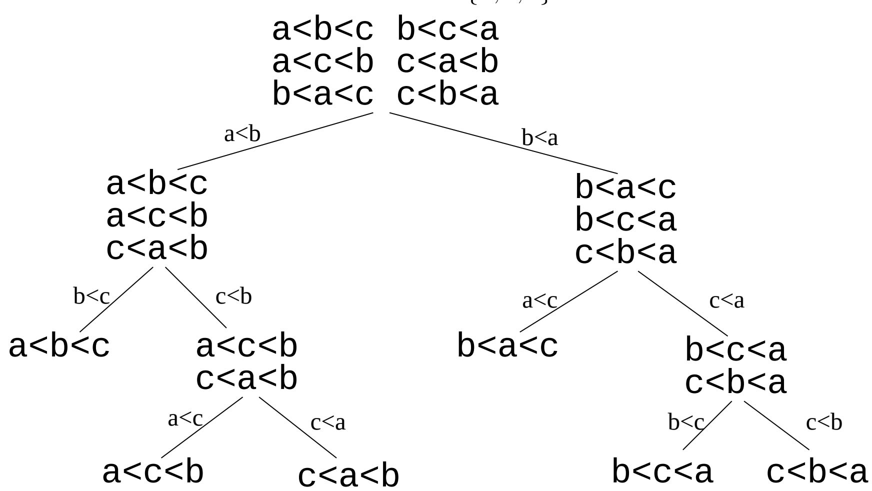
12-0: **Comparison Sorting**

- Comparison sorts work by comparing elements
  - Can only compare 2 elements at a time
  - Check for $<$, $>$, $=$.
- All the sorts we have seen so far (Insertion, Quick, Merge, Heap, etc.) are comparison sorts
- If we know nothing about the list to be sorted, we need to use a comparison sort

12-1: **Decision Trees** Insertion Sort on list $\{a, b, c\}$



12-2: **Decision Trees**

- Every comparison sorting algorithm has a decision tree
- What is the best-case number of comparisons for a comparison sorting algorithm, given the decision tree for the algorithm?

12-3: **Decision Trees**

- Every comparison sorting algorithm has a decision tree
- What is the best-case number of comparisons for a comparison sorting algorithm, given the decision tree for the algorithm?
  - (The depth of the shallowest leaf) + 1
- What is the worst case number of comparisons for a comparison sorting algorithm, given the decision tree for the algorithm?

12-4: **Decision Trees**

- Every comparison sorting algorithm has a decision tree
- What is the best-case number of comparisons for a comparison sorting algorithm, given the decision tree for the algorithm?
  - (The depth of the shallowest leaf) + 1

- What is the worst case number of comparisons for a comparison sorting algorithm, given the decision tree for the algorithm?
  - The height of the tree – (depth of the deepest leaf) + 1

12-5: **Decision Trees**

- What is the largest number of nodes for a tree of depth d?

12-6: **Decision Trees**

- What is the largest number of nodes for a tree of depth d?
  - 2^d
- What is the minimum height, for a tree that has n leaves?

12-7: **Decision Trees**

- What is the largest number of nodes for a tree of depth d?
  - 2^d
- What is the minimum height, for a tree that has n leaves?
  - $\lg n$
- How many leaves are there in a decision tree for sorting n elements?

12-8: **Decision Trees**

- What is the largest number of nodes for a tree of depth d?
  - 2^d
- What is the minimum height, for a tree that has n leaves?
  - $\lg n$
- How many leaves are there in a decision tree for sorting n elements?
  - $n!$
- What is the minimum height, for a decision tree for sorting n elements?

12-9: **Decision Trees**

- What is the largest number of nodes for a tree of depth d?
  - 2^d
- What is the minimum height, for a tree that has n leaves?
  - $\lg n$
- How many leaves are there in a decision tree for sorting n elements?
  - $n!$
- What is the minimum height, for a decision tree for sorting n elements?

