Data Structures and Algorithms

CS245-2016S-17

Shortest Path
Dijkstra's Algorithm

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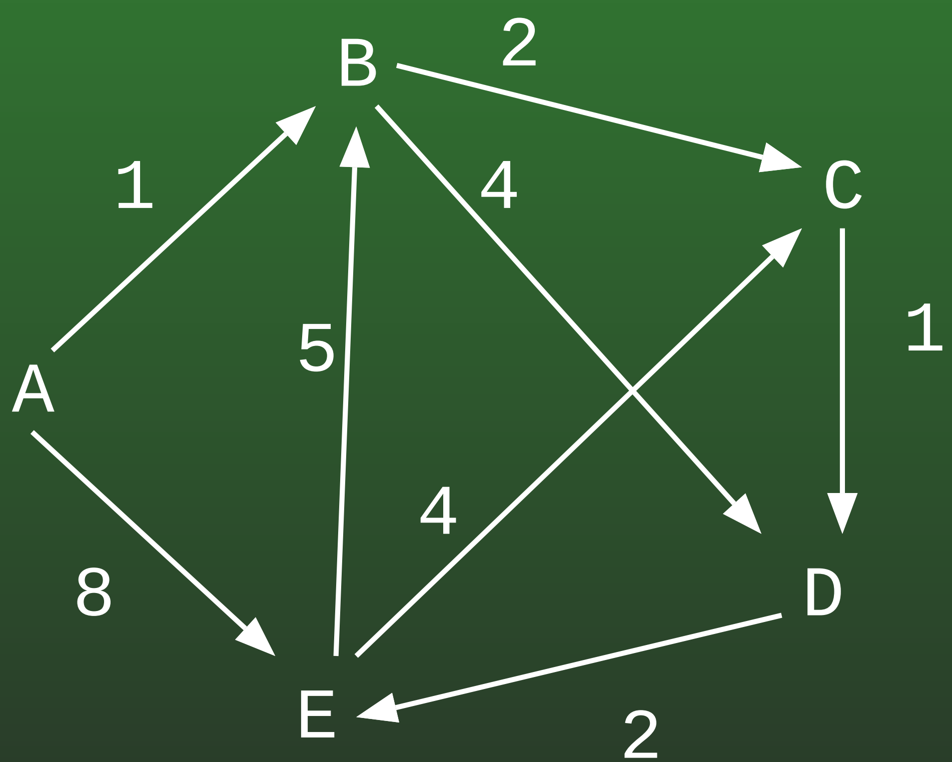
17-0: Computing Shortest Path

- Given a directed weighted graph G (all weights non-negative) and two vertices x and y, find the least-cost path from x to y in G.
  - Undirected graph is a special case of a directed graph, with symmetric edges
- Least-cost path may not be the path containing the fewest edges
  - “shortest path” == “least cost path”
  - “path containing fewest edges” = “path containing fewest edges”

17-1: Shortest Path Example

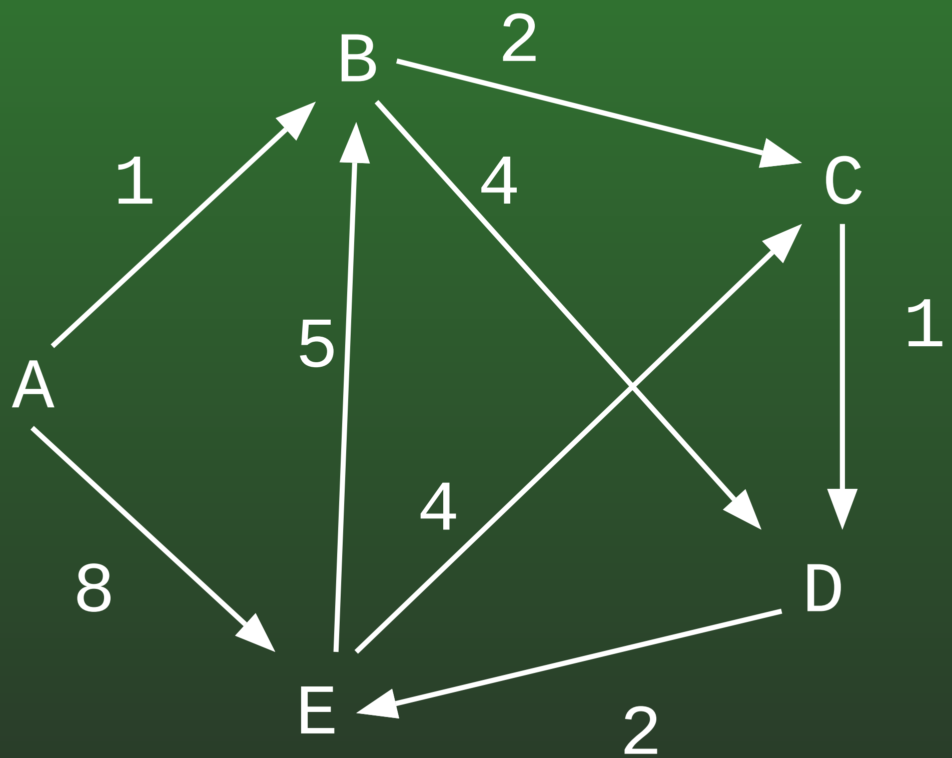
- Shortest path $\neq$ path containing fewest edges



- Shortest Path from A to E?

17-2: Shortest Path Example

- Shortest path $\neq$ path containing fewest edges



- Shortest Path from A to E:
  - A, B, C, D, E

Single Source Shortest Path

- To find the shortest path from vertex x to vertex y, we need (worst case) to find the shortest path from x to *all* other vertices in the graph
  - Why?

17-4: **Single Source Shortest Path**

- To find the shortest path from vertex x to vertex y, we need (worst case) to find the shortest path from x to *all* other vertices in the graph
  - To find the shortest path from x to y, we need to find the shortest path from x to all nodes on the path from x to y
  - Worst case, *all* nodes will be on the path

17-5: Single Source Shortest Path

- If all edges have unit weight ...

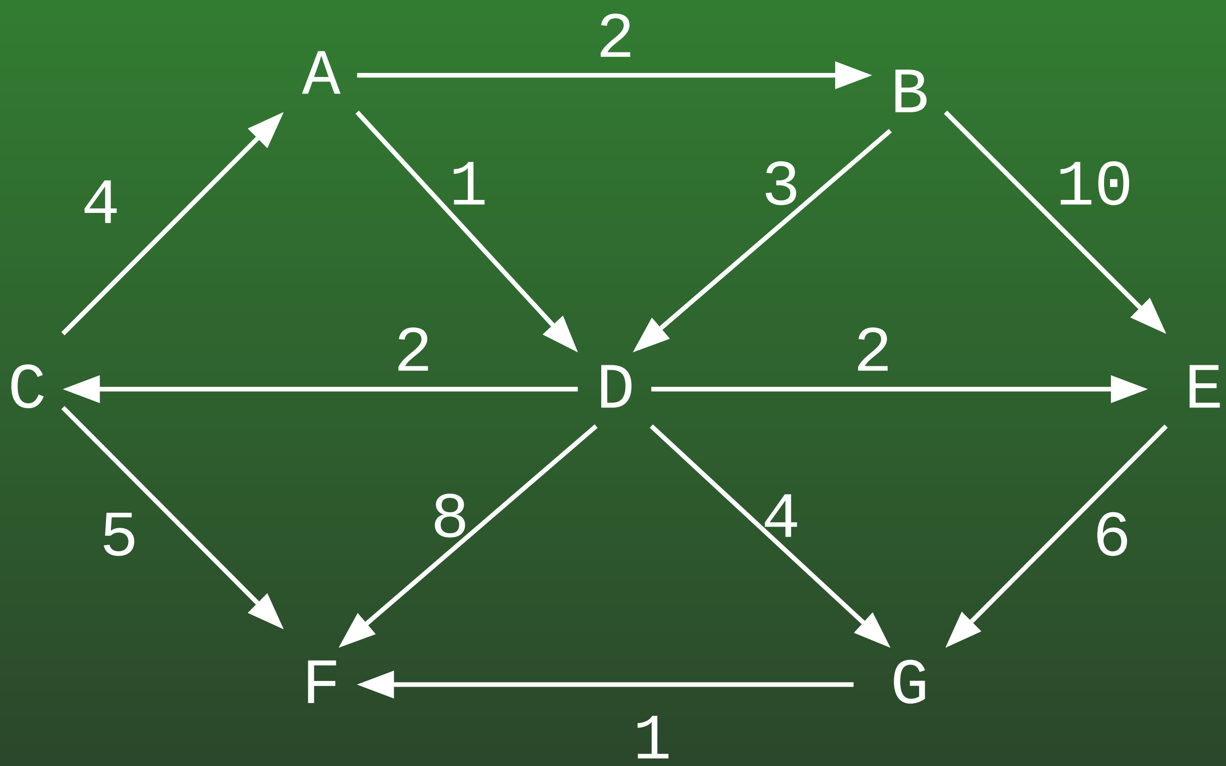
17-6: Single Source Shortest Path

- If all edges have unit weight,
- We can use Breadth First Search to compute the shortest path
- BFS Spanning Tree contains shortest path to each node in the graph
  - Need to do some more work to create & save BFS spanning tree
- When edges have differing weights, this obviously will not work

17-7: Single Source Shortest Path

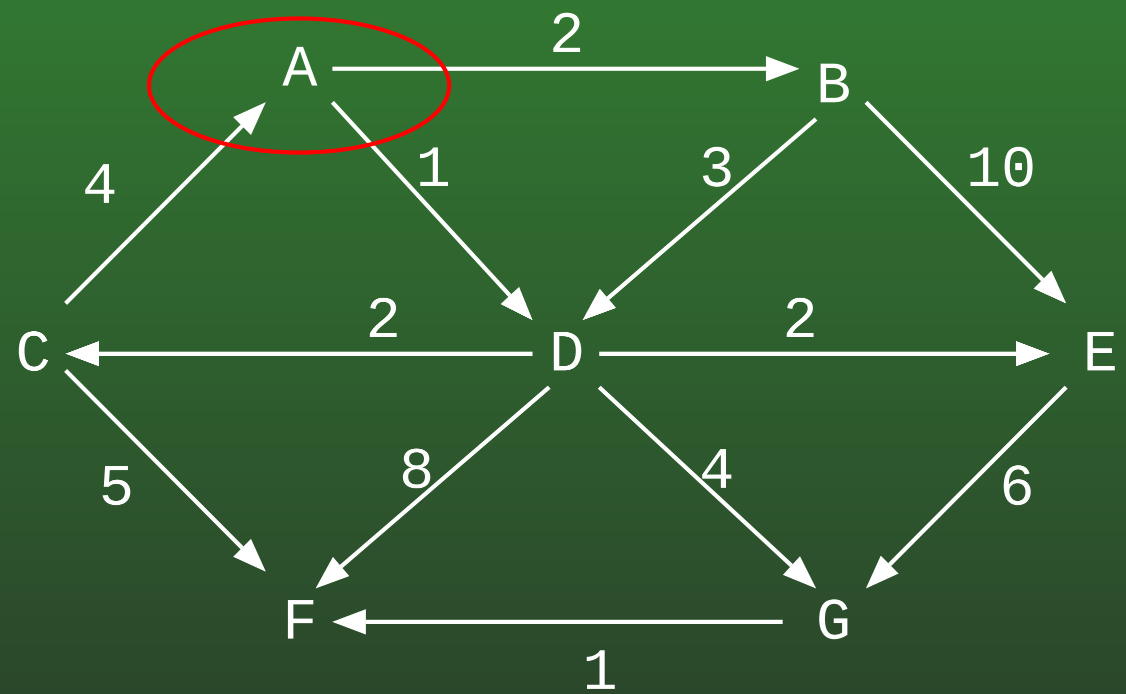
- Divide the vertices into two sets:
  - Vertices whose shortest path from the initial vertex is known
  - Vertices whose shortest path from the initial vertex is not known
- Initially, only the initial vertex is known
- Move vertices one at a time from the unknown set to the known set, until all vertices are known

