On linear programming duality and Landau's characterization of tournament scores[1]

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Abstract.

H. G. Landau has characterized those integer-sequences $S = (s_1, s_2, \ldots, s_n)$ which can arise as score-vectors in an ordinary round-robin tournament among n contestants [15]. If $s_1 \leq s_2 \leq \cdots \leq s_n$, then the relevant conditions are expressed simply by the inequalities:

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2}, \tag{1.1}$$

for $k = 1, 2, \ldots, n$, with equality holding when $k = n$. The necessity of these conditions is fairly obvious, and proofs of their sufficiency have been given using a variety of different methods [1, 2, 4, 8, 20]. The purpose of this note is to exhibit Landau's theorem as an instance of the "duality principle" of linear programming, and to point out that this approach suggests an extension of Landau's result going beyond the well known generalizations due to J. W. Moon [17, 18].

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[1] Based on a talk presented at the 57th Annual Meeting of the Mathematical Association of America, Southeastern Section, held March 31-April 1, 1978, at Clemson University, Clemson, South Carolina.

Background.

In an ordinary round-robin tournament, there are n contestants, each of whom plays exactly one game against each other contestant, and no game is permitted to end in a tie.[2] The results in such a tournament can be represented by an $n \times n$ matrix $T = (t_{ij})$ of zeros and ones, called a tournament matrix, in which $t_{ij} = 1$ if the i-th contestant defeats the j-th contestant, and $t_{ij} = 0$ otherwise. It is easy to see that the set of all $n \times n$ tournament matrices is identical with the set of all integer solutions to the following system of linear relations (in which i, j represent arbitrary distinct indices):

$$(2.1) \qquad t_{ij} \geq 0$$

$$(2.2) \qquad t_{ii} = 0$$

$$(2.3) \qquad t_{ij} + t_{ji} = 1$$

The i-th contestant's score s_i is the total number of games played in which the i-th contestant is the victor. Note that s_i may be obtained by summing the entries in the i-th row of the matrix T :

$$(2.4) \qquad s_i = \sum_{j=1}^{n} t_{ij} .$$

[2]For a survey of results on tournaments and their generalizations, the reader is referred to [10] or [19].

The sequence $S = (s_1, s_2, \ldots, s_n)$ consisting of all the contestants' scores is called the <u>score-vector</u> for the tournament. Clearly, the sum of all the scores in a score-vector must equal the total number of games played in the tournament:

$$(2.5) \qquad \sum_{i=1}^{n} s_i = \binom{n}{2} .$$

Furthermore, any subset of the contestants taken together must score a total number of wins at least as large as the number of games they play with each other; hence, for $k = 1, 2, \ldots, n$, the inequality

$$(2.6) \qquad \sum_{i \in K} s_i \geq \binom{k}{2}$$

must hold for each k-element subset K of $\{1, 2, \ldots, n\}$. Landau's theorem, referred to in the introduction, asserts that these relations (2.5)-(2.6) completely characterize those integer-sequences which are score-vectors.

THEOREM 1 (H. G. Landau [15])

For an arbitrary integer-sequence $S = (s_1, s_2, \ldots, s_n)$ to be the score-vector of some round-robin tournament among n contestants, it is necessary and sufficient that, for $1 \leq k \leq n$, the inequality

$$\sum_{i \in K} s_i \geq \binom{k}{2}$$

shall hold for each k-element subset K of $\{1, 2, \ldots, n\}$, and moreover that strict equality shall hold when $k = n$.

We remark that, in order to test a sequence $S = (s_1, s_2, \ldots, s_n)$ according to the criteria in Theorem 1, it might seem that a check of 2^n inequalities is required; but, as Landau himself pointed out, if the elements $s_1, s_2, \ldots, s_n$ are first arranged in nondescending order, then only n relations actually have to be examined. Thus the crucial relations are indicated in the following

COROLLARY (H. G. Landau [15])

For the integer-sequence $S = (s_1, s_2, \ldots, s_n)$ to be a score-vector, where $s_1 \leq s_2 \leq \cdots \leq s_n$, it is both necessary and sufficient that the inequality

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2}$$

shall hold for each $k \leq n$, with strict equality when $k = n$.

