Data Structures and Algorithms

CS245-2017S-15

Graphs

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15-0: Graphs

- A graph consists of:
  - A set of **nodes** or **vertices** (terms are interchangable)
  - A set of **edges** or **arcs** (terms are interchangable)
- Edges in graph can be either directed or undirected

15-1: Graphs & Edges

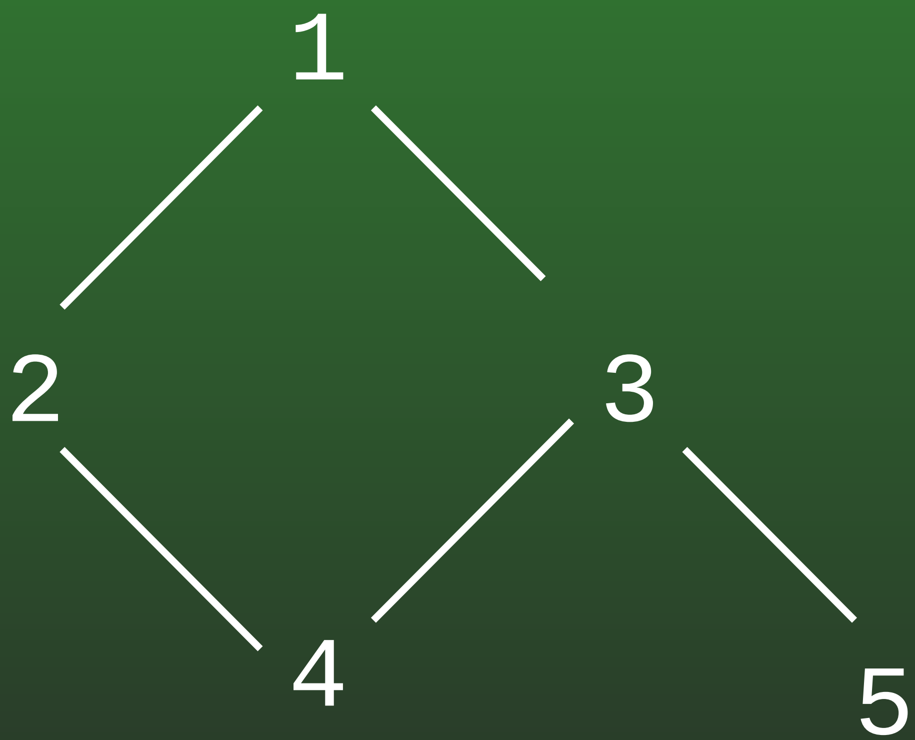
- Edges can be labeled or unlabeled
  - Edge labels are typically the *cost* assoctiated with an edge
  - e.g., Nodes are cities, edges are roads between cities, edge label is the length of road

15-2: Graph Problems

- There are several problems that are “naturally” graph problems
  - Networking problems
  - Route planning
  - etc
- Problems that don’t *seem* like graph problems can also be solved with graphs
  - Register allocation using graph coloring

15-3: Connected Undirected Graph

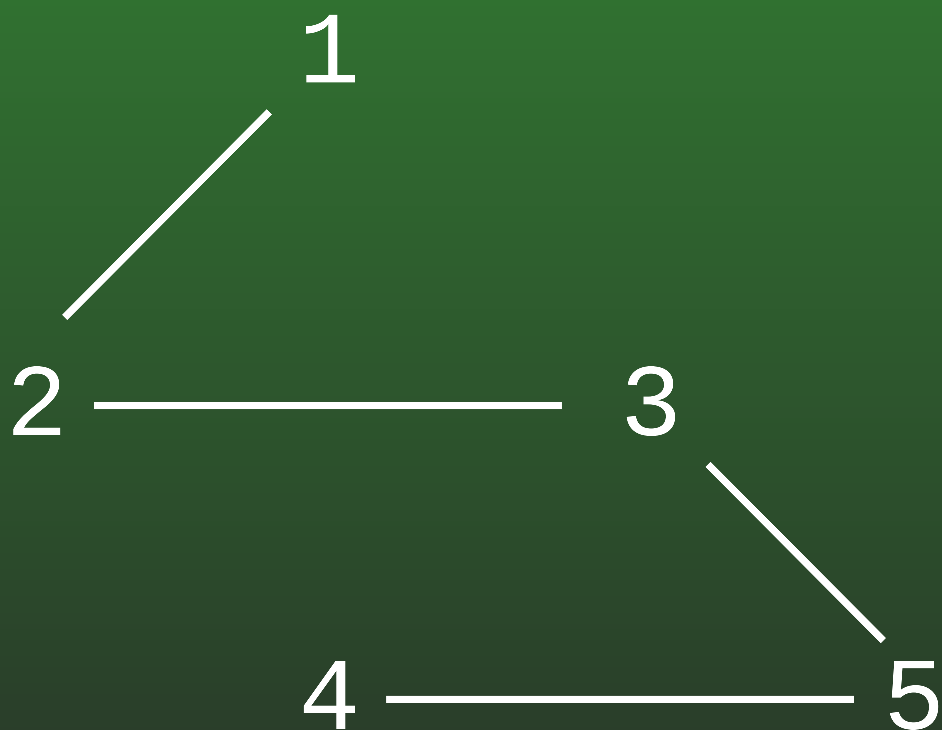
- Path from every node to every other node