17-8: Single Source Shortest Path



- Start with the vertex A

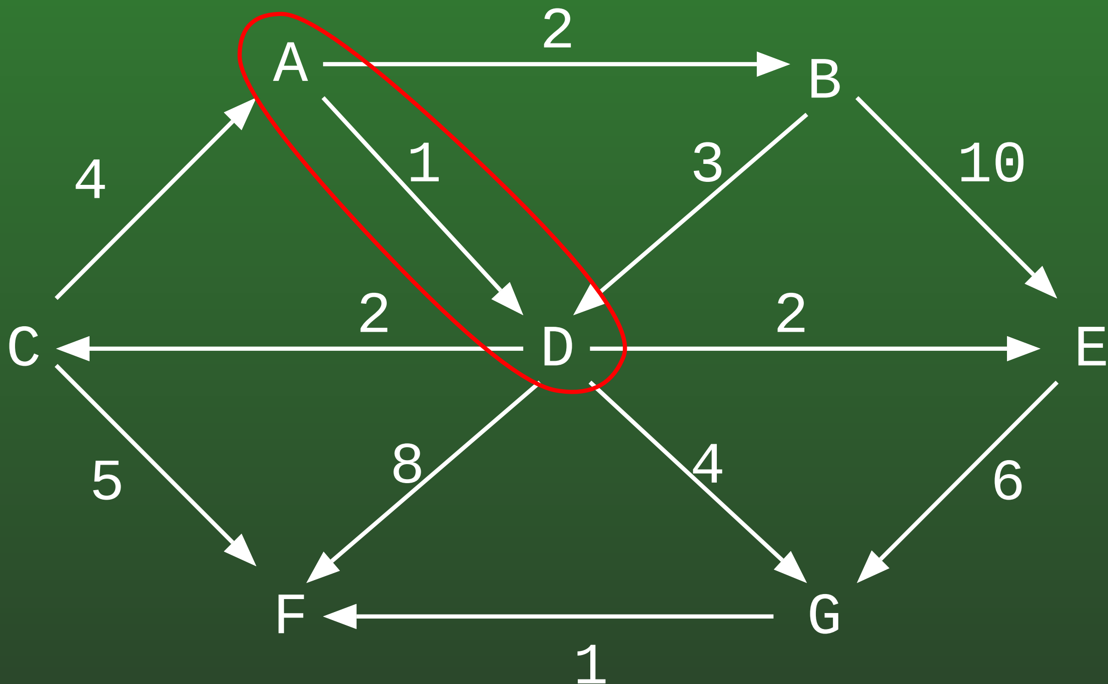
17-9: Single Source Shortest Path



Node	Distance
A	0
B	
C	
D	
E	
F	
G	

- Known vertices are circled in red
- We can now extend the known set by 1 vertex

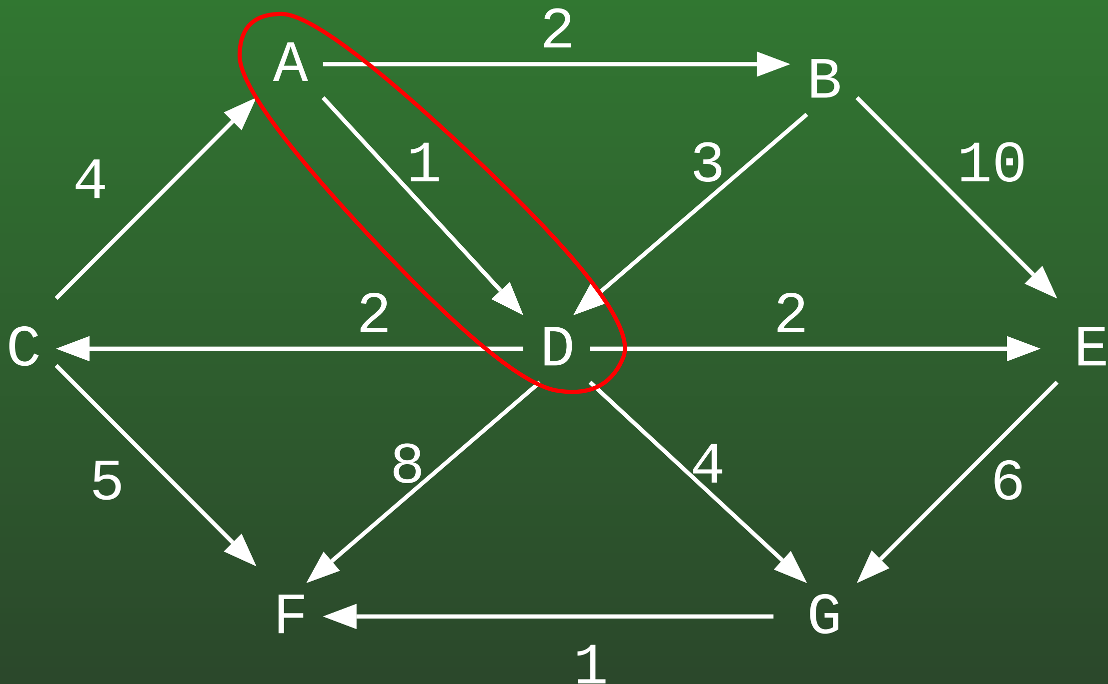
17-10: Single Source Shortest Path



Node	Distance
A	0
B	
C	
D	1
E	
F	
G	

- Why is it safe to add D, with cost 1?

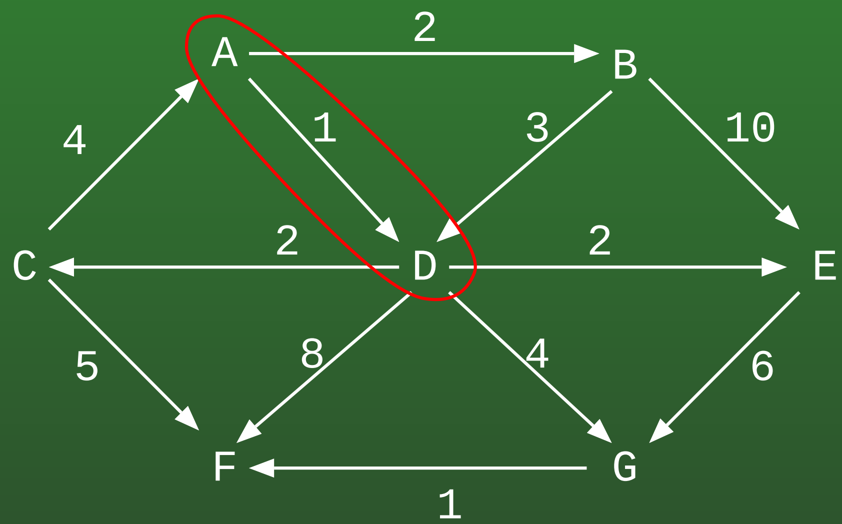
17-11: Single Source Shortest Path



Node	Distance
A	0
B	
C	
D	1
E	
F	
G	

- Why is it safe to add D, with cost 1?
  - Could we do better with a more roundabout path?

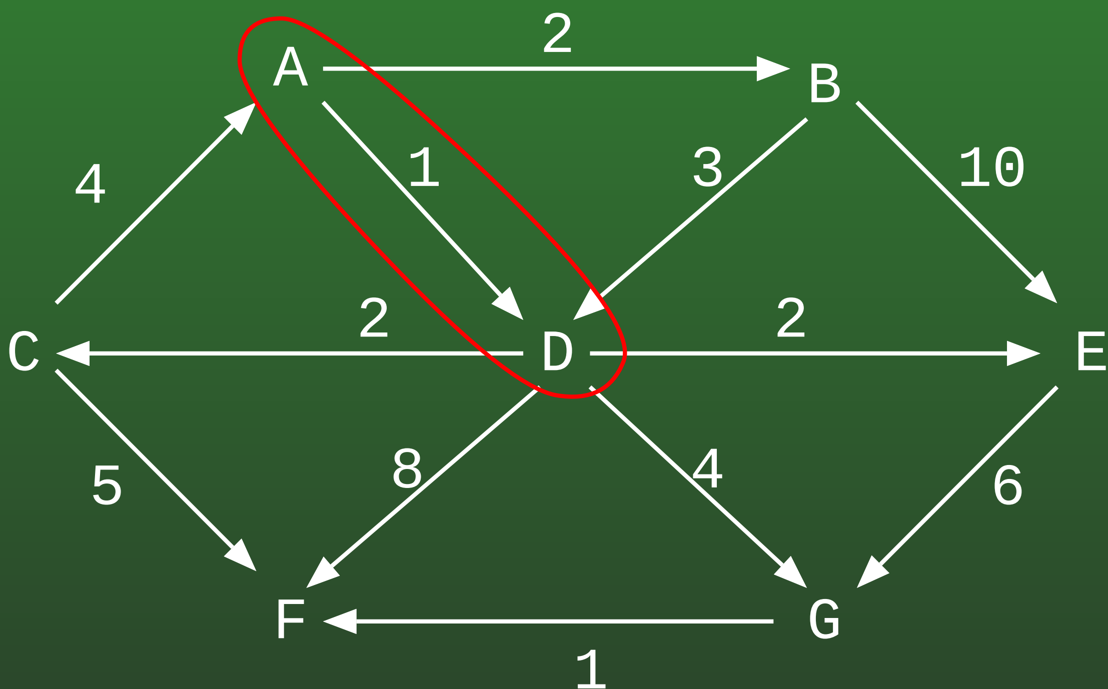
17-12: Single Source Shortest Path



Node	Distance
A	0
B	
C	
D	1
E	
F	
G	

- Why is it safe to add D, with cost 1?
  - Could we do better with a more roundabout path?
  - No – to get to any other node will cost at least 1
  - No negative edge weights, can't do better than 1