Since it is always possible, in at most $\binom{n}{2}$ steps, to rearrange the elements of any n-term sequence so that they appear in non-descending order, this Corollary provides the basis for an <u>efficient algorithm</u> to detect score-vectors.[3] To deduce the Corollary from Theorem 1 is easy: simply observe that the sum of any k elements chosen from a finite set must be at least as large as the sum of the k <u>smallest</u> elements in that set.

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[3]For a recent interesting discussion of algorithmic efficiency, the the reader may consult the popular survey article [16].

J. W. Moon in [17] has extended Landau's theorem by referring to arbitrary real solutions of the system (2.1)-(2.3) as <u>generalized tournaments</u>. Scores for the contestants in a generalized tournament are defined by (2.4) and need not be integers, although such scores still must satisfy the relations (2.5)-(2.6), since these relations actually are algebraic combinations of (2.1)-(2.4). Moon's result, which closely parallels Landau's theorem, may be phrased as follows:

THEOREM 2 (J. W. Moon [17])

For an arbitrary real-sequence $S = (s_1, s_2, \ldots, s_n)$ to be the score-vector of some generalized tournament of size $n \times n$, it is necessary and sufficient that, for $1 \leq k \leq n$, the inequality

$$\sum_{i \in K} s_i \geq \binom{k}{2}$$

shall hold for each k-element subset K of $\{1, 2, \ldots, n\}$, and moreover that strict equality shall hold when $k = n$.

Interestingly, from the point of view of linear programming, as we shall see, Landau's original theorem may be regarded as a somewhat deeper result than the apparently more general theorem due to Moon. The explanation for this opinion is that Landau's theorem rests with greater weight upon a special property of the linear constraint-system (2.1)-(2.4) known as "total unimodularity." The significance of this property for integer linear programming is indicated in the next section.

Duality and unimodularity.

In the argument which follows we shall employ the so-called "duality principle" of linear programming. Complete discussions of this principle may be found in most standard textbooks, such as [7] or [9]. The version needed for our purposes relates the following pair of optimization problems built out of the same data, namely, a $p \times q$ matrix $A = (a_{ij})$, a p-vector $B = (b_i)$, and a q-vector $C = (c_j)$:

Maximum problem	Minimum problem	
Maximize $\sum_{j=1}^{q} c_j x_j$	Minimize $\sum_{i=1}^{p} b_i y_i$	
constrained by	constrained by	
$x_j \geq 0$	$\sum_{i=1}^{p} a_{ij} y_i \geq c_j$	$(1 \leq j \leq q)$
$\sum_{j=1}^{q} a_{ij} x_j \leq b_i$	$y_i \geq 0$	$(1 \leq i \leq p)$

The duality principle asserts that, if the maximum problem is solvable, then the minimum problem also is solvable, and the constrained maximum of $\sum c_j x_j$ equals the constrained minimum of $\sum b_i y_i$.

Besides the duality principle we shall also require certain further facts from linear programming. As is well known, in any linear programming problem, the optimum value of the objective function (if it exists) is attained at a vertex, or "extreme point," of the polyhedral convex set of feasible solutions (see, for example, [7] or [9]). Each such vertex arises as a basic solution of the linear inequalities which define the feasible region; that is, by choosing an appropriate subset of the inequalities, and then solving them simultaneously as if they were linear equations. Accordingly, as a consequence of Cramer's Rule, if the constraint-matrix A in the above pair of optimization problems happens to be totally unimodular (i.e., every square submatrix of A of every order has a determinant equal to 0, $+1$, or -1), and if the given vectors B and C are composed of integers, then in both problems the optimal value of the objective functions will be attained at integral solution-vectors $X = (x_j)$ and $Y = (y_i)$ (see [12]).

In general, it is not easy to tell whether a given matrix A is totally unimodular, although an obvious requirement in view of the definition is that the individual entries a_{ij} must themselves be equal to 0, $+1$, or -1 . A complete characterization of totally unimodular matrices (in terms of forbidden submatrices) has been given by P. Camion in [5]. A simpler criterion which is often useful is the following sufficient condition due to Heller and Tompkins [11].