- Connected

15-4: Connected Undirected Graph

- Path from every node to every other node



- Connected

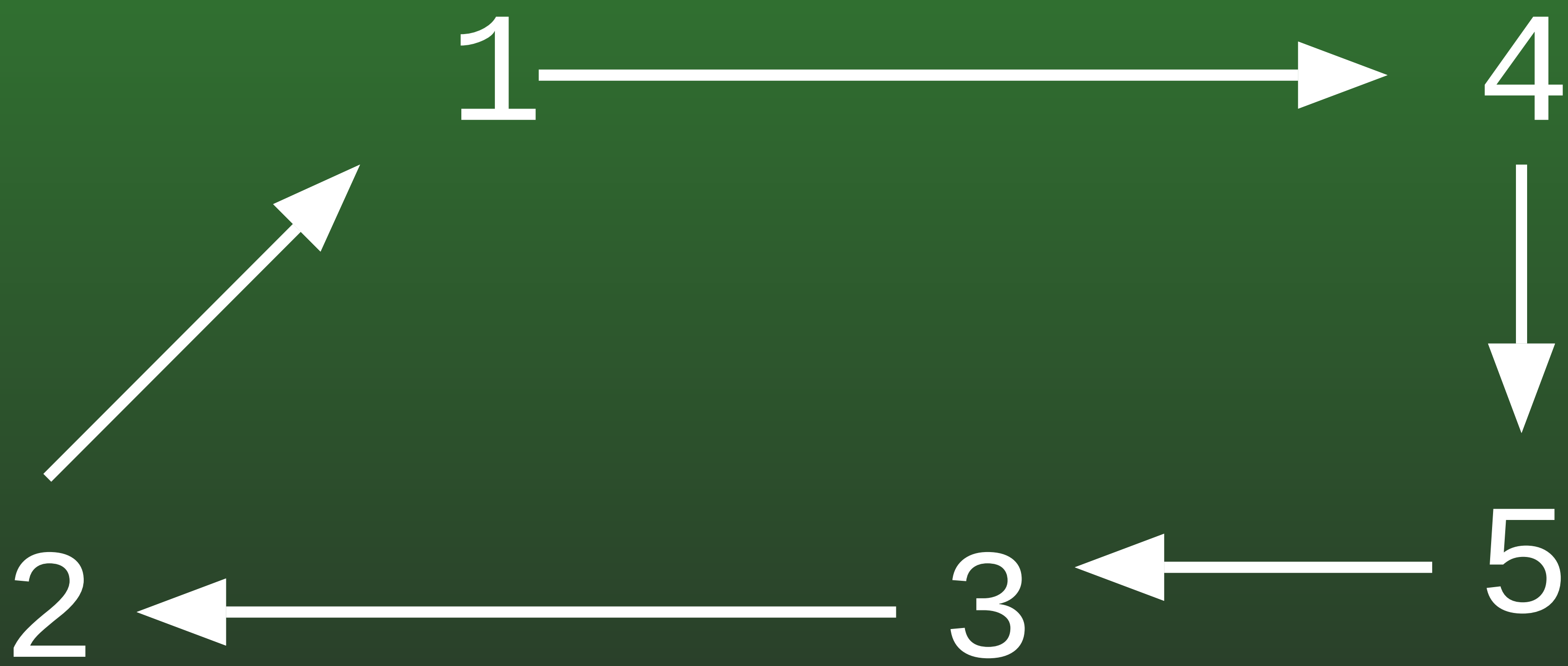
15-5: Connected Undirected Graph

- Path from every node to every other node

- *Not* Connected

15-6: Strongly Connected Graph

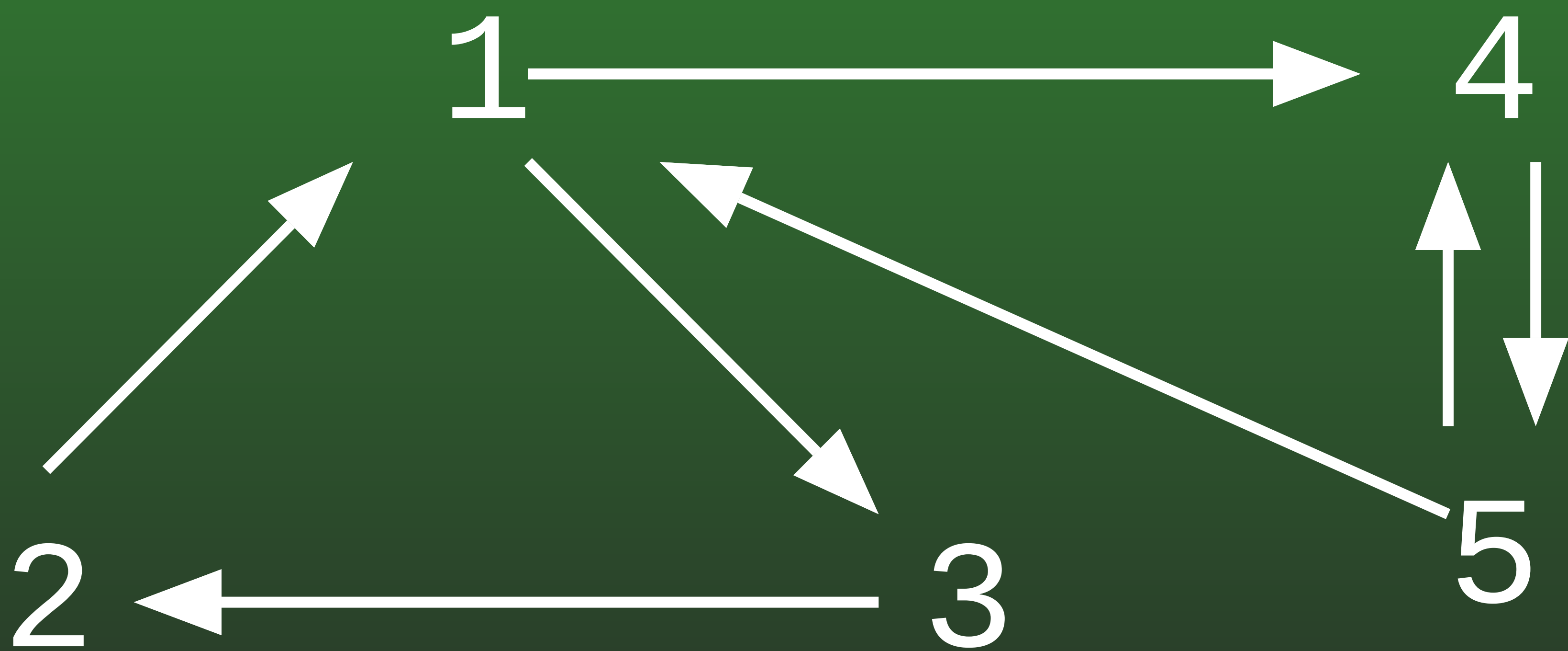
- Directed Path from every node to every other node



