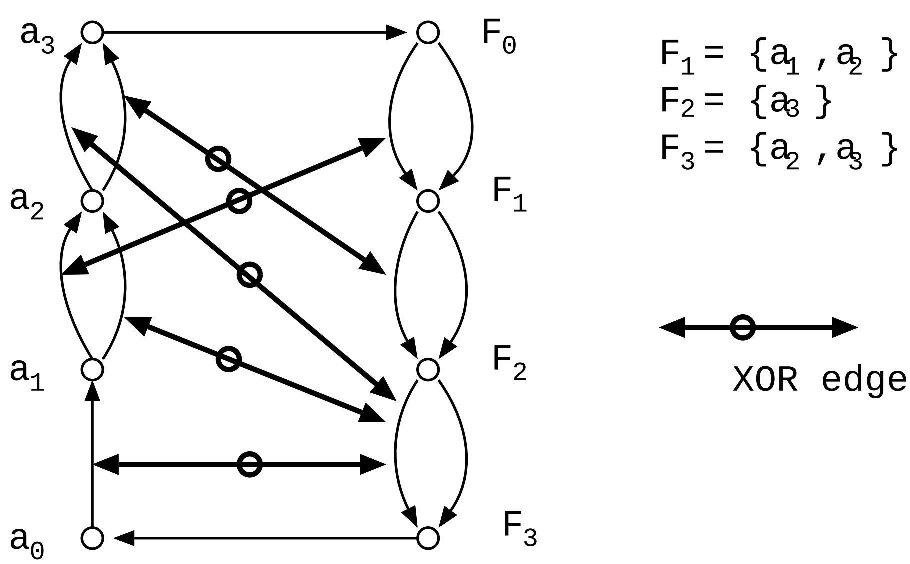


18-70: **Directed Hamiltonian Cycle**

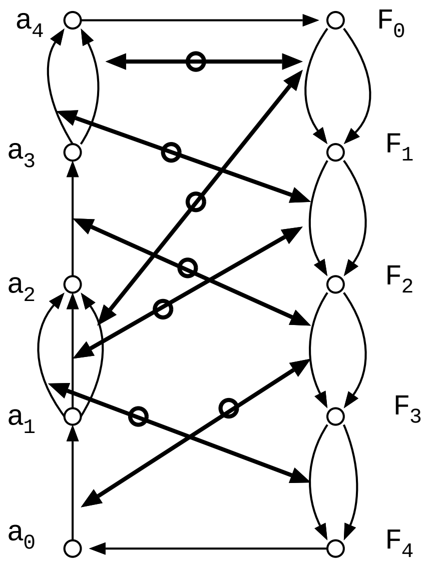
- Each edge (a_{i-1}, a_i) corresponds to some subset that contains a_i. Add an XOR link between this edge and the long edge of the corresponding subset



18-71: **Directed Hamiltonian Cycle**



18-72: **Directed Hamiltonian Cycle**



$F_1 = \{a_2, a_4\}$
$F_2 = \{a_2, a_4\}$
$F_3 = \{a_1, a_3\}$
$F_4 = \{a_2\}$