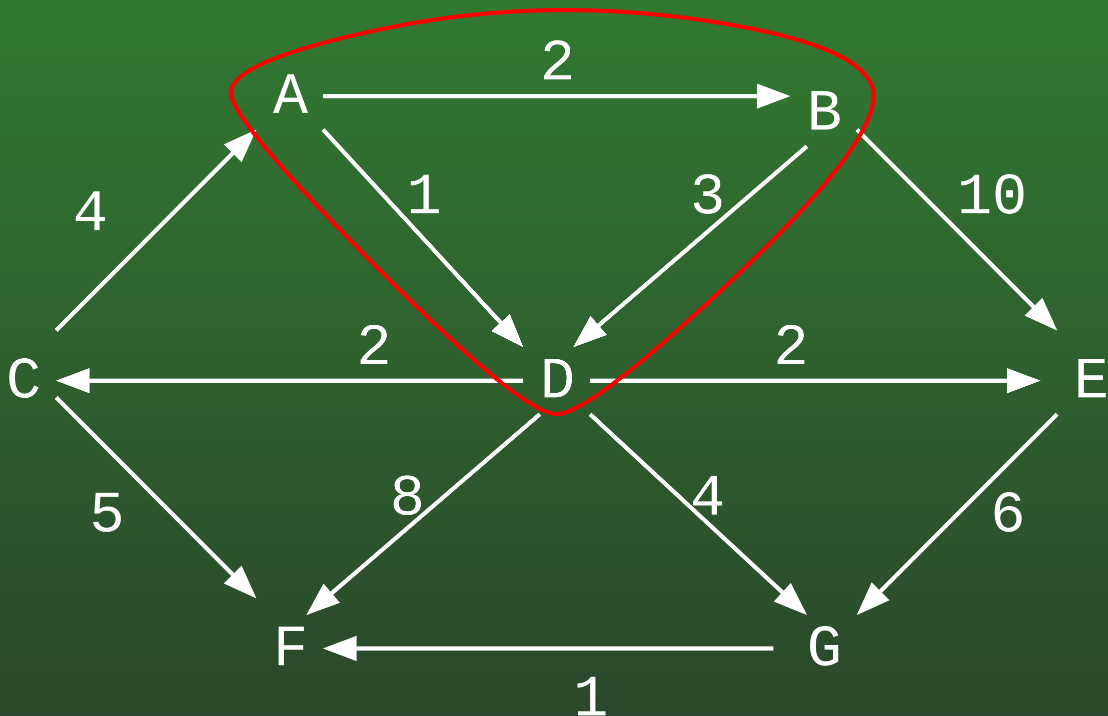
17-13: Single Source Shortest Path



Node	Distance
A	0
B	
C	
D	1
E	
F	
G	

- We can now add another vertex to our known list ...

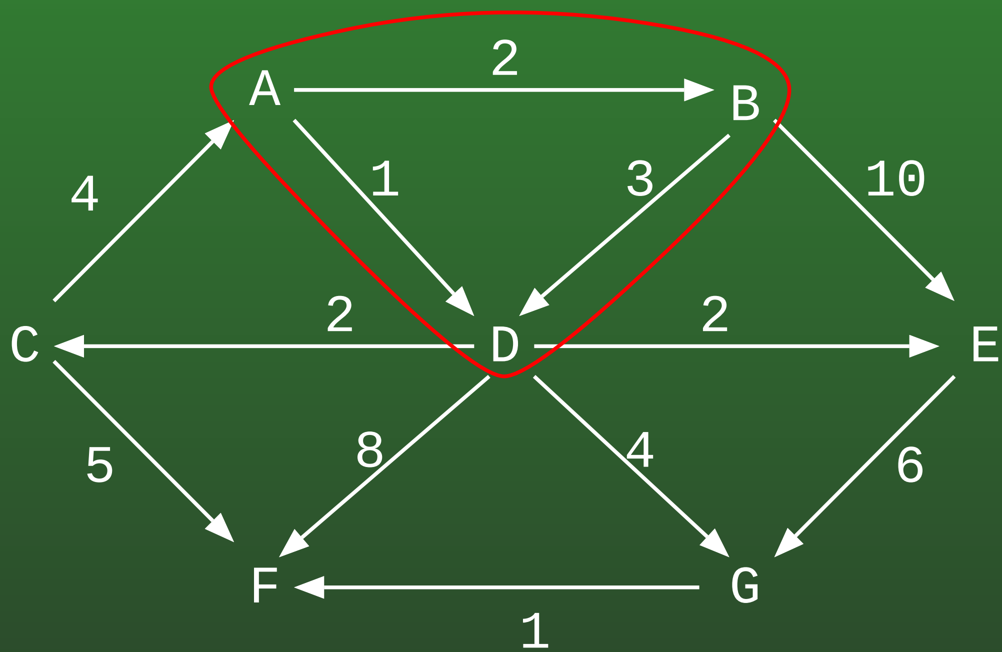
17-14: Single Source Shortest Path



Node	Distance
A	0
B	2
C	
D	1
E	
F	
G	

- How do we know that we could not get to B cheaper than by going through D?

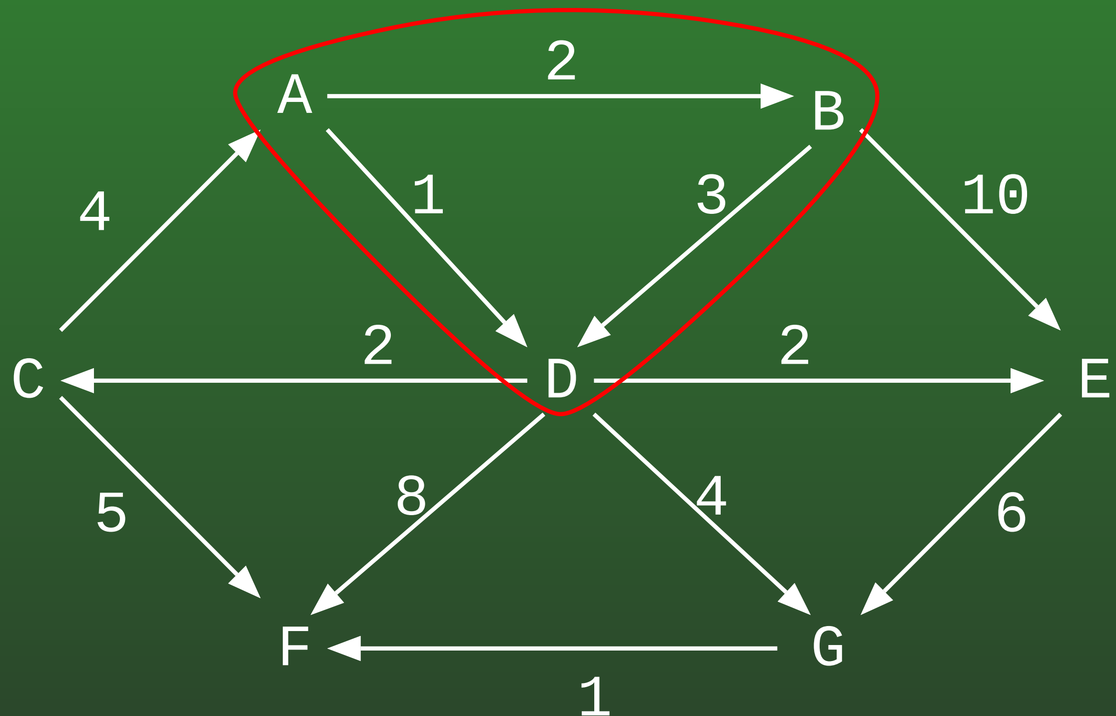
17-15: Single Source Shortest Path



Node	Distance
A	0
B	2
C	
D	1
E	
F	
G	

- How do we know that we could not get to B cheaper than by going through D?
  - Costs 1 to get to D
  - Costs at least 2 to get anywhere from D
    - Cost *at least* (1+2 = 3) to get to B through D

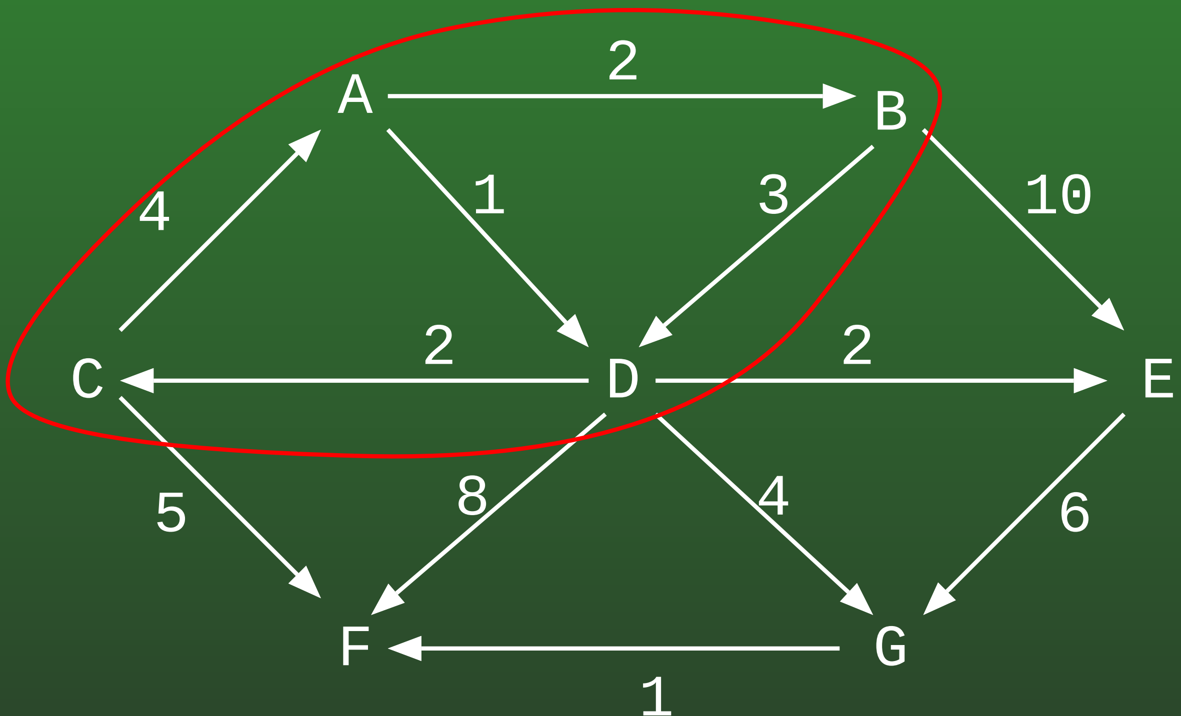
17-16: Single Source Shortest Path



Node	Distance
A	0
B	2
C	
D	1
E	
F	
G	

- Next node we can add ...