- Strongly Connected

15-7: Strongly Connected Graph

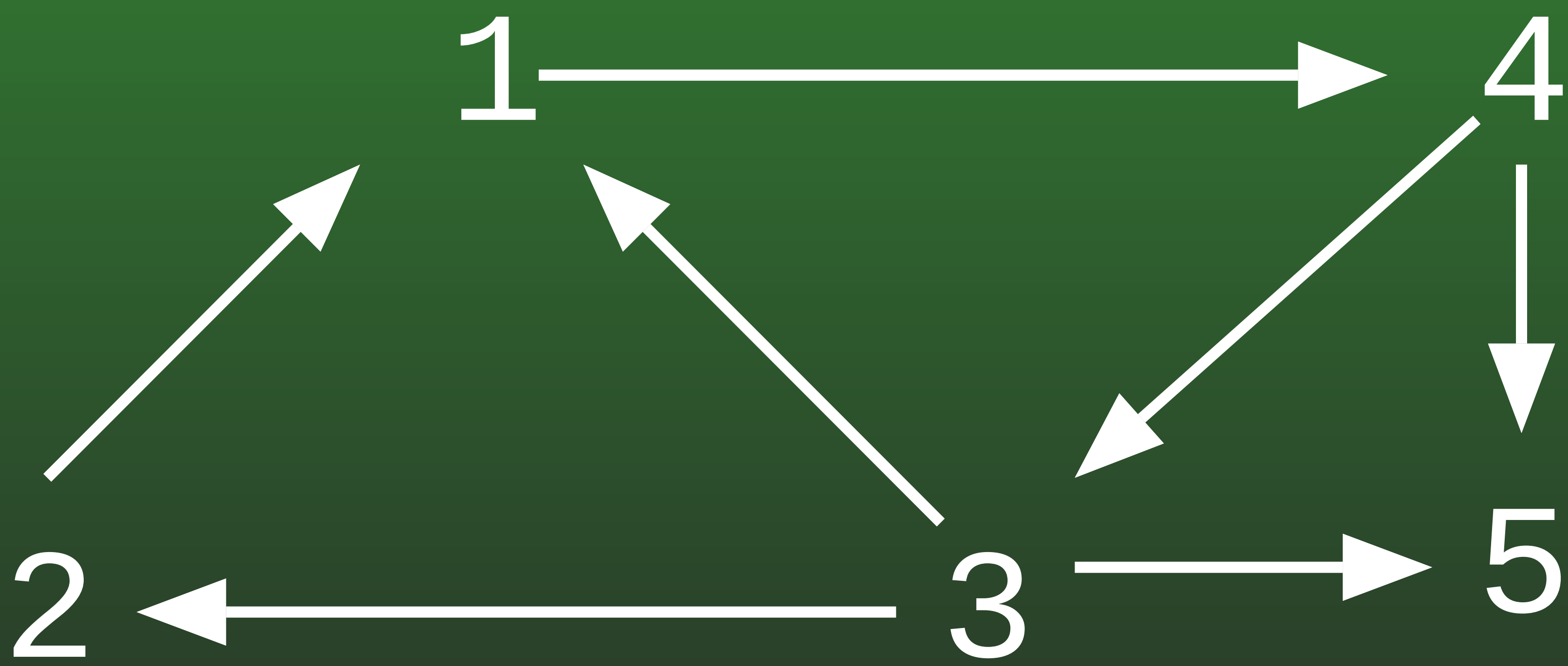
- Directed Path from every node to every other node



- Strongly Connected

15-8: **Strongly Connected Graph**

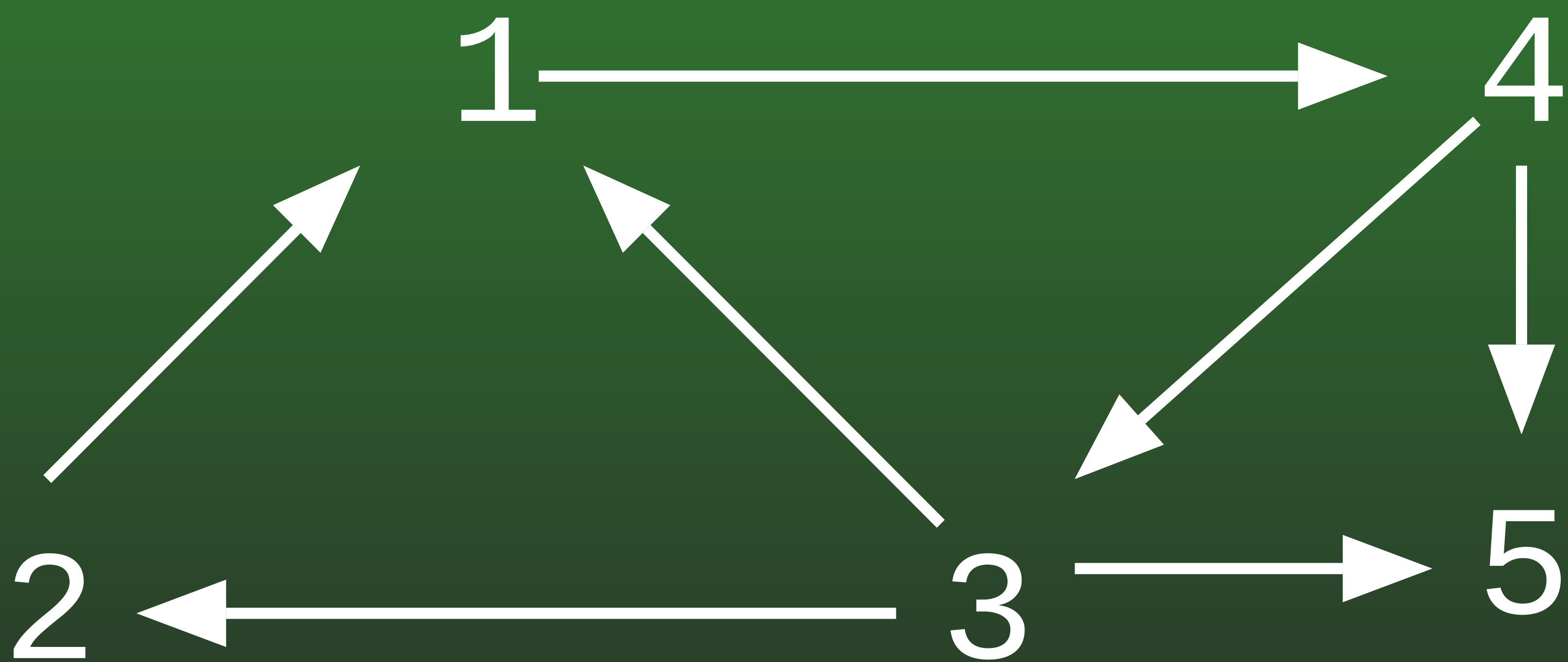
- Directed Path from every node to every other node