- $\lg n!$

12-10: $\lg(n!) \in \Omega(n \lg n)$

$$\begin{aligned}
\lg(n!) &= \lg(n * (n-1) * (n-2) * \ldots * 2 * 1) \\
&= (\lg n) + (\lg(n-1)) + (\lg(n-2)) + \ldots \\
&\quad +(\lg 2) + (\lg 1) \\
&\geq \underbrace{(\lg n) + (\lg(n-1)) + \ldots + (\lg(n/2))}_{n/2 \text{ terms}} \\
&\geq \underbrace{(\lg n/2) + (\lg(n/2)) + \ldots + \lg(n/2)}_{n/2 \text{ terms}} \\
&= (n/2)\lg(n/2) \\
&\in \Omega(n \lg n)
\end{aligned}$$

12-11: **Sorting Lower Bound**

- All comparison sorting algorithms can be represented by a decision tree with $n!$ leaves
- Worst-case number of comparisons required by a sorting algorithm represented by a decision tree is the height of the tree
- A decision tree with $n!$ leaves must have a height of at least $n \lg n$
- All comparison sorting algorithms have worst-case running time $\Omega(n \lg n)$

12-12: **Counting Sort**

- Sorting a list of n integers
- We know all integers are in the range $0 \ldots m$
- We can potentially sort the integers faster than $n \lg n$
- Keep track of a "Counter Array" C:
  - $C[i]$ = # of times value i appears in the list

Example: 3 1 3 5 2 1 6 7 8 1

1	2	3	4	5	6	7	8	9

12-13: **Counting Sort Example**

`3135216781`

0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9

12-14: **Counting Sort Example**

135216781

0	0	0	1	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9

12-15: **Counting Sort Example**

35216781

0	1	0	1	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9

12-16: **Counting Sort Example**

5216781

0	1	0	2	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9

12-17: **Counting Sort Example**

216781

0	1	0	2	0	1	0	0	0	0
0	1	2	3	4	5	6	7	8	9

12-18: **Counting Sort Example**

16781

0	1	1	2	0	1	0	0	0	0
0	1	2	3	4	5	6	7	8	9

12-19: **Counting Sort Example**

6781

0	2	1	2	0	1	0	0	0	0
0	1	2	3	4	5	6	7	8	9

12-20: **Counting Sort Example**

781

0	1	2	3	4	5	6	7	8	9
0	2	1	2	0	1	1	0	0	0

12-21: **Counting Sort Example**

81

0	1	2	3	4	5	6	7	8	9
0	2	1	2	0	1	1	1	0	0

12-22: **Counting Sort Example**

1

0	1	2	3	4	5	6	7	8	9
0	2	1	2	0	1	1	1	1	0

12-23: **Counting Sort Example**

0	1	2	3	4	5	6	7	8	9
0	3	1	2	0	1	1	1	1	0

12-24: **Counting Sort Example**

0	1	2	3	4	5	6	7	8	9
0	3	1	2	0	1	1	1	1	0

1 1 1 2 3 3 5 6 7 8 12-25: $\Theta()$ **of Counting Sort**

- What its the running time of Counting Sort?
- If the list has n elements, all of which are in the range $0 \ldots m$:

12-26: $\Theta()$ **of Counting Sort**

- What its the running time of Counting Sort?
- If the list has n elements, all of which are in the range $0 \ldots m$:
    - Running time is $\Theta(n + m)$
- What about the $\Omega(n \lg n)$ bound for all sorting algorithms?

12-27: $\Theta()$ **of Counting Sort**

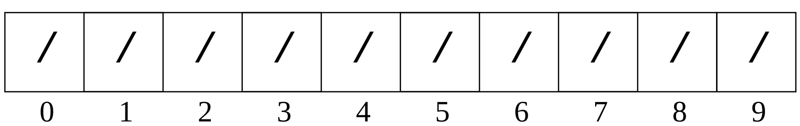
- What its the running time of Counting Sort?
- If the list has n elements, all of which are in the range $0 \ldots m$:
  - Running time is $\Theta(n + m)$
- What about the $\Omega(n \lg n)$ bound for all sorting algorithms?
  - For *Comparison Sorts*, which allow for sorting arbitrary data. What happens when m is very large?

12-28: **Binsort**

- Counting Sort will need some modification to allow us to sort *records* with integer keys, instead of just integers.
- Binsort is much like Counting Sort, except that in each index i of the counting array C:
  - Instead of storing the *number* of elements with the value i, we store a *list* of all elements with the value i.