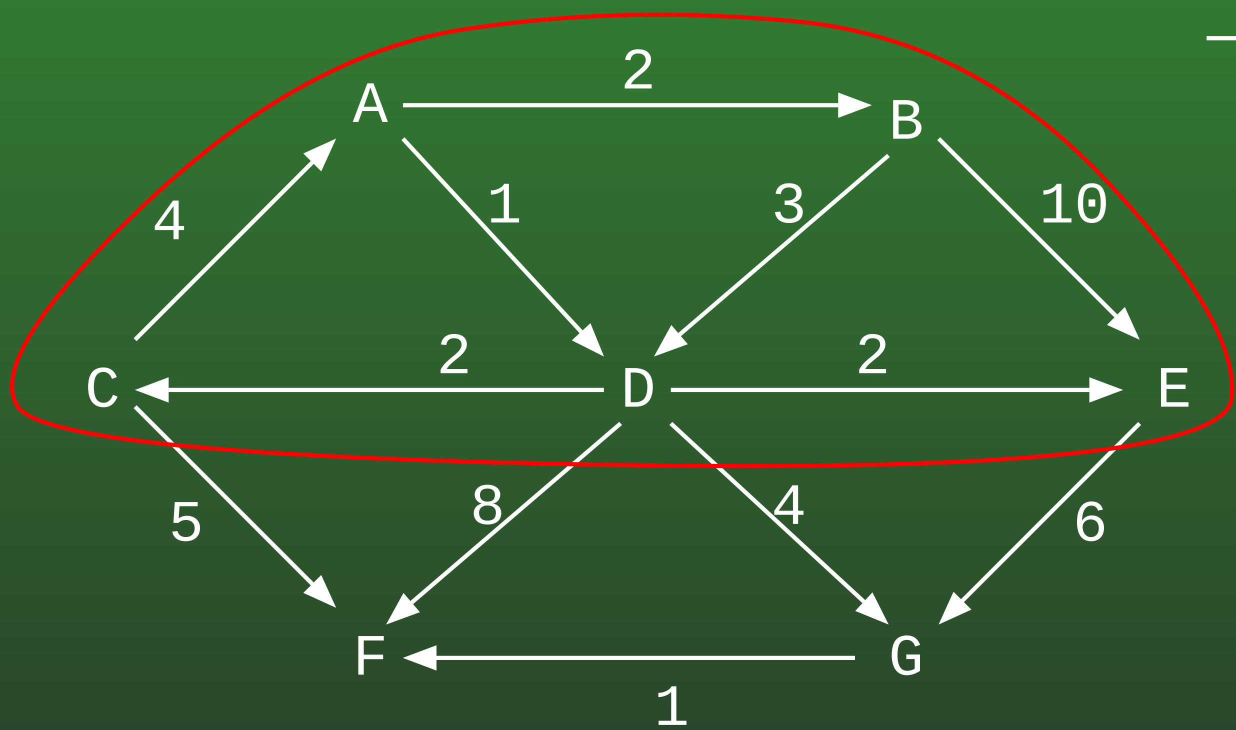
17-17: Single Source Shortest Path



Node	Distance
A	0
B	2
C	3
D	1
E	
F	
G	

- (We also could have added E for this step)
- Next vertex to add to Known ...

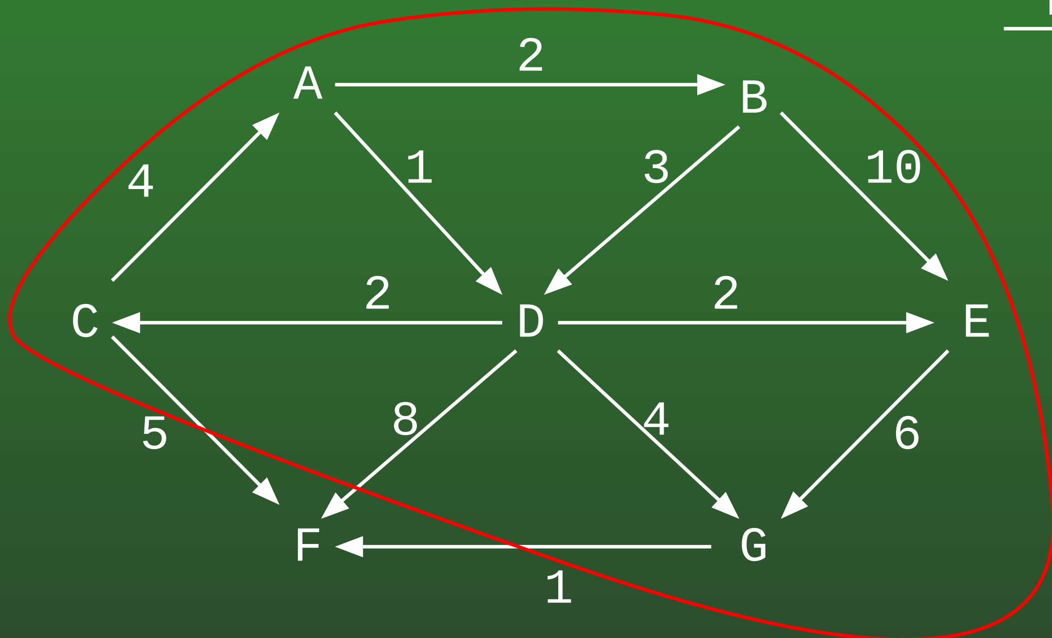
17-18: Single Source Shortest Path



Node	Distance
A	0
B	2
C	3
D	1
E	3
F	
G	

- Cost to add F is 8 (through C)
- Cost to add G is 5 (through D)

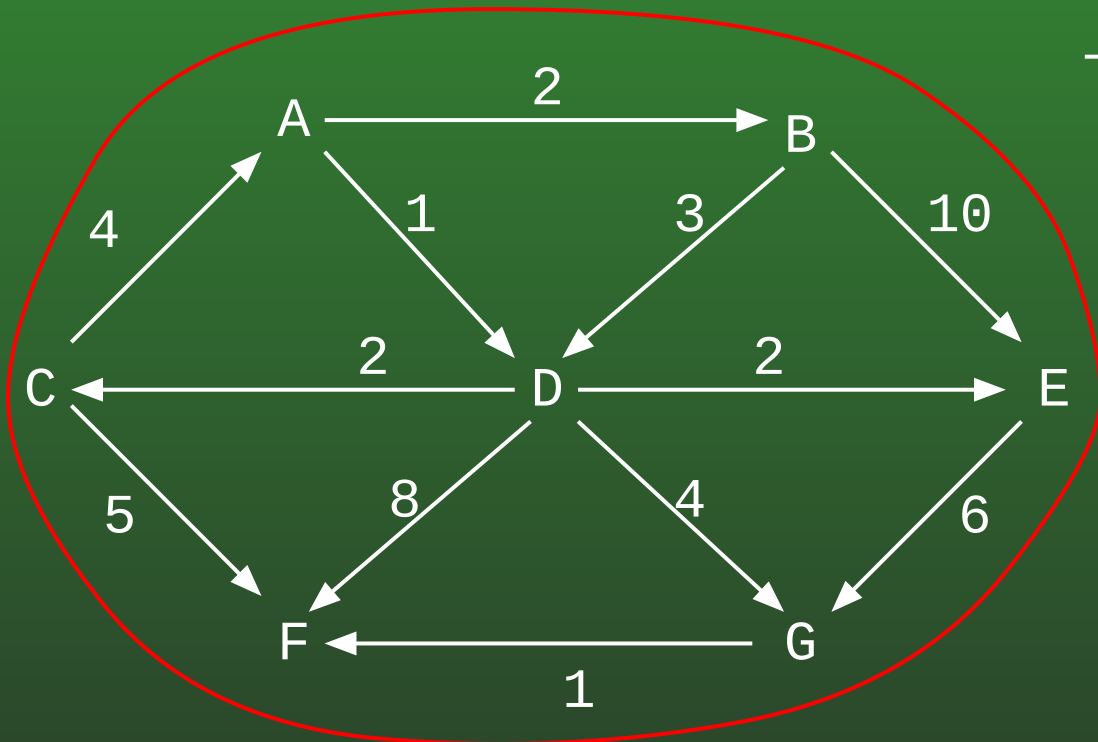
17-19: Single Source Shortest Path



Node	Distance
A	0
B	2
C	3
D	1
E	3
F	
G	5

- Last node ...

17-20: Single Source Shortest Path



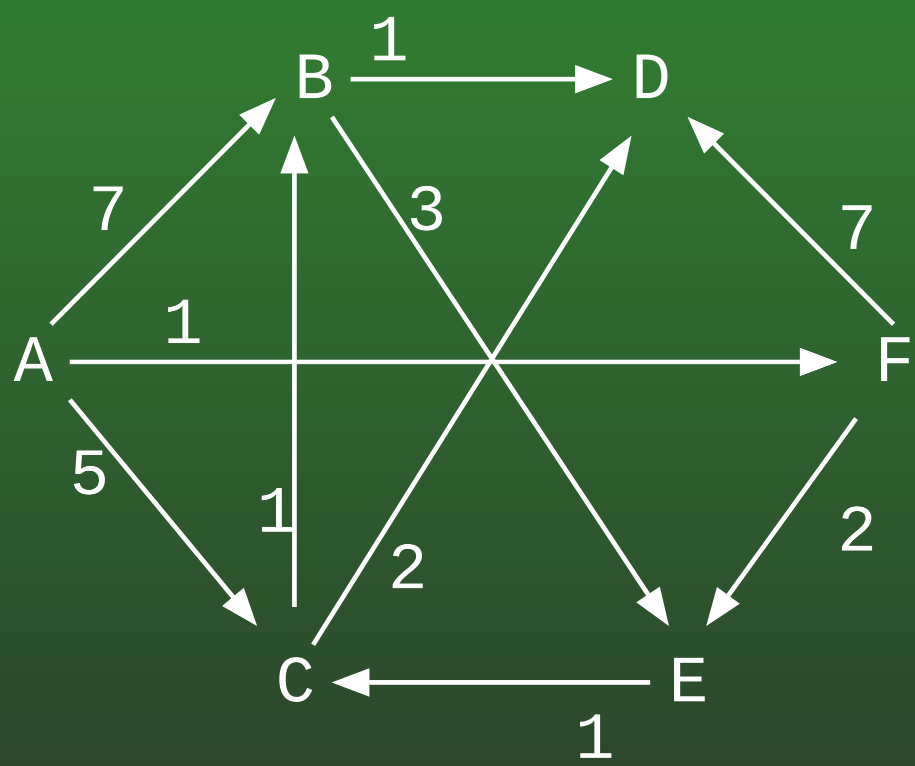
Node	Distance
A	0
B	2
C	3
D	1
E	3
F	6
G	5

- We now know the length of the shortest path from A to all other vertices in the graph