- Not Strongly Connected

15-9: Weakly Connected Graph

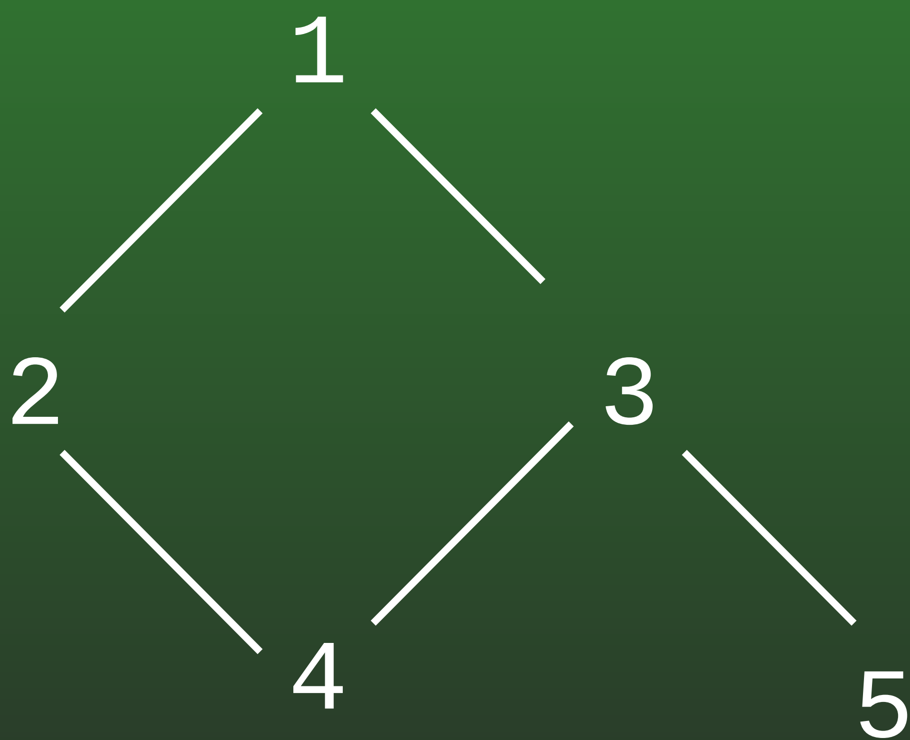
- Directed graph w/ connected backbone



- Weakly Connected

Cycles in Graphs

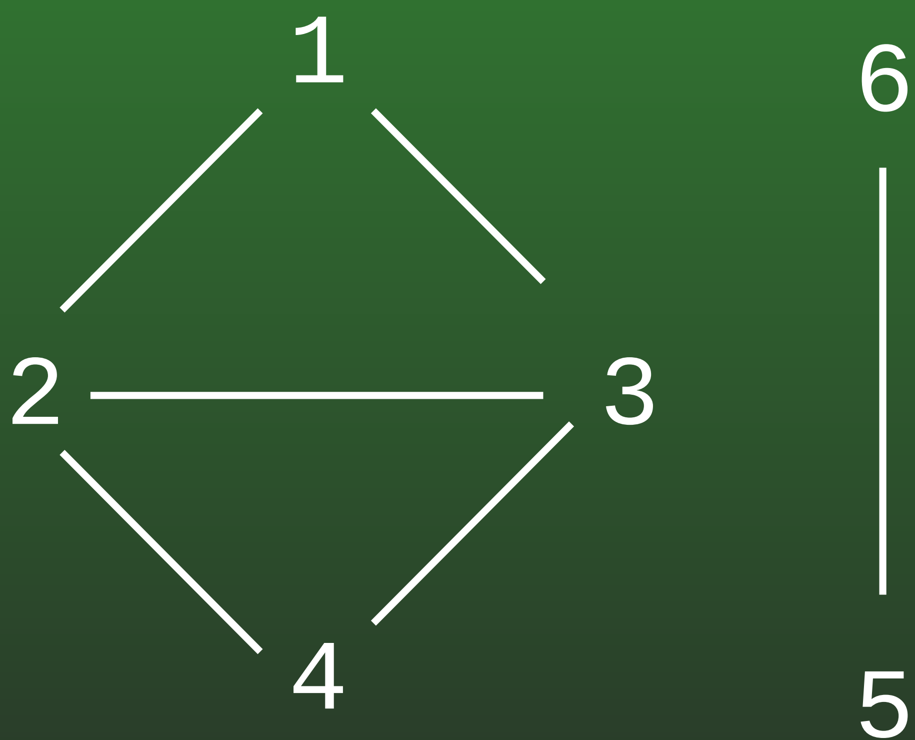
- Undirected cycles



- Contains an undirected cycle

Cycles in Graphs

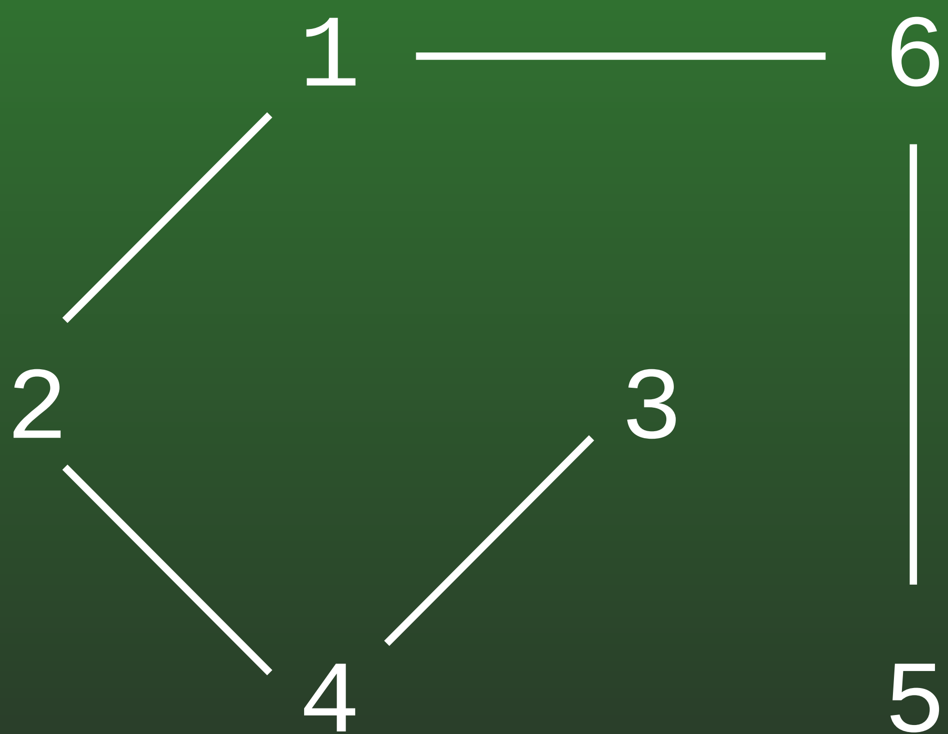
- Undirected cycles



- Contains an undirected cycle

15-12: Cycles in Graphs

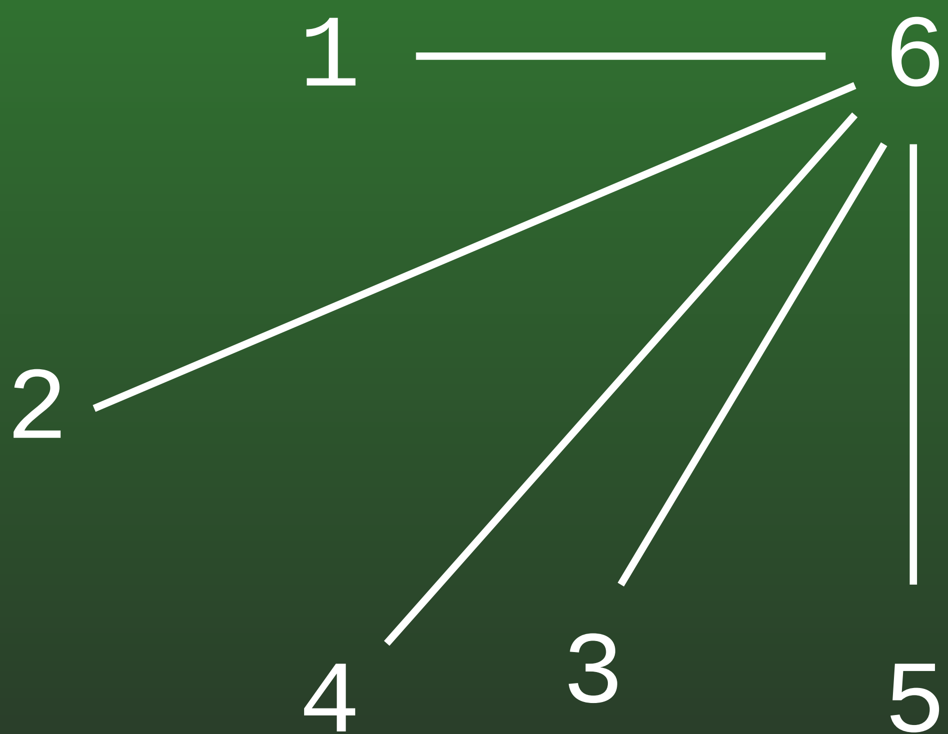
- Undirected cycles



- Contains *no* undirected cycle

15-13: Cycles in Graphs

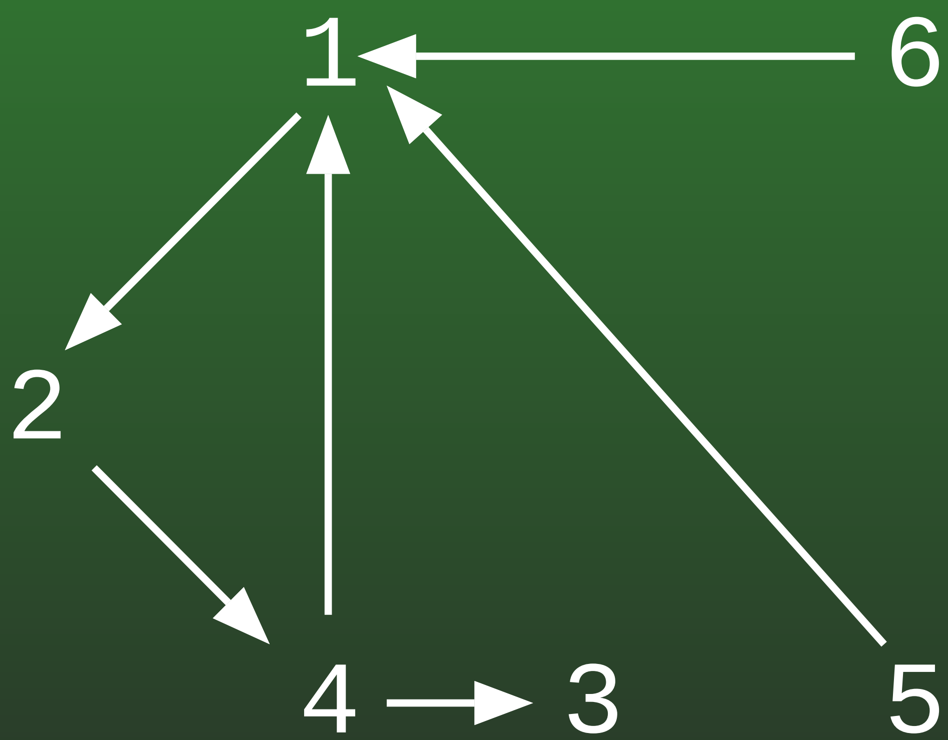
- Undirected cycles



- Contains *no* undirected cycle

15-14: Cycles in Graphs

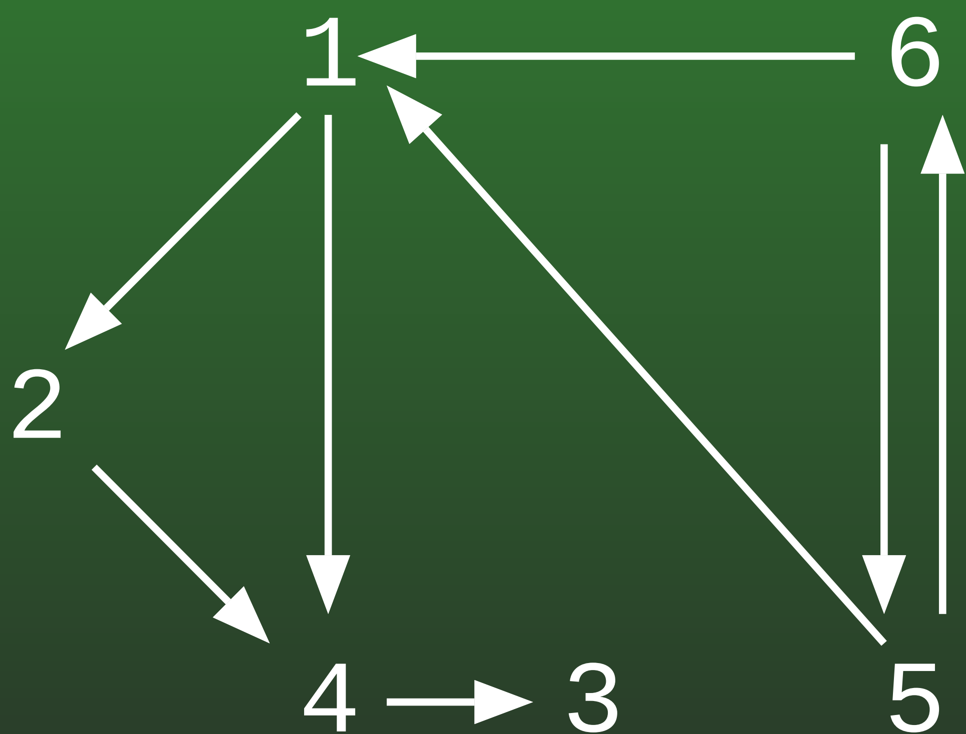
- Directed cycles



- Contains a directed cycle

15-15: Cycles in Graphs

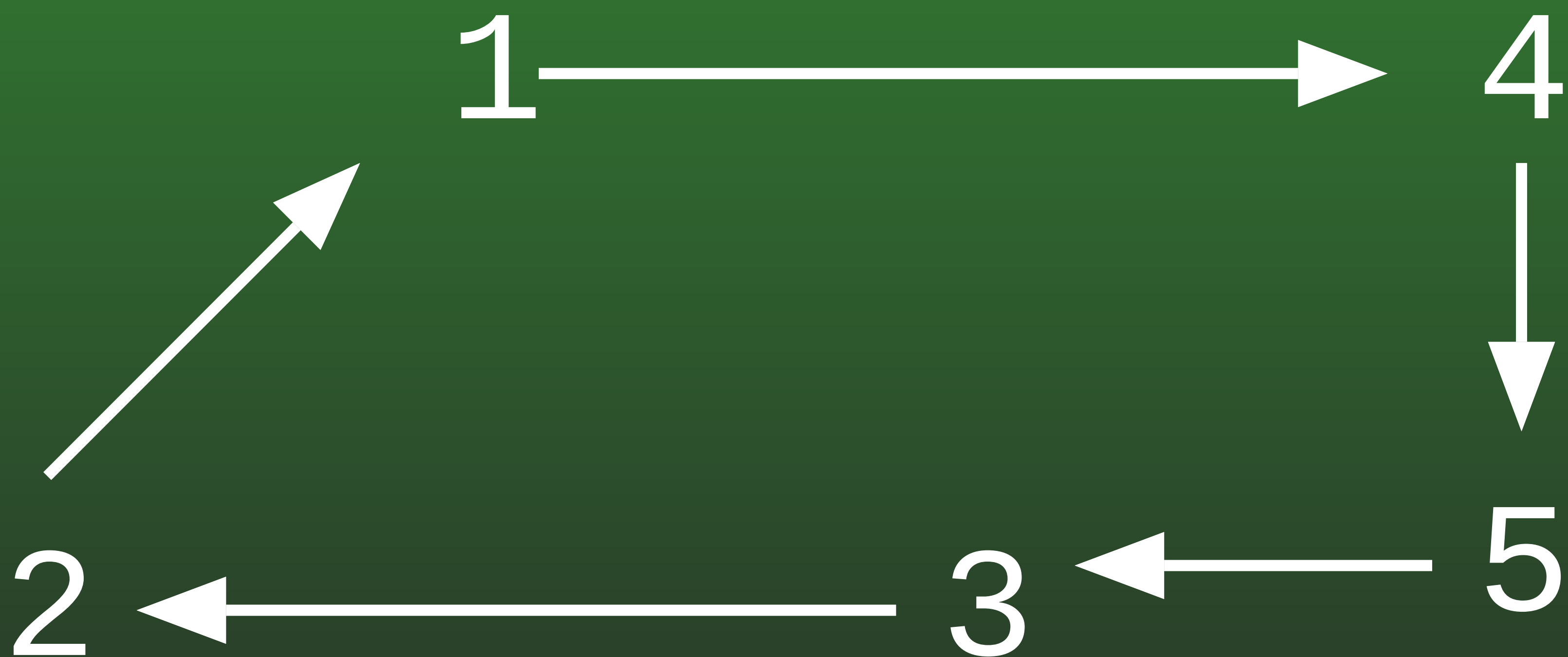