THEOREM 3 (Heller and Tompkins [11])

In order for the matrix $A = (a_{ij})$ to be totally unimodular, the following three conditions are sufficient:

(1) Each entry a_{ij} is 0, $+1$, or -1 .

(2) At most two nonzero entries appear in any column of A .

(3) The rows of A can be partitioned into two subsets R_1 and R_2 such that:

(i) If a column contains two nonzero entries with the same sign, then one entry is in a row of R_1 and one entry is in a row of R_2.

(ii) If a column contains two nonzero entries of opposite sign, then both entries are in rows of R_1, or both entries are in rows of R_2.

For later reference we note that, in stating this criterion, the words "row" and "column" could be interchanged throughout, since it is obvious from the definition that a matrix A is totally unimodular if and only if its transpose A^t is also totally unimodular. With these results freshly in mind, we proceed to our proof of the Moon and Landau theorems.

Proof of Landau's theorem.

Assume that $S = (s_1, s_2, \ldots, s_n)$ is an arbitrary real sequence satisfying the relations (2.5)-(2.6). We wish to show, first, that there exists a real $n \times n$ matrix $T = (t_{ij})$ satisfying (2.1)-(2.4); and further, that if S happens to be composed of integers, then T may be assumed to consist of integers as well. The first statement yields Moon's theorem (Theorem 2), while the second assertion gives the original theorem of Landau (Theorem 1). To achieve their proofs, we consider the following linear programming problem:

$$\text{Maximize} \quad z = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij}$$

subject to the constraints

(4.1) $\quad x_{ij} \geq 0 , \quad \text{for } 1 \leq i, j \leq n$

(4.2) $\quad x_{ii} \leq 0 , \quad \text{for } 1 \leq i \leq n$

(4.3) $\quad x_{ij} + x_{ji} \leq 1 , \quad \text{for } 1 \leq i < j \leq n$

(4.4) $\quad \sum_{j=1}^{n} x_{ij} \leq s_i , \quad \text{for } 1 \leq i \leq n .$

Notice that these constraints have at least one feasible solution (e.g., the zero matrix) since the inequalities (2.6) imply that the numbers s_i are all nonnegative; and since the set of all feasible solutions is evidently closed and bounded, an optimal solution must exist. Indeed, by adding all inequalities of type (4.4), we can see from (2.5) that $\max z \leq \binom{n}{2}$. Let us now show that in fact $\max z = \binom{n}{2}$. It is for this purpose that we utilize the principle of duality.

Consider the following minimum problem (which is the dual of the maximum problem above):

$$\text{Minimize} \qquad y = \sum_{i=1}^{n} s_i u_i + \sum_{i<j} v_{ij}$$

subject to the constraints

$$(4.5) \qquad u_i \geq 0 , \qquad \text{for } 1 \leq i \leq n$$

$$(4.6) \qquad v_{ij} \geq 0 , \qquad \text{for } 1 \leq i \leq j \leq n$$

$$(4.7) \qquad u_i + v_{ij} \geq 1 , \qquad \text{for } 1 \leq i < j \leq n$$

$$(4.8) \qquad u_j + v_{ij} \geq 1 , \qquad \text{for } 1 \leq i < j \leq n$$

$$(4.9) \qquad u_i + v_{ii} \geq 1 , \qquad \text{for } 1 \leq i \leq n .$$

By the fundamental duality principle of linear programming, we know that $\min y = \max z$. So let us now show that $\min y < \binom{n}{2}$ is impossible.

Suppose, on the contrary, that we did have $\min y < \binom{n}{2}$. Then there would have to exist a solution-vector $(\bar{u}_i, \bar{v}_{ij})$ satisfying the constraints (4.5)-(4.9) such that

$$(4.10) \qquad y = \sum_{i=1}^{n} s_i \bar{u}_i + \sum_{i<j} \bar{v}_{ij} < \binom{n}{2} ,$$

and we may assume (as explained in the preceeding section) that this vector $(\bar{u}_i, \bar{v}_{ij})$ is an extreme point of the polyhedral convex set defined by the constraints (4.5)-(4.9). For this polyhedron, the extreme points are particularly easy to describe.