17-21: Dijkstra’s Algorithm

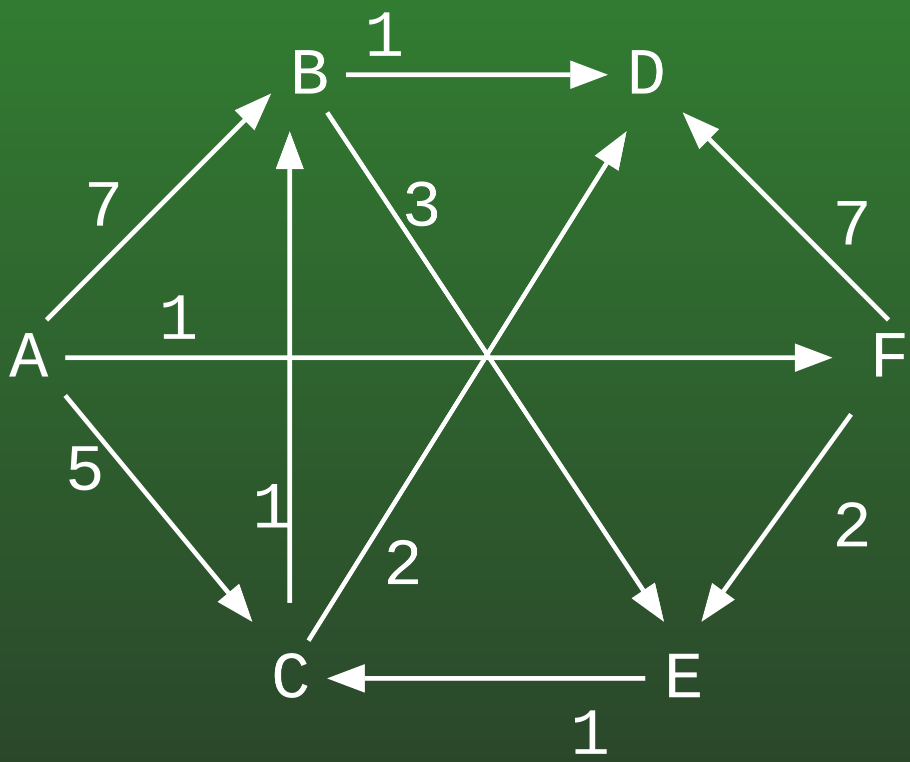
- Keep a table that contains, for each vertex
  - Is the distance to that vertex known?
  - What is the best distance we’ve found so far?
- Repeat:
  - Pick the smallest unknown distance
  - mark it as known
  - update the distance of all unknown neighbors of that node
- Until all vertices are known

Dijkstra's Algorithm Example



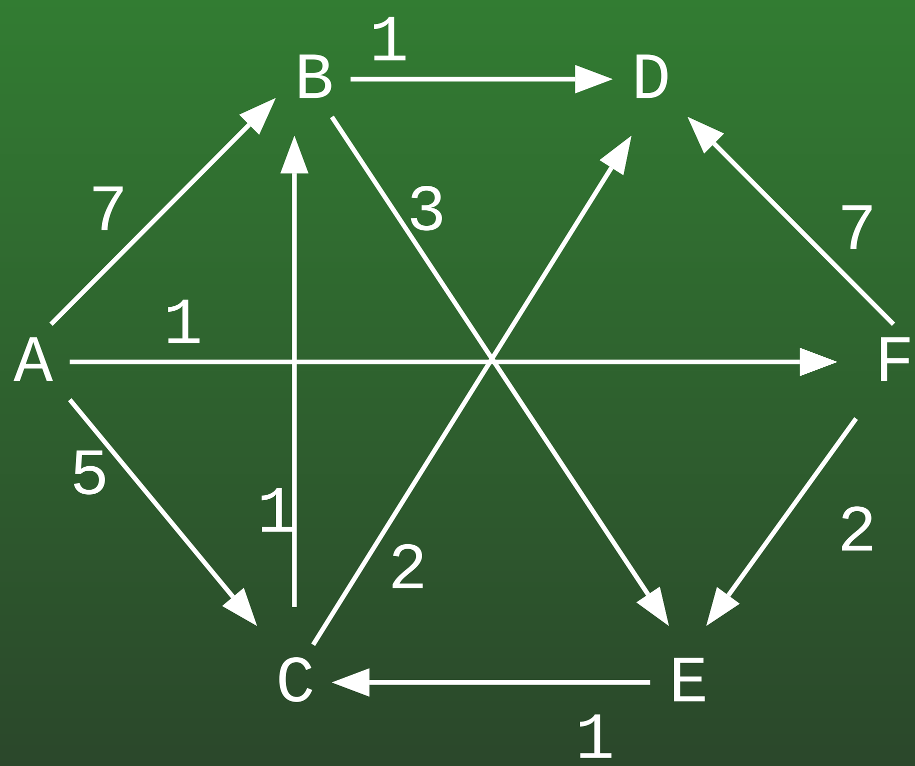
Node	Known	Distance
A	false	0
B	false	∞
C	false	∞
D	false	∞
E	false	∞
F	false	∞

17-23: Dijkstra's Algorithm Example



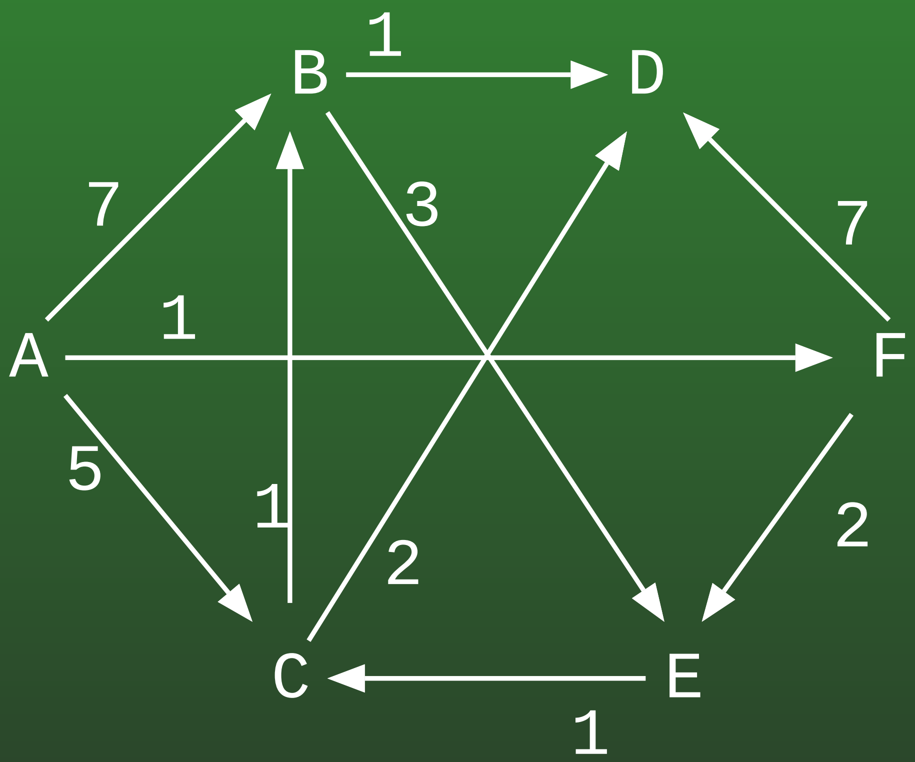
Node	Known	Distance
A	true	0
B	false	7
C	false	5
D	false	∞
E	false	∞
F	false	1

17-24: Dijkstra's Algorithm Example



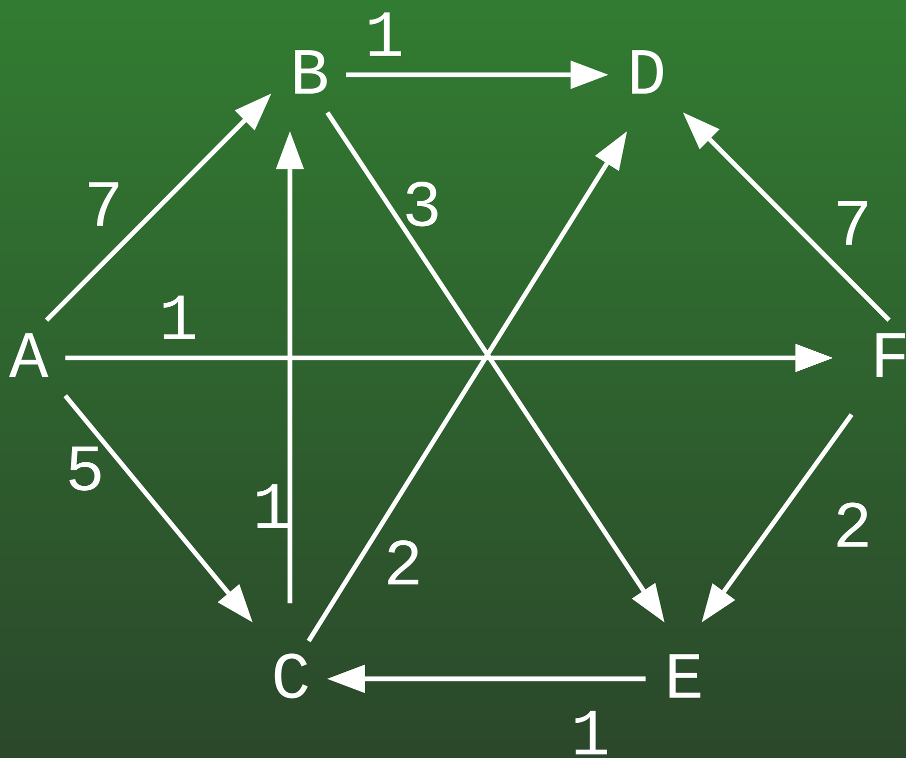
Node	Known	Distance
A	true	0
B	false	7
C	false	5
D	false	8
E	false	3
F	true	1

17-25: Dijkstra's Algorithm Example



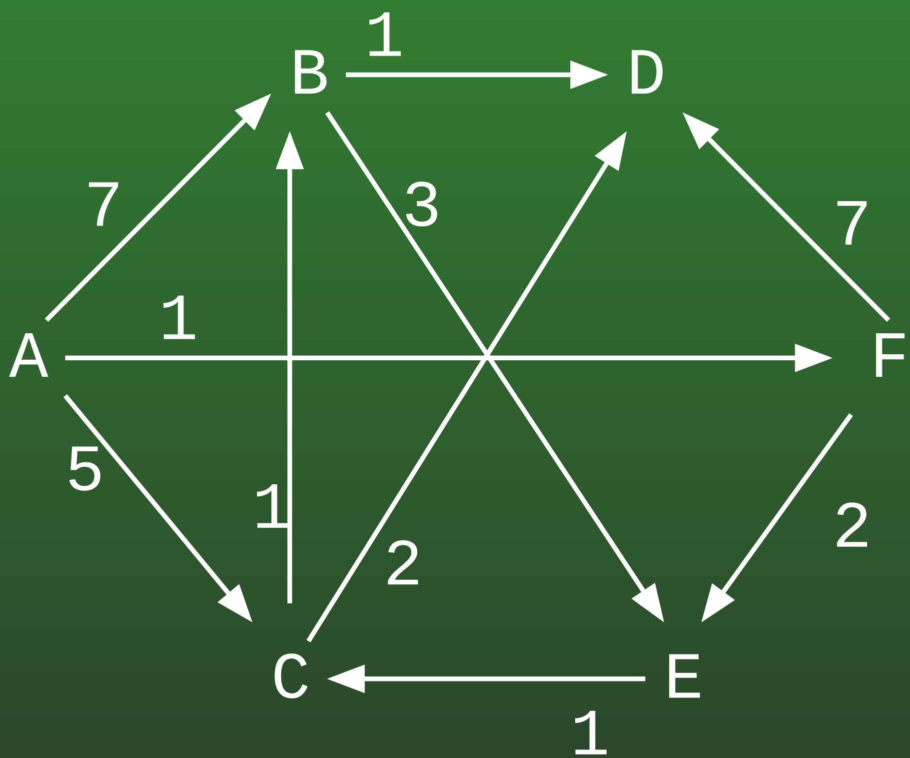
Node	Known	Distance
A	true	0
B	false	7
C	false	4
D	false	8
E	true	3
F	true	1