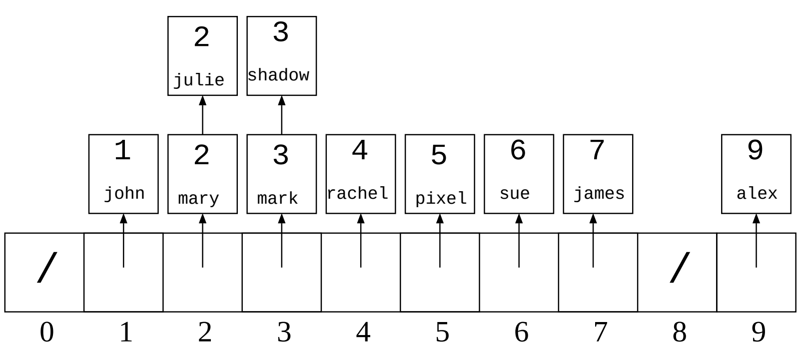
12-29: **Binsort Example**

3	1	2	6	2	4	5	3	9	7	key
mark	john	mary	sue	julie	rachel	pixel	shadow	alex	james	data



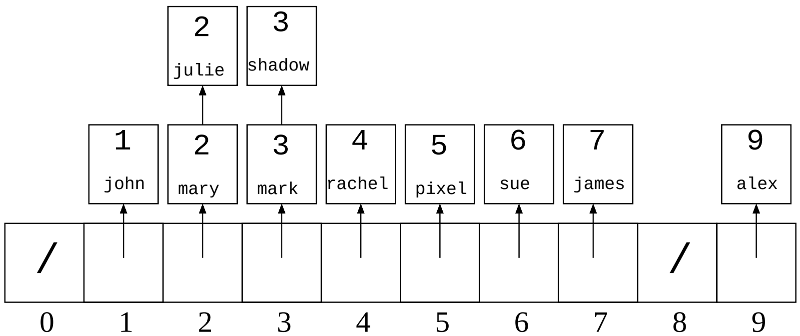
12-30: **Binsort Example**

3	1	2	6	2	4	5	3	9	7	key
mark	john	mary	sue	julie	rachel	pixel	shadow	alex	james	data



12-31: **Binsort Example**

1	2	2	3	3	4	5	6	7	9	key
john	mary	julie	mark	shadow	rachel	pixel	sue	james	alex	data

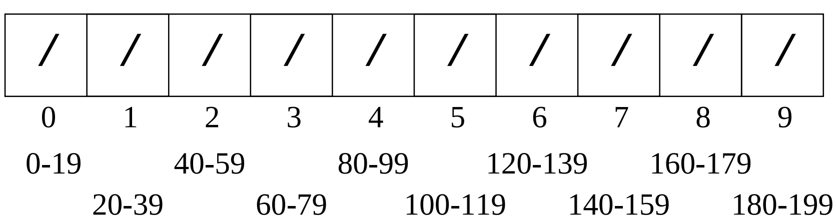


12-32: **Bucket Sort**

- Expand the “bins” in Bin Sort to “buckets”
- Each bucket holds a range of key values, instead of a single key value
- Elements in each bucket are sorted.

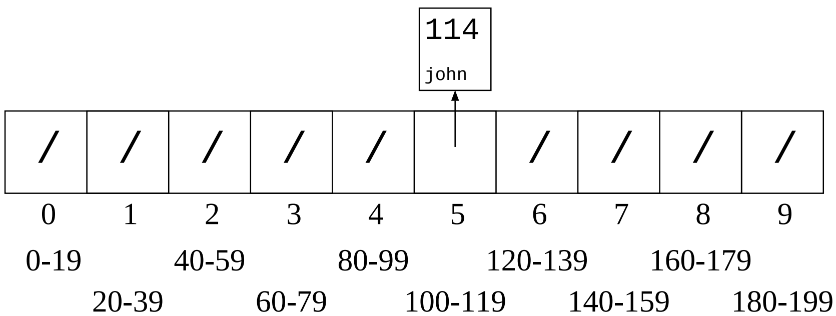
12-33: **Bucket Sort Example**

114	26	50	180	44	111	4	95	196	170	key
john	mary	julie	mark	shadow	rachel	pixel	sue	james	alex	data



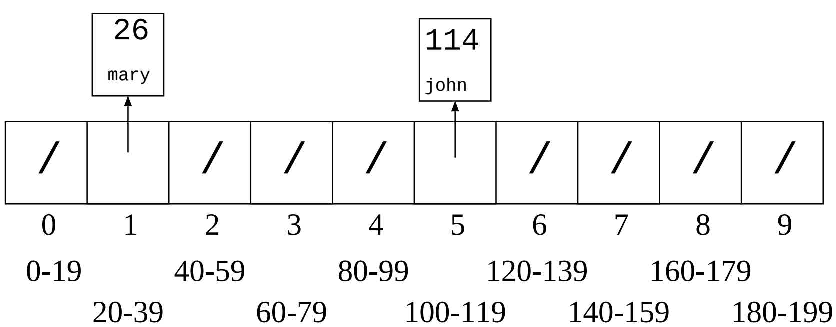
12-34: **Bucket Sort Example**

	26	50	180	44	111	4	95	196	170	key
	mary	julie	mark	shadow	rachel	pixel	sue	james	alex	data