- Directed cycles



- Contains a directed cycle

15-16: Cycles in Graphs

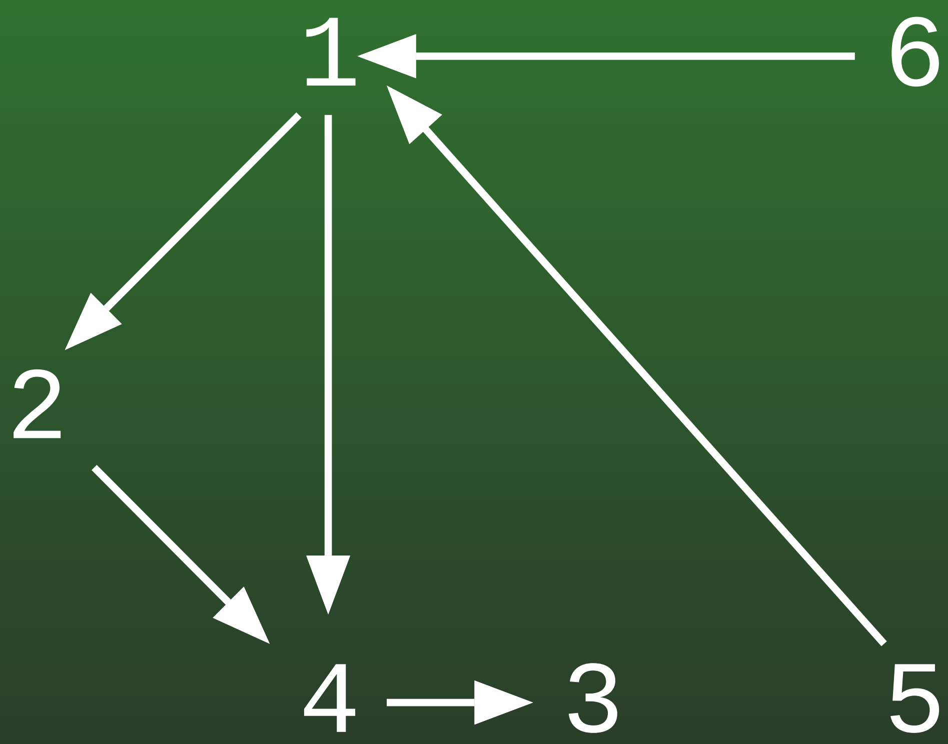
- Directed cycles



- Contains a directed cycle

15-17: Cycles in Graphs

- Directed cycles



- Contains *no* directed cycle

15-18: Cycles & Connectivity

- Must a connected, undirected graph contain a cycle?

15-19: Cycles & Connectivity

- Must a connected, undirected graph contain a cycle?
  - No.
- Can an unconnected, undirected graph contain a cycle?

Cycles & Connectivity

- Must a connected, undirected graph contain a cycle?
  - No.
- Can an unconnected, undirected graph contain a cycle?
  - Yes.
- Must a strongly connected graph contain a cycle?

15-21: Cycles & Connectivity

- Must a connected, undirected graph contain a cycle?
  - No.
- Can an unconnected, undirected graph contain a cycle?