Dijkstra's Algorithm Example



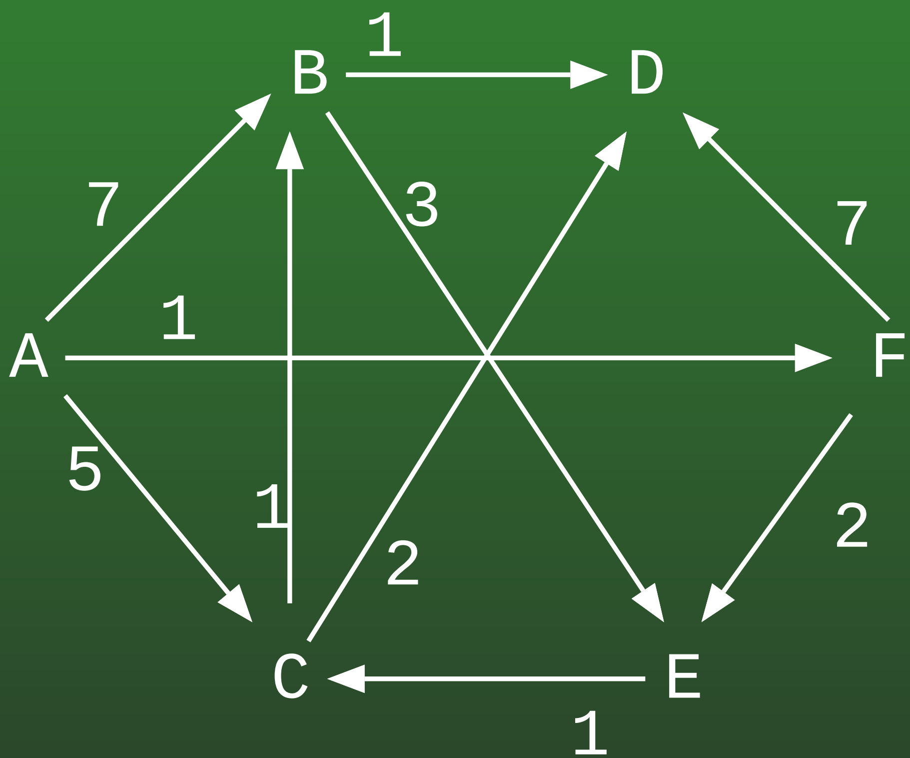
Node	Known	Distance
A	true	0
B	false	5
C	true	4
D	false	6
E	true	3
F	true	1

17-27: Dijkstra's Algorithm Example



Node	Known	Distance
A	true	0
B	true	5
C	true	4
D	false	6
E	true	3
F	true	1

17-28: Dijkstra's Algorithm Example



Node	Known	Distance
A	true	0
B	true	5
C	true	4
D	true	6
E	true	3
F	true	1

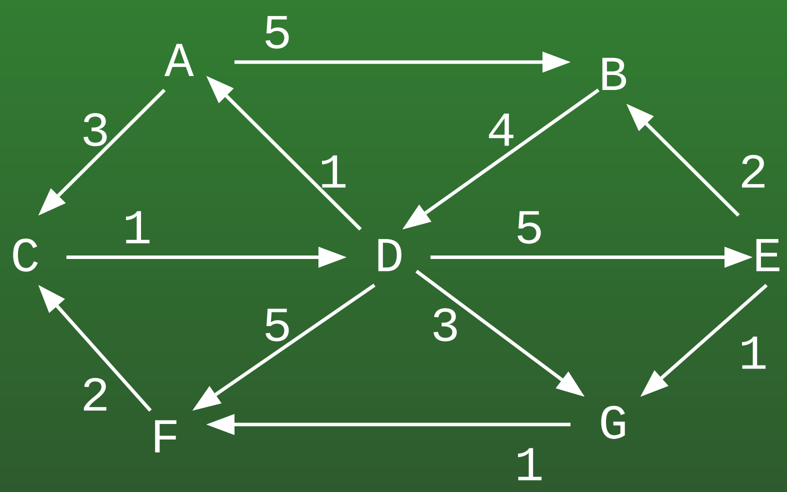
17-29: Dijkstra's Algorithm

- After Dijkstra's algorithm is complete:
  - We know the *length* of the shortest path
  - We do not know *what* the shortest path is
- How can we modify Dijstra's algorithm to compute the path?

17-30: Dijkstra's Algorithm

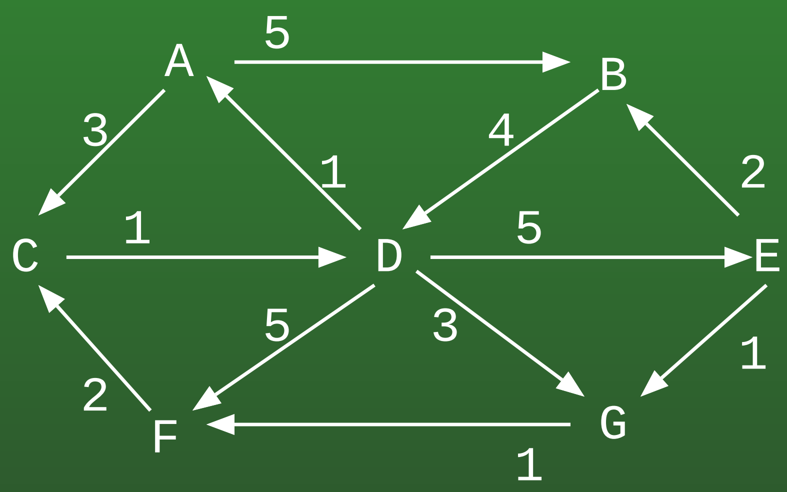
- After Dijkstra's algorithm is complete:
  - We know the *length* of the shortest path
  - We do not know *what* the shortest path is
- How can we modify Dijstra's algorithm to compute the path?
  - Store not only the distance, but the immediate parent that led to this distance

17-31: Dijkstra's Algorithm Example



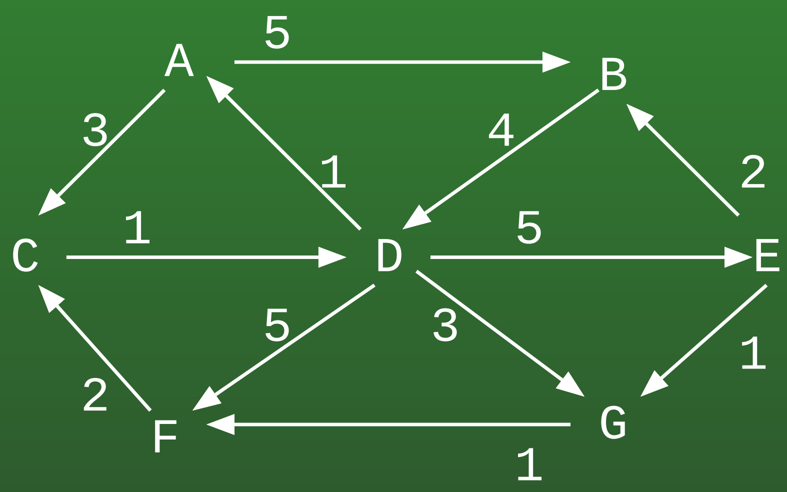
Node	Known	Dist	Path
A	false	0	
B	false	∞	
C	false	∞	
D	false	∞	
E	false	∞	
F	false	∞	
G	false	∞	

17-32: Dijkstra's Algorithm Example



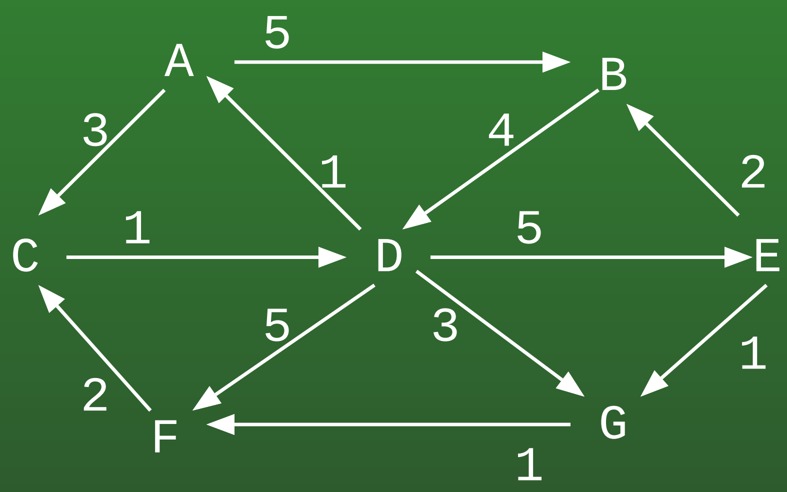
Node	Known	Dist	Path
A	true	0	
B	false	5	A
C	false	3	A
D	false	∞	
E	false	∞	
F	false	∞	
G	false	∞	

17-33: Dijkstra's Algorithm Example



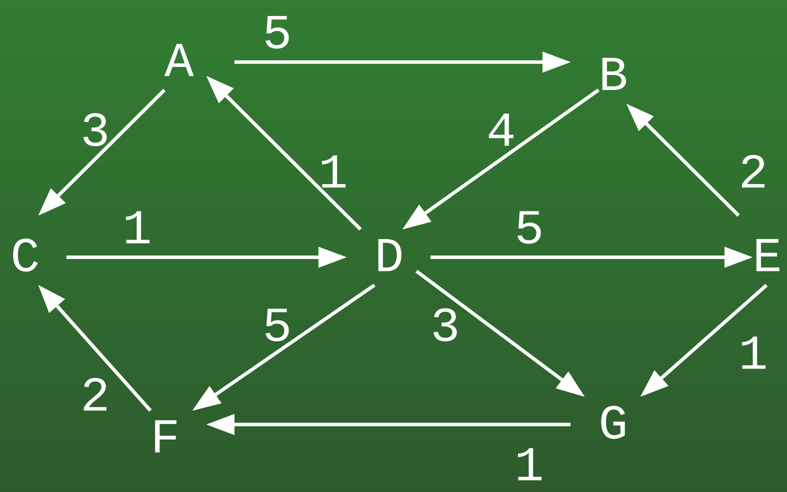
Node	Known	Dist	Path
A	true	0	
B	false	5	A
C	true	3	A
D	false	4	C
E	false	∞	
F	false	∞	
G	false	∞	