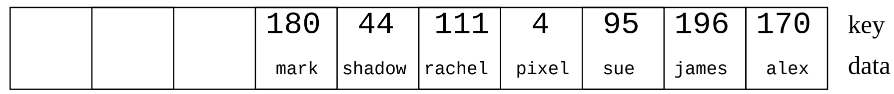


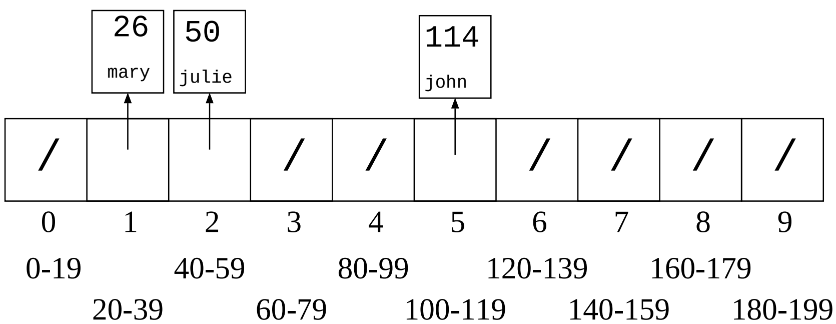
12-35: **Bucket Sort Example**

		50	180	44	111	4	95	196	170	key
		julie	mark	shadow	rachel	pixel	sue	james	alex	data