LEMMA

If $(\bar{u}_i, \bar{v}_{ij})$ is any extreme point of the convex polyhedron defined by (4.5)-(4.9), then:

(i) The components of $(\bar{u}_i, \bar{v}_{ij})$ are zeros and ones.

(ii) If $K = \left\{ i : \bar{u}_i = 1 \right\}$, then $\bar{v}_{ij} = 0$ if and only if $i \in K$ and $j \in K$.

Proof of Lemma: It is evident from an inspection of the constraints (4.5)-(4.9) that the vector $(\bar{u}_i, \bar{v}_{ij})$ cannot be extremal if it contains any entries larger than 1, since all such entries can be either increased or decreased by a small amount without violating any of the constraints. Thus to prove (i) it suffices to show that $(\bar{u}_i, \bar{v}_{ij})$ cannot be extremal unless it is composed entirely of integers. But this fact follows at once from Cramer's Rule and the "unimodular property" of the constraints (4.5)-(4.9). (The criterion of Theorem 3 may be used to detect this property.) Alternatively, a direct argument inspired by Hoffman and Kuhn [13] may be given as follows.

Suppose $(\bar{u}_i, \bar{v}_{ij})$ is a vector satisfying (4.5)-(4.9) which contains some non-integral entries. Then for $e \neq 0$ let $(u_i', v_{ij}')^e$ be the vector defined by

$$u_i' = \begin{cases} \bar{u}_i & \text{if } \bar{u}_i \text{ is an integer,} \\ \bar{u}_i + e & \text{otherwise,} \end{cases}$$

$$v_{ij}' = \begin{cases} \bar{v}_{ij} & \text{if } \bar{v}_{ij} \text{ is an integer,} \\ \bar{v}_{ij} - e & \text{otherwise.} \end{cases}$$

Evidently both $(u_i',v_{ij}')^e$ and $(u_i',v_{ij}')^{-e}$ will satisfy (4.5)-(4.9) for a sufficiently small choice of $e > 0$, since the sum of two numbers cannot equal an integer if exactly one of them is non-integral. Now since $(\bar{u}_i,\bar{v}_{ij})$ can be written as

$$(\bar{u}_i,\bar{v}_{ij}) = \tfrac{1}{2}(u_i',v_{ij}')^e + \tfrac{1}{2}(u_i',v_{ij}')^{-e} ,$$

we see that the vector $(\bar{u}_i,\bar{v}_{ij})$ is not extremal, which proves (i). Property (ii) is now apparent from an inspection of the constraints. ∎

From this Lemma we see that inequality (4.10) may be rewritten as

$$(4.11) \qquad y = \sum_{i \in K} s_i + \left[\binom{n}{2} - \binom{k}{2} \right] < \binom{n}{2} ,$$

where k denotes the cardinality of the set $K = \{ i : \bar{u}_i = 1 \}$. But this relation is clearly inconsistent with our hypothesis that the given sequence $S = (s_1,s_2,\ldots,s_n)$ satisfies (2.6). This contradiction shows that $\min y \not< \binom{n}{2}$, and so we must indeed have $\min y = \max z = \binom{n}{2}$.

Having established the existence of an $n \times n$ matrix $X = (x_{ij})$ satisfying (4.1)-(4.4) with $\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} = \binom{n}{2}$, we note that this matrix X must satisfy all of the constraints (4.2)-(4.4) as actual equations. For otherwise, if any one of the relations (4.2)-(4.4) held for X as a strict inequality, then addition of all of those constraints would yield the relation

$$2\left(\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} \right) < n(n-1) ,$$

thereby contradicting the choice of X. Finally, since we may also assume that the matrix X is an extreme point of the convex polyhedron defined by (4.1)-(4.4), then in case the sequence $S = (s_1, s_2, \ldots, s_n)$ was composed entirely of integers, we may invoke the "unimodular property" once again to infer that X is actually a matrix of zeros and ones. This shows that X represents a (generalized) tournament having the given sequence S as its score-vector. The proof of Landau's theorem is now complete, and with it the proof of Moon's generalization.[4]

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[4] The argument presented here is similar in spirit and in certain details to the proof via linear programming of a theorem on systems of distinct representatives due to Hoffman and Kuhn [13], and to a proof by the author of a theorem due to D. R. Fulkerson which characterizes permutation matrices [6]. Still other combinatorial theorems whose proofs follow this same pattern are treated by various authors in [14].