17-34: Dijkstra's Algorithm Example



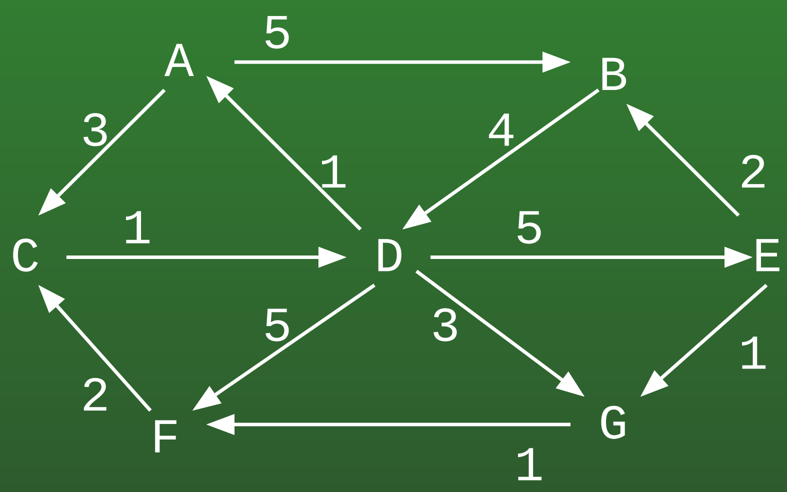
Node	Known	Dist	Path
A	true	0	
B	false	5	A
C	true	3	A
D	true	4	C
E	false	9	D
F	false	9	D
G	false	7	D

17-35: Dijkstra's Algorithm Example



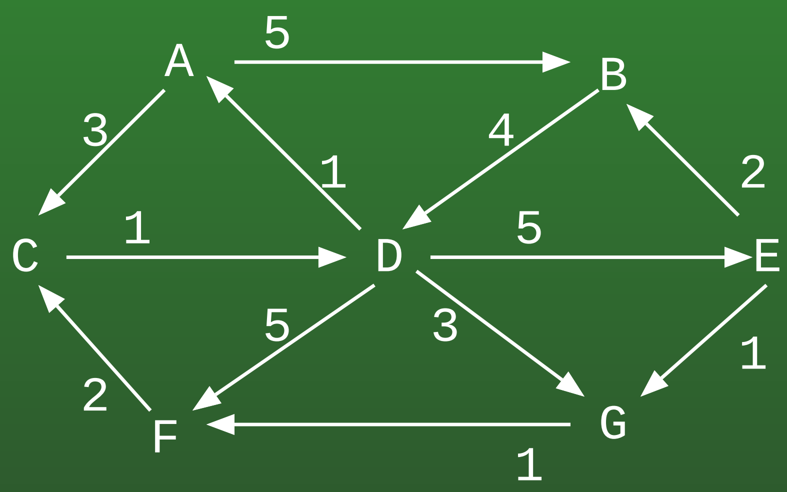
Node	Known	Dist	Path
A	true	0	
B	true	5	A
C	true	3	A
D	true	4	C
E	false	9	D
F	false	9	D
G	false	7	D

17-36: Dijkstra's Algorithm Example



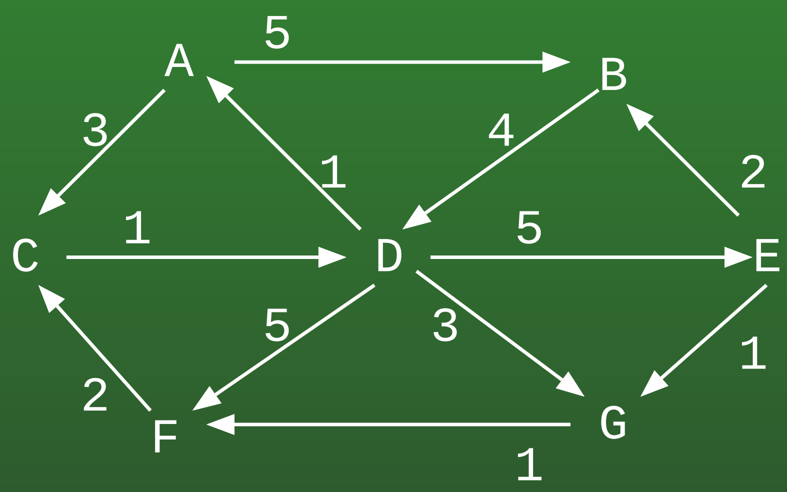
Node	Known	Dist	Path
A	true	0	
B	true	5	A
C	true	3	A
D	true	4	C
E	false	9	D
F	false	8	G
G	true	7	D

17-37: Dijkstra's Algorithm Example



Node	Known	Dist	Path
A	true	0	
B	true	5	A
C	true	3	A
D	true	4	C
E	false	9	D
F	true	8	G
G	true	7	D

17-38: Dijkstra's Algorithm Example



Node	Known	Dist	Path
A	true	0	
B	true	5	A
C	true	3	A
D	true	4	C
E	true	9	D
F	true	8	G
G	true	7	D

17-39: Dijkstra's Algorithm

- Given the "path" field, we can construct the shortest path
  - Work backward from the end of the path
  - Follow the "path" pointers until the start node is reached
    - We can use a sentinel value in the "path" field of the initial node, so we know when to stop

Dijkstra Code

```
void Dijkstra(Edge G[], int s, tableEntry T[]) {
  int i, v;
  Edge e;
  for(i=0; i<G.length; i++) {
    T[i].distance = Integer.MAX_VALUE;
    T[i].path = -1;
    T[i].known = false;
  }
  T[s].distance = 0;
  for (i=0; i < G.length; i++) {
    v = minUnknownVertex(T);
    T[v].known = true;
    for (e = G[v]; e != null; e = e.next) {
      if (T[e.neighbor].distance >
            T[v].distance + e.cost) {
        T[e.neighbor].distance = T[v].distance + e.cost;
        T[e.neighbor].path = v;
      }
    }
  }
}
```

17-41: minUnknownVertex

- Calculating minimum distance unknown vertex:

```
int minUnknownVertex(tableEntry T[]) {
  int i;
  int minVertex = -1;
  int minDistance = Integer.MAX_VALUE;
  for (i=0; i < T.length; i++) {
    if ((!T[i].known) &&
        (T[i].distance < MinDistance)) {
      minVertex = i;
      minDistance = T[i].distance;
    }
  }
  return minVertex;
}
```

Dijkstra Running Time

- Time for initialization:

```
for(i=0; i<G.length; i++) {
  T[i].distance = Integer.MAX_VALUE;
  T[i].path = -1;
  T[i].known = false;
}
T[s].distance = 0;
```

Dijkstra Running Time

- Time for initialization:

```
for(i=0; i<G.length; i++) {
  T[i].distance = Integer.MAX_VALUE;
  T[i].path = -1;
  T[i].known = false;
}
T[s].distance = 0;
```

- $\Theta(V)$

17-44: Dijkstra Running Time

- Total time for all calls to minUnknownVertex, and setting T[v].known = true (for all iterations of the loop)

```
for (i=0; i < G.length; i++) {
  v = minUnknownVertex(T);       <  These two lines
  T[v].known = true;             <  ---------------
  for (e = G[v]; e != null; e = e.next) {
    if (T[e.neighbor].distance >
          T[v].distance + e.cost) {
      T[e.neighbor].distance = T[v].distance + e.cost;
      T[e.neighbor].path = v;
    }
  }
}
```

17-45: Dijkstra Running Time

- Total time for all calls to minUnknownVertex, and setting T[v].known = true (for all iterations of the loop)

```
for (i=0; i < G.length; i++) {
  v = minUnknownVertex(T);          <  These two lines
  T[v].known = true;                <  ---------------
  for (e = G[v]; e != null; e = e.next) {
    if (T[e.neighbor].distance >
          T[v].distance + e.cost) {
      T[e.neighbor].distance = T[v].distance + e.cost;
      T[e.neighbor].path = v;
    }
  }
}
```

- $\Theta(V^2)$

 # Dijkstra Running Time

- Total # of times the if statement will be executed:

```
 for (i=0; i < G.length; i++) {
   v = minUnknownVertex(T);
   T[v].known = true;
   for (e = G[v]; e != null; e = e.next) {
|>    if (T[e.neighbor].distance >
|>         T[v].distance + e.cost) {
|>     T[e.neighbor].distance = T[v].distance + e.cost;
|>     T[e.neighbor].path = v;
     }
   }
 }
```

17-47: Dijkstra Running Time

- Total # of times the if statement will be executed:

```
for (i=0; i < G.length; i++) {
  v = minUnknownVertex(T);
  T[v].known = true;
  for (e = G[v]; e != null; e = e.next) {
|>     if (T[e.neighbor].distance >
|>          T[v].distance + e.cost) {
|>      T[e.neighbor].distance = T[v].distance + e.cost;
|>      T[e.neighbor].path = v;
    }
  }
}
```

- E

17-48: Dijkstra Running Time

- Total running time for all iterations of the inner for statement:

```
for (i=0; i < G.length; i++) {
   v = minUnknownVertex(T);
   T[v].known = true;
|>  for (e = G[v]; e != null; e = e.next) {
|>    if (T[e.neighbor].distance >
|>          T[v].distance + e.cost) {
|>     T[e.neighbor].distance = T[v].distance + e.cost;
|>     T[e.neighbor].path = v;
     }
   }
}
```

17-49: Dijkstra Running Time

- Total running time for all iterations of the inner for statement:

```
for (i=0; i < G.length; i++) {
  v = minUnknownVertex(T);
  T[v].known = true;
|>  for (e = G[v]; e != null; e = e.next) {
|>    if (T[e.neighbor].distance >
|>         T[v].distance + e.cost) {
|>     T[e.neighbor].distance = T[v].distance + e.cost;
|>     T[e.neighbor].path = v;
    }
  }
}
```

- $\Theta(V + E)$
  - Why $\Theta(V + E)$ and not just $\Theta(E)$?

Dijkstra Running Time

- Total running time:
- Sum of:
  - Time for initialization
  - Time for executing all calls to `minUnknownVertex`
  - Time for executing all distance / path updates
- = $\Theta(V + V^2 + (V + E)) = \Theta(V^2)$

17-51: Improving Dijkstra

- Can we do better than $\Theta(V^2)$
- For *dense* graphs, we can't do better
  - To ensure that the shortest path to all vertices is computed, need to look at all edges in the graph
  - A dense graph can have $\Theta(V^2)$ edges
- For *sparse* graphs, we can do better
  - Where should we focus our attention?