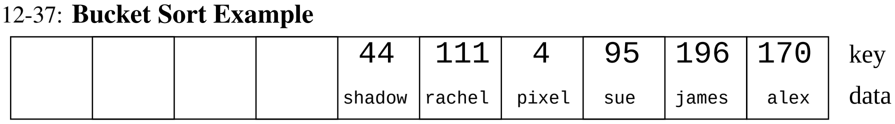


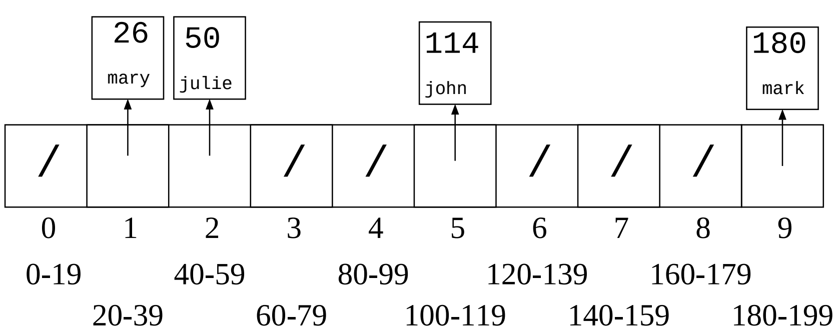
12-36: **Bucket Sort Example**



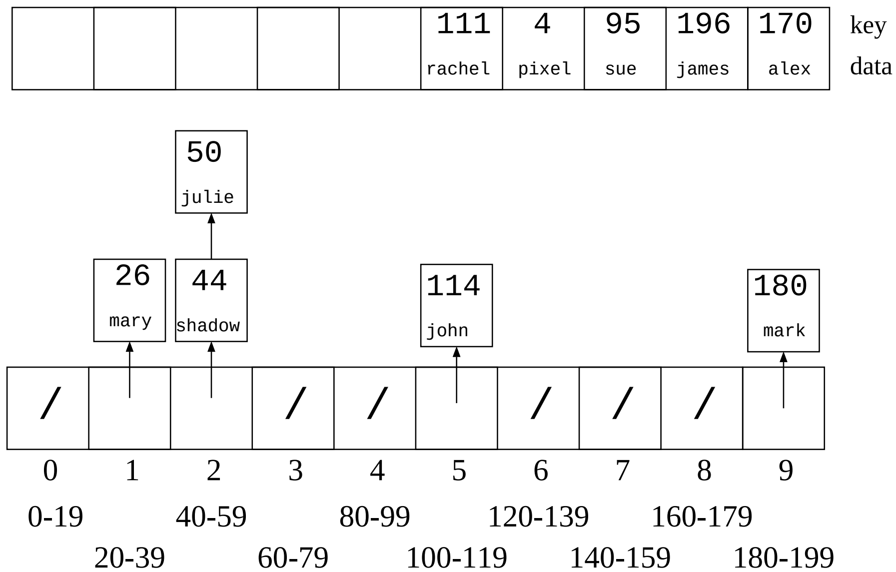


12-37: **Bucket Sort Example**

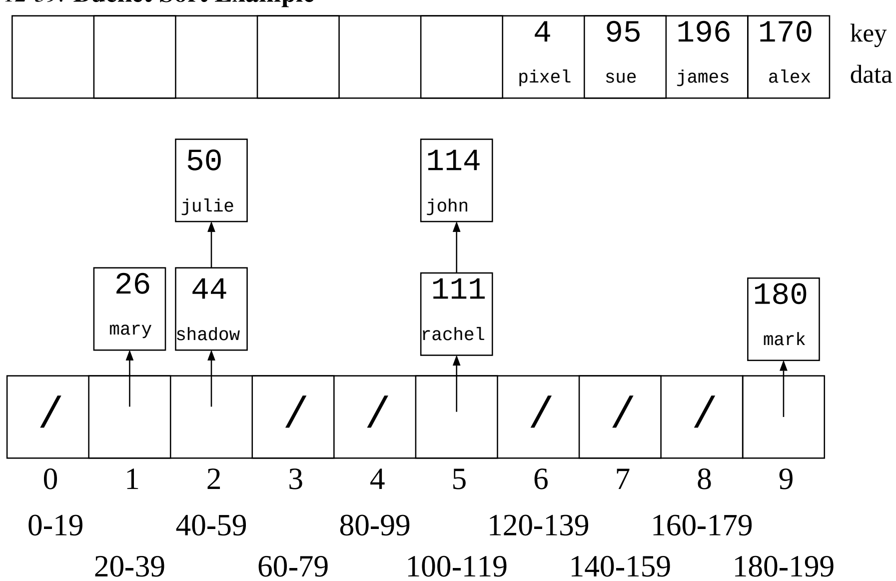




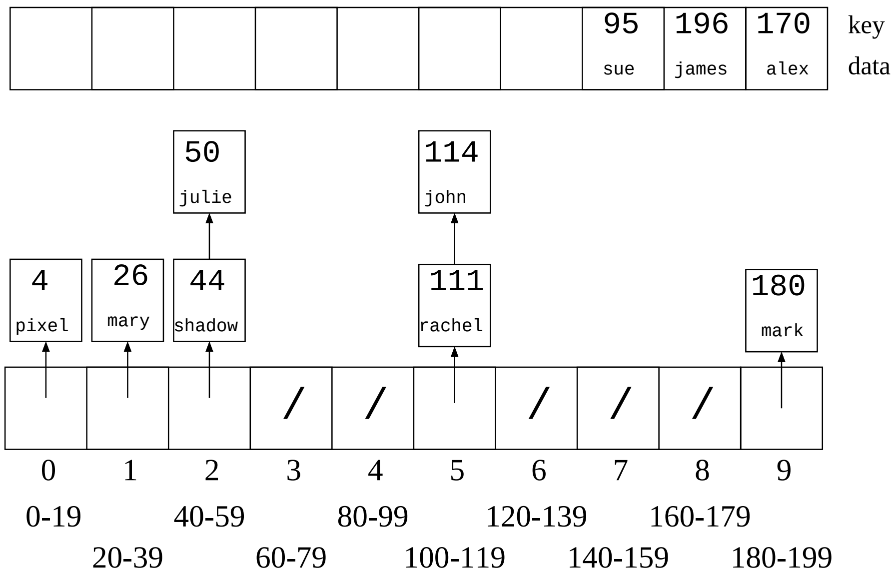
12-38: **Bucket Sort Example**



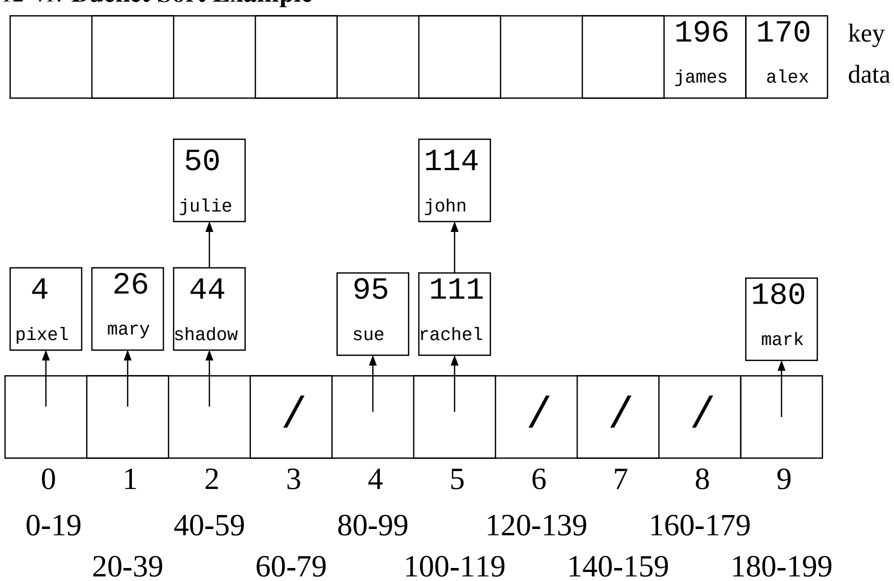
12-39: **Bucket Sort Example**



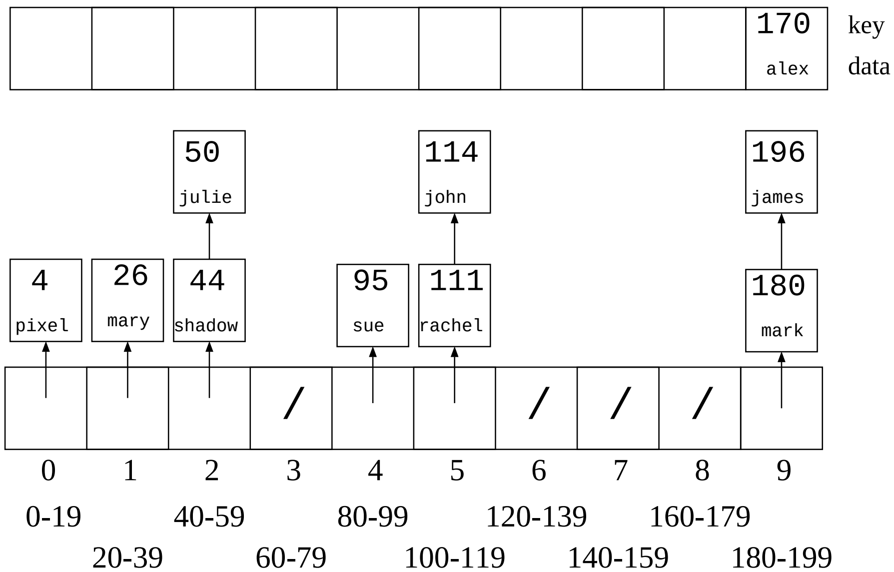
12-40: **Bucket Sort Example**



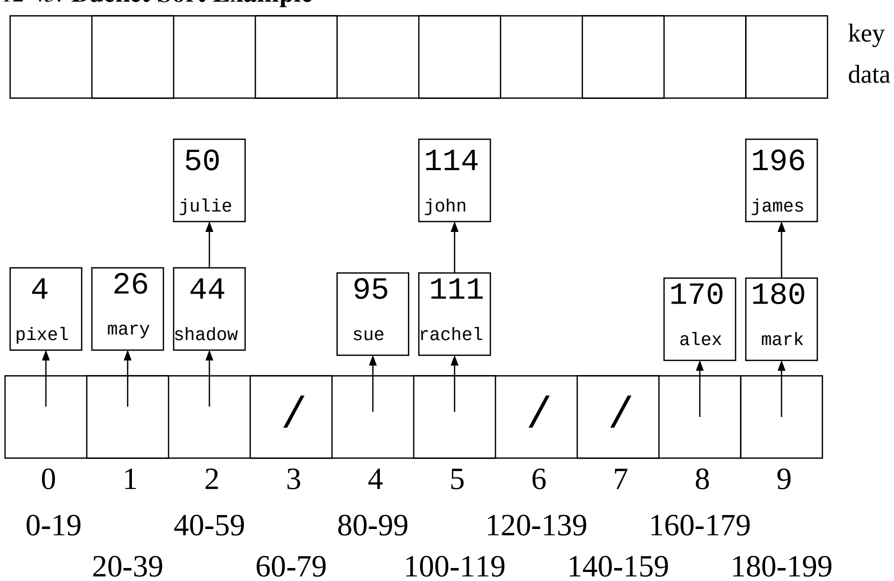
12-41: **Bucket Sort Example**



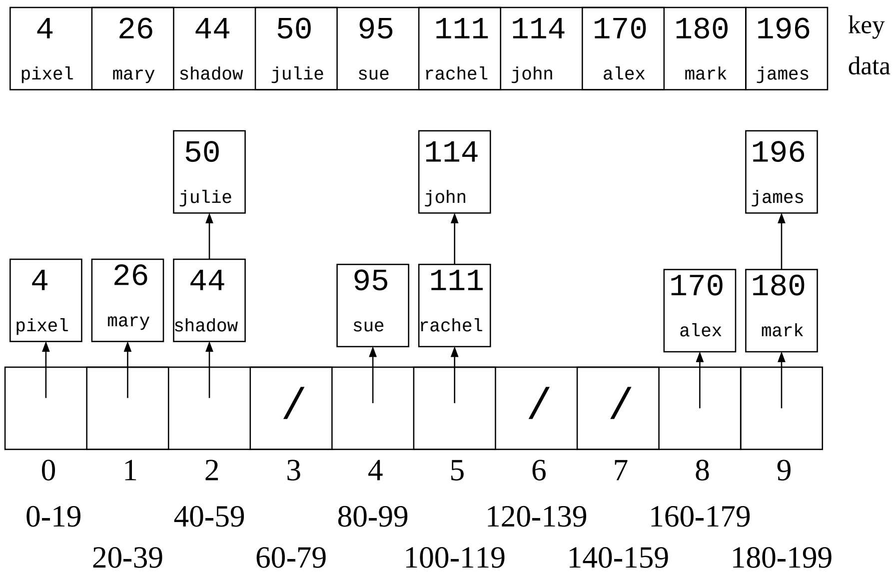
12-42: **Bucket Sort Example**



12-43: **Bucket Sort Example**



12-44: **Bucket Sort Example**



12-45: **Counting Sort Revisited**

- We're going to look at counting sort again
- For the moment, we will assume that our array is indexed from $1 \ldots n$ (where n is the number of elements in the list) instead of being indexed from $0 \ldots n-1$, to make the algorithm easier to understand
- Later, we will go back and change the algorithm to allow for an index between $0 \ldots n-1$

12-46: **Counting Sort Revisited**