  - Yes.
- Must a strongly connected graph contain a cycle?
  - Yes! (why?)

15-22: Cycles & Connectivity

- If a graph is weakly connected, and contains a cycle, must it be strongly connected?

15-23: Cycles & Connectivity

- If a graph is weakly connected, and contains a cycle, must it be strongly connected?
  - No.

Cycles & Connectivity

- If a graph is weakly connected, and contains a cycle, must it be strongly connected?
  - No.
- If a graph contains a cycle which contains all nodes, must the graph be strongly connected?

Cycles & Connectivity

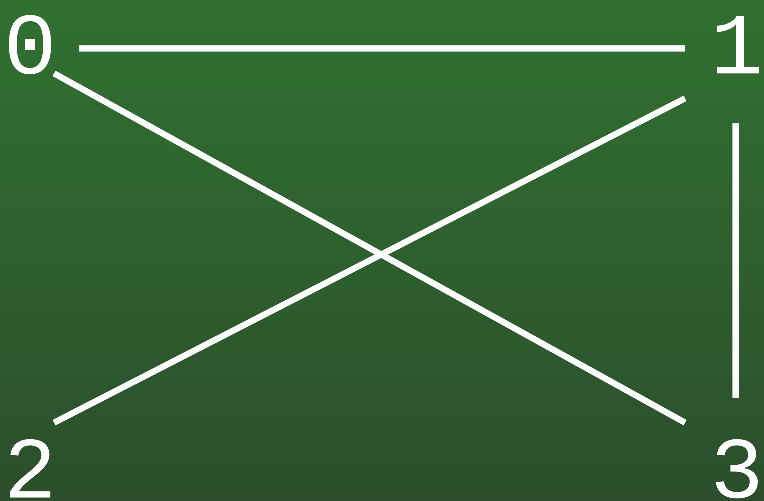
- If a graph is weakly connected, and contains a cycle, must it be strongly connected?
  - No.
- If a graph contains a cycle which contains all nodes, must the graph be strongly connected?
  - Yes. (why?)