Improving Dijkstra

- Can we do better than $\Theta(V^2)$
- For *dense* graphs, we can't do better
  - To ensure that the shortest path to all vertices is computed, need to look at all edges in the graph
  - A dense graph can have $\Theta(V^2)$ edges
- For *sparse* graphs, we can do better
  - Where should we focus our attention?
  - Finding the unknown vertex with minimum cost!

17-53: Improving Dijkstra

- To improve the running time of Dijkstra:
  - Place all of the vertices on a min-heap
    - Key value for min-heap = distance of vertex from initial
  - While min-heap is not empty:
    - Pop smallest value off min-heap
    - Update table
- Problems with this method?

Improving Dijkstra

- To improve the running time of Dijkstra:
  - Place all of the vertices on a min-heap
    - Key value for min-heap = distance of vertex from initial
  - While min-heap is not empty:
    - Pop smallest value off min-heap
    - Update table
- Problems with this method?
  - When we update the table, we need to rearrange the heap

17-55: Rearranging the heap

- Store a pointer for each vertex back into the heap
- When we update the table, we need to do a decrease-key operation
- Decrease-key can take up to time $O(\lg V)$.
- (Examples!)

Rearranging the heap

- Total time:
  - $O(V)$ remove-mins – $O(V \lg V)$
  - $O(E)$ dercrease-keys – $O(E \lg V)$
  - Total time: $O(V \lg V + E \lg V) \in O(E \lg V)$

17-57: Improving Dijkstra

- Store vertices in heap
- When we update the table, we need to rearrange the heap
- Alternate Solution:
  - When the cost of a vertex decreases, add a *new copy* to the heap

17-58: Improving Dijkstra

- Create a new priority queue, add start node
- While the queue is not empty:
  - Remove the vertex v with the smallest distance in the heap
  - If v is not known
    - Mark v as known
    - For each neigbor w of v
      - If distance[w] > distance[v] + cost((v, w))
      - Set distance[w] = distance[v] + cost((v, w))
      - Add w to priority queue with priority distance[w]

17-59: Improved Dijkstra Time

- Each vertex can be added to the heap once for each incoming edge
- Size of the heap can then be up to $\Theta(E)$
  - E inserts, on heap that can be up to size E
  - E delete-mins, on heap that can be upto to size E
- Total: $\Theta(E \lg E) \in \Theta(E \lg V)$

17-60: Improved? Dijkstra Time

- Don't use priority queue, running time is $\Theta(V^2)$
- Do use a prioroty queue, running time is $\Theta(E \lg E)$
- Which is better?

17-61: Improved? Dijkstra Time

- Don’t use priority queue, running time is $\Theta(V^2)$
- Do use a prioroty queue, running time is $\Theta(E \lg E)$
- Which is better?
  - For dense graphs, ($E \in \Theta(V^2)$), $\Theta(V^2)$ is better
  - For sparse graphs ($E \in \Theta(V)$), $\Theta(E \lg E)$ is better

17-62: Improved! Dijkstra Time

- If we use a data structre called a Fibonacci heap instead of a standard heap, we can implement decrease-key in constant time (on average).
- Total time:
  - $O(V)$ remove-mins – $O(V \lg V)$
  - $O(E)$ dercrease-keys – $O(E)$ (each decrease key takes O(1) on average)
  - Total time: $O(V \lg V + E)$

Negative Edges

- What if our graph has negative-weight edges?
  - Think of the cost of the edge as the amount of energy consumed for a segment of road
  - A downhill segment could have negative energy consumed for a hybrid
- Will Dijkstra's algorithm still work correctly?
  - Examples

Negative Edges

- What happens if there is a negative-weight cycle?
- What does the shortest path even mean?

Negative Edges

- What happens if there is a negative-weight cycle?
- What does the shortest path even mean?
  - Finding shortest paths in graphs that contain negative edges, assume that there are no negative weight cycles
  - Hybrid example

All-Source Shortest Path

- What if we want to find the shortest path from all vertices to all other vertices?
- How can we do it?

All-Source Shortest Path

- What if we want to find the shortest path from all vertices to all other vertices?
- How can we do it?
  - Run Dijktra's Algorithm V times
  - How long will this take?
  - What about negative edges?

All-Source Shortest Path

- What if we want to find the shortest path from all vertices to all other vertices?
- How can we do it?
  - Run Dijktra's Algorithm V times
  - How long will this take?
    - $\Theta(VE \lg E)$ (using priority queue)
      - for sparse graphs, $\Theta(V^2 \lg V)$
      - for dense graphs, $\Theta(V^3 \lg V)$
    - $\Theta(V^3)$ (not using a priority queue)
  - What about negative edges?
    - Doesn't work correctly

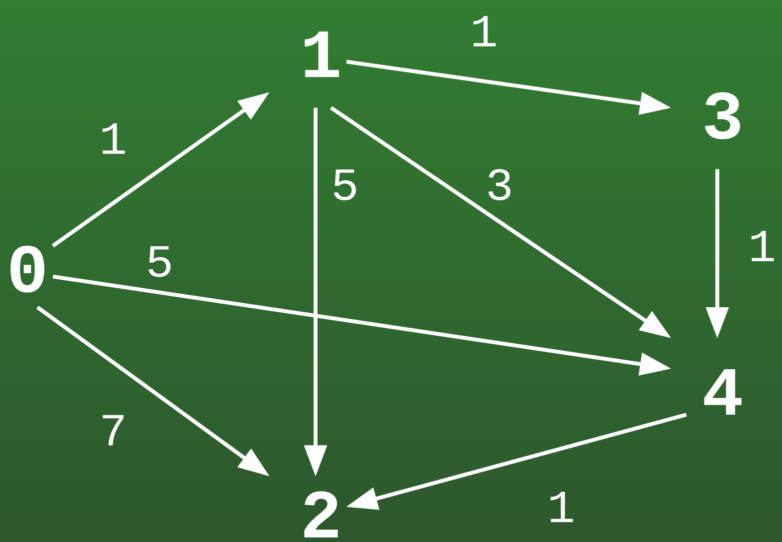
Floyd's Algorithm

- Alternate solution to all pairs shortest path
- Yields $\Theta(V^3)$ running time for all graphs
- Works for graphs with negative edges
- Can detect negative-weight cycles

17-70: Floyd's Algorithm

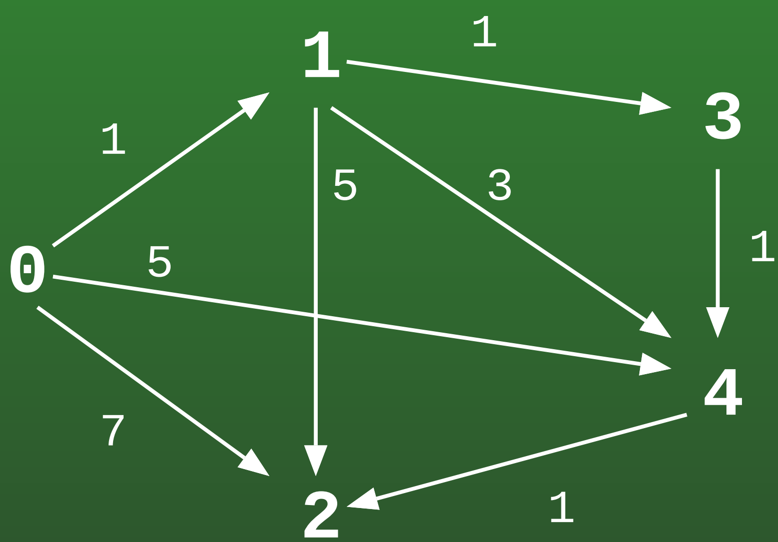
- Vertices numbered from 0..(n-1)
- k-path from vertex v to vertex u is a path whose intermediate vertices (other than v and u) contain only vertices numbered less than or equal to k
- -1-path is a direct link

17-71: k-path Examples



- Shortest -1-path from 0 to 4: 5
- Shortest 0-path from 0 to 4: 5
- Shortest 1-path from 0 to 4: 4
- Shortest 2-path from 0 to 4: 4
- Shortest 3-path from 0 to 4: 3

17-72: **k-path Examples**



- Shortest -1-path from 0 to 2: 7
- Shortest 0-path from 0 to 2: 7
- Shortest 1-path from 0 to 2: 6
- Shortest 2-path from 0 to 2: 6
- Shortest 3-path from 0 to 2: 6
- Shortest 4-path from 0 to 2: 4

17-73: Floyd's Algorithm

- Shortest n-path = Shortest path
- Shortest -1-path:
  - ∞ if there is no direct link
  - Cost of the direct link, otherwise

Floyd's Algorithm

- Shortest n-path = Shortest path
- Shortest -1-path:
  - ∞ if there is no direct link
  - Cost of the direct link, otherwise
- If we could use the shortest k-path to find the shortest $(k+1)$ path, we would be set

Floyd's Algorithm

- Shortest k-path from v to u either goes through vertex k, or it does not
- If not:
  - Shortest k-path = shortest $(k-1)$-path
- If so:
  - Shortest k-path = shortest $k-1$ path from v to k, followed by the shortest $k-1$ path from k to w

Floyd's Algorithm

- If we had the shortest k-path for all pairs (v,w), we could obtain the shortest $k+1$-path for all pairs
  - For each pair v, w, compare:
    - length of the k-path from v to w
    - length of the k-path from v to k appended to the k-path from k to w
  - Set the $k+1$ path from v to w to be the minimum of the two paths above

Floyd's Algorithm

- Let $D_k[v, w]$ be the length of the shortest k-path from v to w.
- $D_0[v, w]$ = cost of arc from v to w (∞ if no direct link)
- $D_k[v, w]$ = MIN($D_{k-1}[v, w]$, $D_{k-1}[v, k] + D_{k-1}[k, w]$)
- Create D_{-1}, use D_{-1} to create D_0, use D_0 to create D_1, and so on – until we have D_{n-1}

Floyd's Algorithm

- Use a doubly-nested loop to create D_k from D_{k-1}
  - Use the same array to store D_{k-1} and D_k – just overwrite with the new values
- Embed this loop in a loop from 1..k

17-79: Floyd's Algorithm

```
Floyd(Edge G[], int D[][]) {
  int i,j,k

  Initialize D, D[i][j] = cost from i to j

  for (k=0; k<G.length; k++;
    for(i=0; i<G.length; i++)
      for(j=0; j<G.length; j++)
        if ((D[i][k] != Integer.MAX_VALUE) &&
            (D[k][j] != Integer.MAX_VALUE) &&
            (D[i][j] > (D[i,k] + D[k,j])))
          D[i][j] = D[i][k] + D[k][j]
}
```

17-80: Floyd's Algorithm

- We've only calculated the *distance* of the shortest path, not the path itself
- We can use a similar strategy to the PATH field for Dijkstra to store the path
  - We will need a 2-D array to store the paths: P[i][j] = last vertex on shortest path from i to j