- Create the array $C[]$, such that $C[i]$ = # of times key i appears in the array.
- Modify $C[]$ such that $C[i]$ = the *index* of key i in the sorted array. (assume no duplicate keys, for now)
- If $x \notin A$, we don't care about $C[x]$

12-47: **Counting Sort Revisited**

- Create the array $C[]$, such that $C[i]$ = # of times key i appears in the array.
- Modify $C[]$ such that $C[i]$ = the *index* of key i in the sorted array. (assume no duplicate keys, for now)
- If $x \notin A$, we don't care about $C[x]$

```
for(i=1; i<C.length; i++)
  C[i] = C[i] + C[i-1];
```

- Example: 3 1 2 4 9 8 7

12-48: **Counting Sort Revisited**

- Once we have a modified C, such that $C[i]$ = index of key i in the array, how can we use C to sort the array?

12-49: **Counting Sort Revisited**

- Once we have a modified C, such that $C[i]$ = index of key i in the array, how can we use C to sort the array?

```
for (i=1; i <= n; i++)
   B[C[A[i].key()]] = A[i];
for (i=1; i <= n; i++)
   A[i] = B[i];
```

- Example: 3 1 2 4 9 8 7

12-50: **Counting Sort & Duplicates**

- If a list has duplicate elements, and we create C as before:

```
for(i=1; i <= n; i++)
   C[A[i].key()]++;
for(i=1; i < C.length; i++)
  C[i] = C[i] + C[i-1];
```

What will the value of $C[i]$ represent?

12-51: **Counting Sort & Duplicates**

- If a list has duplicate elements, and we create C as before:

```
for(i=1; i <= n; i++)
   C[A[i].key()]++;
for(i=1; i < C.length; i++)
  C[i] = C[i] + C[i-1];
```

What will the value of $C[i]$ represent?

  - The *last* index in A where element i could appear.

12-52: **(Almost) Final Counting Sort**

```
for(i=1; i <= n; i++)
   C[A[i].key()]++;
for(i=1; i < C.length; i++)
  C[i] = C[i] + C[i-1];

for (i=1; i <= n; i++) {
   B[C[A[i].key()]] = A[i];
   C[A[i].key()]--;
}
for (i=1; i <= n; i++)
   A[i] = B[i];
```

- Example: 3 1 2 4 2 2 9 1 6

12-53: **(Almost) Final Counting Sort**

```
for(i=1; i <= n; i++)
   C[A[i].key()]++;
for(i=1; i<C.length; i++)
  C[i] = C[i] + C[i-1];

for (i=1; i <= n; i++) {
   B[C[A[i].key()]] = A[i];
   C[A[i].key()]--;
}
for (i=1; i <= n; i++)
   A[i] = B[i];
```

- Example: 3 1 2 4 2 2 9 1 6
- Is this a Stable sorting algorithm?