Graph Representations

- Adjacency Matrix
- Represent a graph with a two-dimensional array G
  - $G[i][j] = 1$ if there is an edge from node i to node j
  - $G[i][j] = 0$ if there is no edge from node i to node j
- If graph is undirected, matrix is symmetric
- Can represent edges labeled with a cost as well:
  - $G[i][j] =$ cost of link between i and j
  - If there is no direct link, $G[i][j] = \infty$

Adjacency Matrix

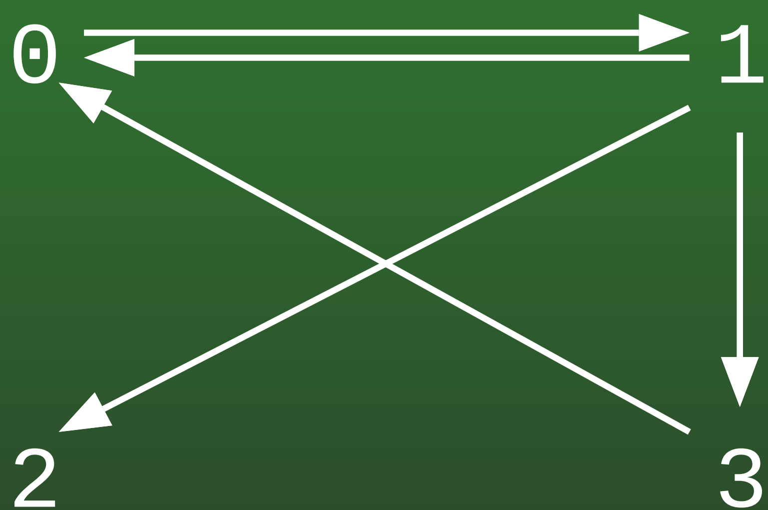
- Examples:



	0	1	2	3
0	0	1	0	1
1	1	0	1	1
2	0	1	0	0
3	1	1	0	0

15-28: Adjacency Matrix

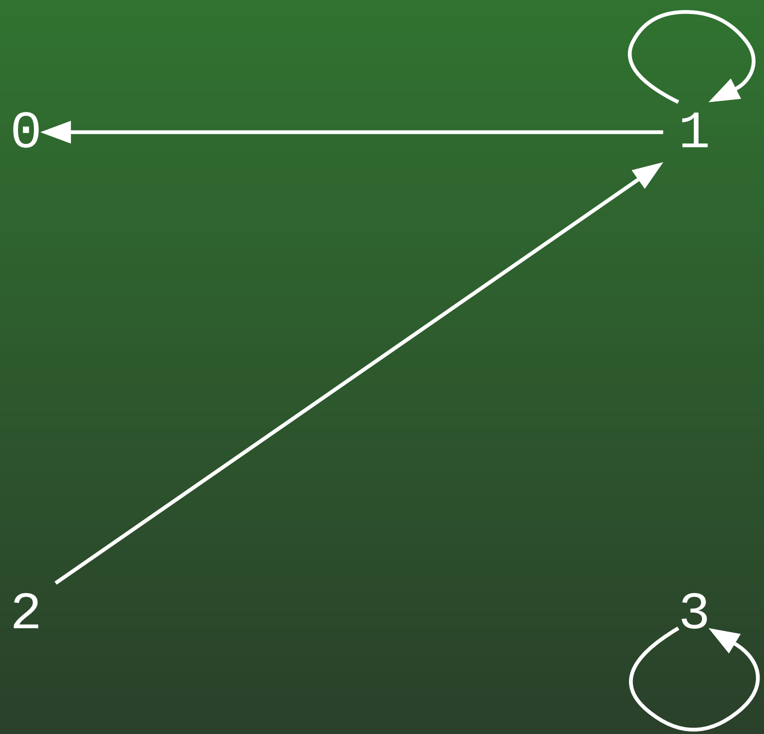
- Examples:



	0	1	2	3
0	0	1	0	0
1	1	0	1	1
2	0	0	0	0
3	1	0	0	0

Adjacency Matrix

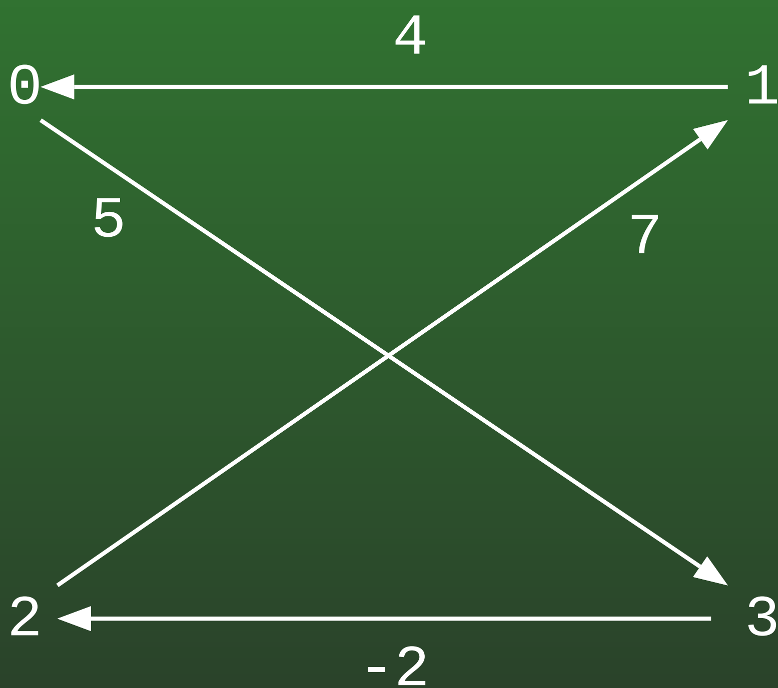
- Examples:



	0	1	2	3
0	0	0	0	0
1	1	1	0	0
2	0	1	0	0
3	0	0	0	1

Adjacency Matrix

- Examples:



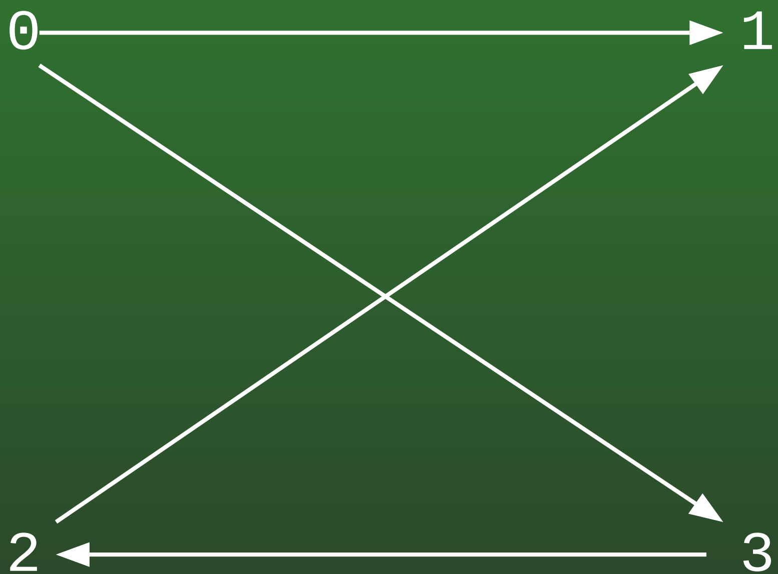
	0	1	2	3
0	∞	∞	∞	5
1	4	∞	∞	∞
2	∞	7	∞	∞
3	∞	∞	-2	∞