Generalization to C-tournaments.

The approach taken in the preceeding argument may be followed in a more general setting. Let $C = (c_{ij})$ be any upper-triangular $n \times n$ matrix of nonnegative integers, and consider the set of all integer solutions $T = (t_{ij})$ to the following linear system:

(5.1) $\quad t_{ij} \geq 0 , \quad \text{for } 1 \leq i, j \leq n$

(5.2) $\quad t_{ii} = c_{ii} , \quad \text{for } 1 \leq i \leq n$

(5.3) $\quad t_{ij} + t_{ji} = c_{ij} , \quad \text{for } 1 \leq i < j \leq n .$

Such an integer solution-matrix T will be called a C-tournament, since it is plausable to interpret T as a record of the wins and losses in an expanded type of tournament competition where the i-th contestant plays an arbitrary predetermined number of games against the j-th contestant. For example, C-tournaments include the so-called "n-partite tournaments" introduced by Moon in [18]. (An n-partite tournament differs from an ordinary tournament in that there are n nonempty sets of players $P_1, P_2, \ldots , P_n$, and two of the players compete if and only if they do not belong to the same set P_i .) Scores for the contestants in a C-tournament are defined in the same way as for ordinary tournaments, and it is clear that by modifying our proof of Landau's theorem in only a few details, we can immediately obtain a characterization for the score-vectors which may arise from a given choice of the matrix C .

THEOREM 4

For an arbitrary integer-sequence $S = (s_1, s_2, \ldots, s_n)$ to be the score-vector of some C-tournament among n contestants, where C is a given $n \times n$ upper-triangular matrix of nonnegative integers, it is both necessary and sufficient that the inequality

$$\sum_{i \in K} s_i \geq \sum_{i \in K} \sum_{j \in K} c_{ij}$$

shall hold for each subset K of $\{1, 2, \ldots, n\}$, and moreover that strict equality shall hold when $K = \{1, 2, \ldots, n\}$.

We remark that Theorem 4, which is the main new result of this note, reduces to Landau's theorem (Theorem 1) in case C is the upper-triangular matrix of zeros and ones in which $c_{ij} = 1$ if and only if $i < j$. Theorem 4 also encompasses the characterization of score-vectors for n-partite tournaments which was obtained with different methods by Moon in [18]. Finally we mention that Theorem 4 extends to "generalized" C-tournaments simply by dropping the requirement that the matrices C and T must be composed of integers.

It seems likely that several other problems which arise in the theory of tournaments may be ammenable to the methods of linear programming illustrated here. One example which suggests itself is the problem of determining the number of "upsets" which can occur in a tournament having a prescribed score-vector S. (An <u>upset</u> occurs when one contestant defeats another whose record of wins is better (or at least no worse).) For ordinary tournaments this problem was completely solved by D. R. Fulkerson in [8].

Earlier H. J. Ryser in [20] had obtained an explicit formula for the minimum number of upsets that must occur in any tournament with score-vector $S = (s_1, s_2, \ldots, s_n)$, where $s_1 \leq s_2 \leq \cdots \leq s_n$, namely:

$$\sum_{i \in J} [s_i - (i - 1)] ,$$

where $J = \{ i : s_i \geq i - 1 \}$. Although Ryser's methods were completely combinatorial, one can hardly help noticing that this problem asks for the optimum of a certain linear function defined over a convex polyhedron which, in view of the "unimodular property," can have only integral vertices.

This paper was written during Spring 1978 while the author was Visiting Associate Professor in the Mathematics Department at Emory University. The author expresses thanks to Professors C. M. Bang and H. Sharp, Jr. for introducing him to the subject of tournaments and for providing a pre-publication copy of [2].

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