12-54: **(Almost) Final Counting Sort**

```
for(i=1; i <= n; i++)
   C[A[i].key()]++;
for(i=1; i < C.length; i++)
  C[i] = C[i] + C[i-1];

for (i = n; i>=1; i++) {
   B[C[A[i].key()]] = A[i];
   C[A[i].key()]--;
}

for (i=1; i < n; i++)
   A[i] = B[i];
```

- How would we change this algorithm if our arrays were indexed from $0 \ldots n-1$ instead of $1 \ldots n$?

12-55: **Final (!) Counting Sort**

```
for(i=0; i < A.length; i++)
  C[A[i].key()]++;
for(i=1; i < C.length; i++)
  C[i] = C[i] + C[i-1];

for (i=A.length - 1; i>=0; i++) {
   C[A[i].key()]--;
   B[C[A[i].key()]] = A[i];
}

for (i=0; i < A.length; i++)
   A[i] = B[i];
```

12-56: **Radix Sort**

- Sort a list of numbers one digit at a time
  - Sort by 1st digit, then 2nd digit, etc

- Each sort can be done in linear time, using counting sort

- First Try: Sort by most significant digit, then the next most significant digit, and so on
  - Need to keep track of a lot of sublists

12-57: **Radix Sort** Second Try:

- Sort by *least significant* digit first
- Then sort by next-least significant digit, using a Stable sort

  . . .
- Sort by most significant digit, using a Stable sort

At the end, the list will be completely sorted. Why?

12-58: **Radix Sort**

- If (most significant digit of x) ¡
  (most significant digit of y),

  then x will appear in A before y.

12-59: **Radix Sort**

- If (most significant digit of x) ¡
  (most significant digit of y),

  then x will appear in A before y.
  - Last sort was by the most significant digit

12-60: **Radix Sort**

- If (most significant digit of x) ¡
  (most significant digit of y),

  then x will appear in A before y.
  - Last sort was by the most significant digit
- If (most significant digit of x) =
  (most significant digit of y) and

  (second most significant digit of x) ¡
  (second most significant digit of y),

  then x will appear in A before y.

12-61: **Radix Sort**

- If (most significant digit of x) ¡
  (most significant digit of y),

  then x will appear in A before y.
  - Last sort was by the most significant digit
- If (most significant digit of x) =
  (most significant digit of y) and

  (second most significant digit of x) ¡
  (second most significant digit of y),

  then x will appear in A before y.
  - After next-to-last sort, x is before y. Last sort does not change relative order of x and y

12-62: **Radix Sort**

Original List

982	414	357	495	500	904	645	777	716	637	149	913	817	493	730	331	201

Sorted by Least Significant Digit

500	730	331	201	982	493	913	414	904	645	495	716	357	777	637	817	149

Sorted by Second Least Significant Digit

500	201	904	913	414	716	817	730	331	637	645	149	357	777	982	493	495

Sorted by Most Significant Digit

149	201	331	357	414	493	495	500	637	645	716	730	777	817	904	913	982

12-63: **Radix Sort**

- We do not need to use a single digit of the key for each of our counting sorts
  - We could use 2-digit chunks of the key instead
  - Our C array for each counting sort would have 100 elements instead of 10

12-64: **Radix Sort**

Original List

9823	4376	2493	1055	8502	4333	1673	8442	8035	6061	7004	3312	4409	2338

Sorted by Least Significant Base-100 Digit (last 2 base-10 digits)

8502	7004	4409	3312	9823	4333	8035	2338	8442	1055	6061	1673	4376	2493

Sorted by Most Significant Base-100 Digit (first 2 base-10 digits)

1055	1673	2338	2493	3312	4333	4376	4409	6061	7004	8035	8442	8502	9823

12-65: **Radix Sort**

- "Digit" does not need to be base ten

- For any value r:
  - Sort the list based on (key $\%$ r)
  - Sort the list based on ((key / r) $\%$ r))
  - Sort the list based on ((key / r^2) $\%$ r))
  - Sort the list based on ((key / r^3) $\%$ r))

    ...
  - Sort the list based on
    ((key / $r^{\log_k(\text{largest value in array})}$) $\%$ r))
- Code on other screen