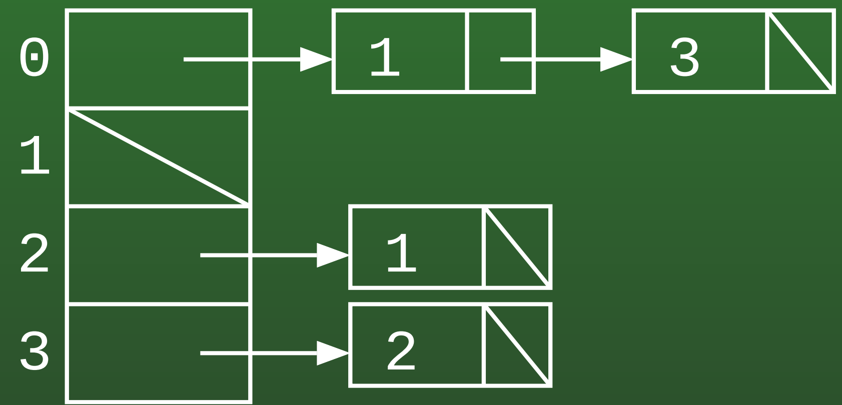
15-31: Graph Representations

- Adjacency List
- Maintain a linked-list of the neighbors of every vertex.
  - n vertices
  - Array of n lists, one per vertex
  - Each list i contains a list of all vertices adjacent to i.

15-32: Adjacency List

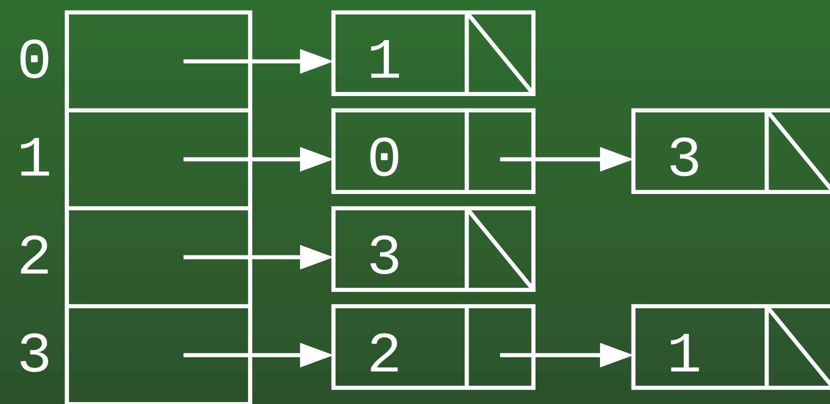
- Examples:





15-33: Adjacency List

- Examples:



- Note – lists are not always sorted

15-34: Sparse vs. Dense

- Sparse graph – relatively few edges
- Dense graph – lots of edges
- Complete graph – contains all possible edges
  - These terms are fuzzy. “Sparse” in one context may or may not be “sparse” in a different context

15-35: Nodes with Labels

- If nodes are labeled with strings instead of integers
  - Internally, nodes are still represented as integers
  - Need to associate string labels & vertex numbers
    - Vertex number $\rightarrow$ label
    - Label $\rightarrow$ vertex number

15-36: Nodes with Labels

- Vertex numbers $\longrightarrow$ labels

15-37: Nodes with Labels

- Vertex numbers $\rightarrow$ labels
  - Array
    - Vertex numbers are indices into array
    - Data in array is string label

15-38: Nodes with Labels

- Labels $\longrightarrow$ vertex numbers

15-39: Nodes with Labels

- Labels $\rightarrow$ vertex numbers
  - Use a hash table
    - Key is the vertex label
    - Data is vertex number

Examples!

15-40: Topological Sort

- Directed Acyclic Graph, Vertices $v_1 \ldots v_n$
- Create an ordering of the vertices
  - If there a path from v_i to v_j, then v_i appears before v_j in the ordering
- Example: Prerequisite chains

15-41: Topological Sort

- Which node(s) could be first in the topological ordering?

15-42: Topological Sort

- Which node(s) could be first in the topological ordering?
  - Node with no incident (incoming) edges

15-43: Topological Sort

- Pick a node v_k with no incident edges
- Add v_k to the ordering
- Remove v_k and all edges from v_k from the graph
- Repeat until all nodes are picked.

15-44: Topological Sort

- How can we find a node with no incident edges?
- Count the incident edges of all nodes

15-45: Topological Sort

```
for (i=0; i < NumberOfVertices; i++)
   NumIncident[i] = 0;

for(i=0; i < NumberOfVertices; i++)
   each node k adjacent to i
      NumIncident[k]++
```

15-46: Topological Sort

```
for(i=0; i < NumberOfVertices; i++)
   NumIncident[i] = 0;

for(i=0; i < NumberOfVertices; i++)
  for(tmp=G[i]; tmp != null; tmp=tmp.next())
    NumIncident[tmp.neighbor()]++
```

15-47: Topological Sort

- Create NumIncident array
- Repeat
  - Search through NumIncident to find a vertex v with NumIncident[v] == 0
  - Add v to the ordering
  - Decrement NumIncident of all neighbors of v
  - Set NumIncident[v] = -1
- Until all vertices have been picked

15-48: Topological Sort

- In a graph with V vertices and E edges, how long does this version of topological sort take?

15-49: Topological Sort

- In a graph with V vertices and E edges, how long does this version of topological sort take?
  - $\Theta(V^2 + E) = \Theta(V^2)$
    - Since $E \in O(V^2)$

15-50: Topological Sort

- Where are we spending “extra” time

15-51: Topological Sort

- Where are we spending “extra” time
  - Searching through NumIncident each time looking for a vertex with no incident edges
  - Keep around a set of all nodes with no incident edges
  - Remove an element v from this set, and add it to the ordering
  - Decrement NumIncident for all neighbors of v
    - If NumIncident[k] is decremented to 0, add k to the set.
  - How do we implement the set of nodes with no incident edges?

Topological Sort

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  - Searching through NumIncident each time looking for a vertex with no incident edges
  - Keep around a set of all nodes with no incident edges
  - Remove an element v from this set, and add it to the ordering
  - Decrement NumIncident for all neighbors of v
    - If NumIncident[k] is decremented to 0, add k to the set.
  - How do we implement the set of nodes with no incident edges?
    - Use a stack

15-53: Topological Sort

- Examples!!
  - Graph
  - Adjacency List
  - NumIncident
  - Stack