Data Structures and Algorithms

CS245-2017S-23

NP-Completeness and Undecidablity

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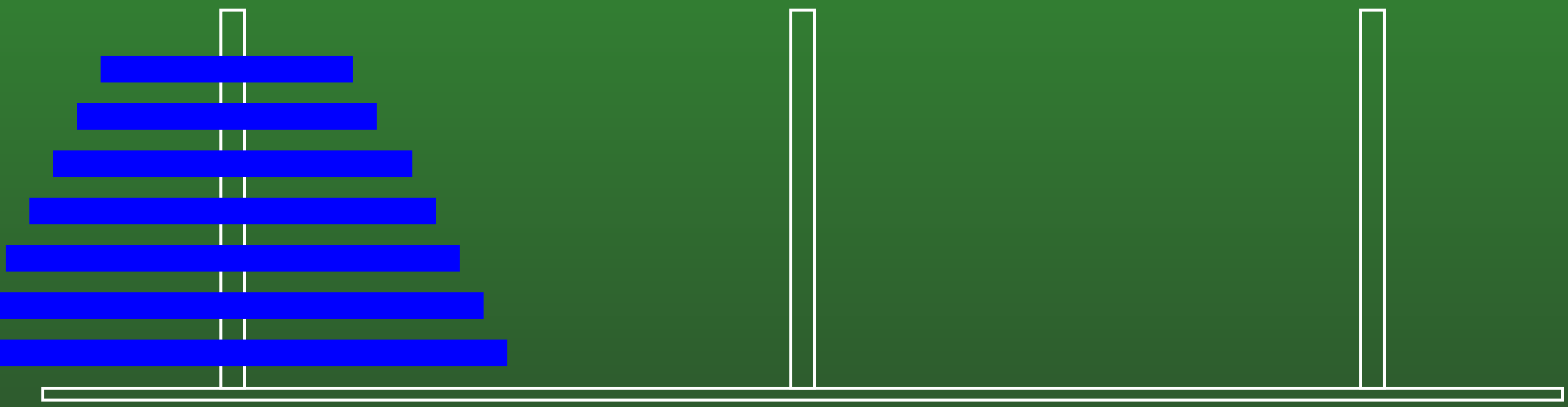
23-0: Hard Problems

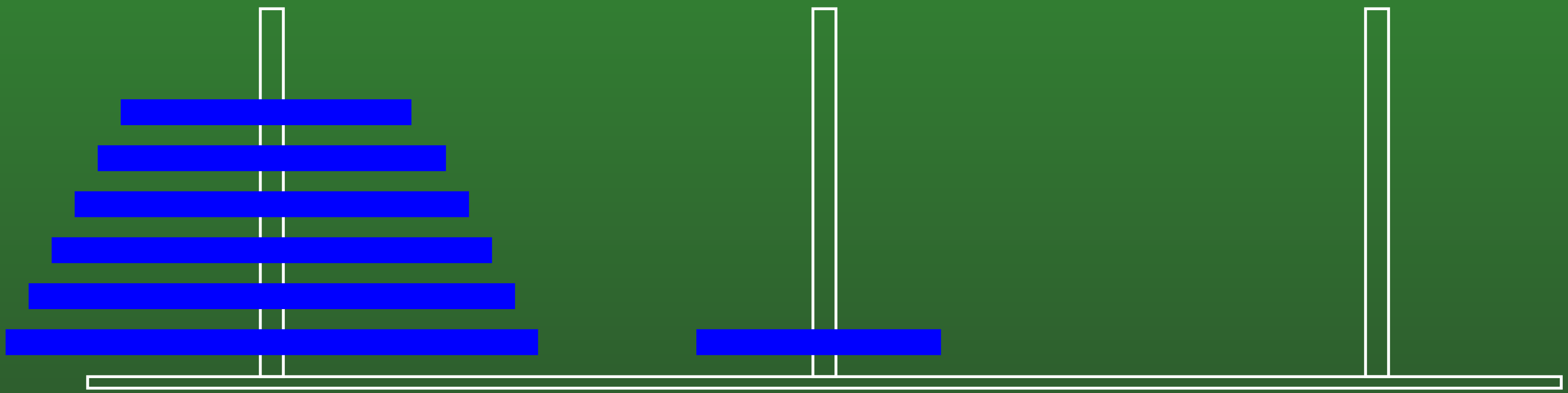
- Some algorithms take exponential time
  - Simple version of Fibonacci
  - Faster versions of Fibonacci that take linear time
- Some *Problems* take exponential time
  - *All* algorithms that solve the problem take exponential time
  - Towers of Hanoi

23-1: Towers of Hanoi



- Move one disk at a time
- Never place a larger disk on a smaller disk

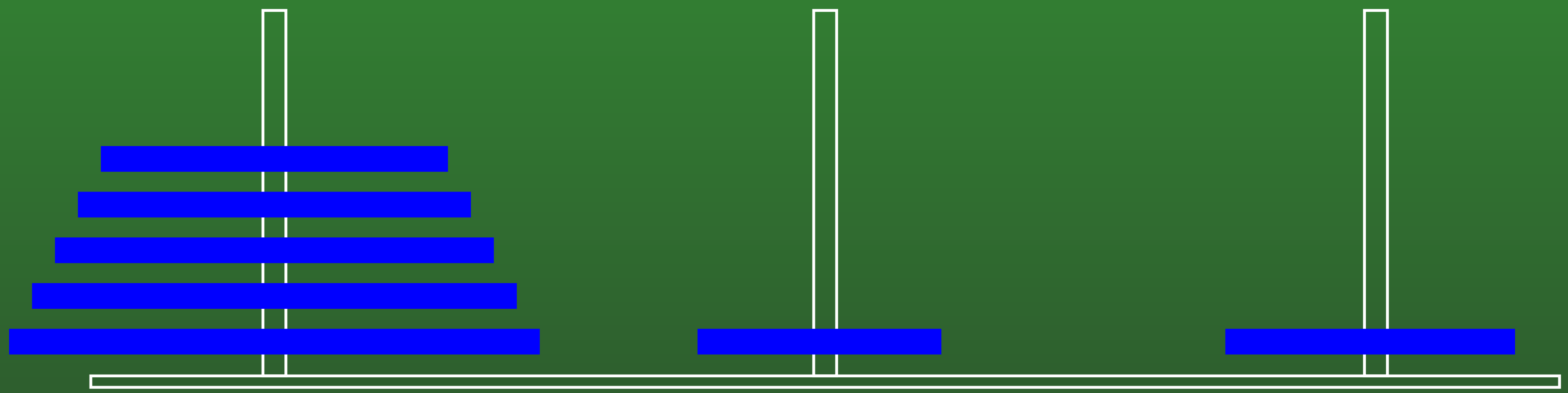
23-2: Towers of Hanoi



- Move one disk at a time
- Never place a larger disk on a smaller disk

Moves = 1

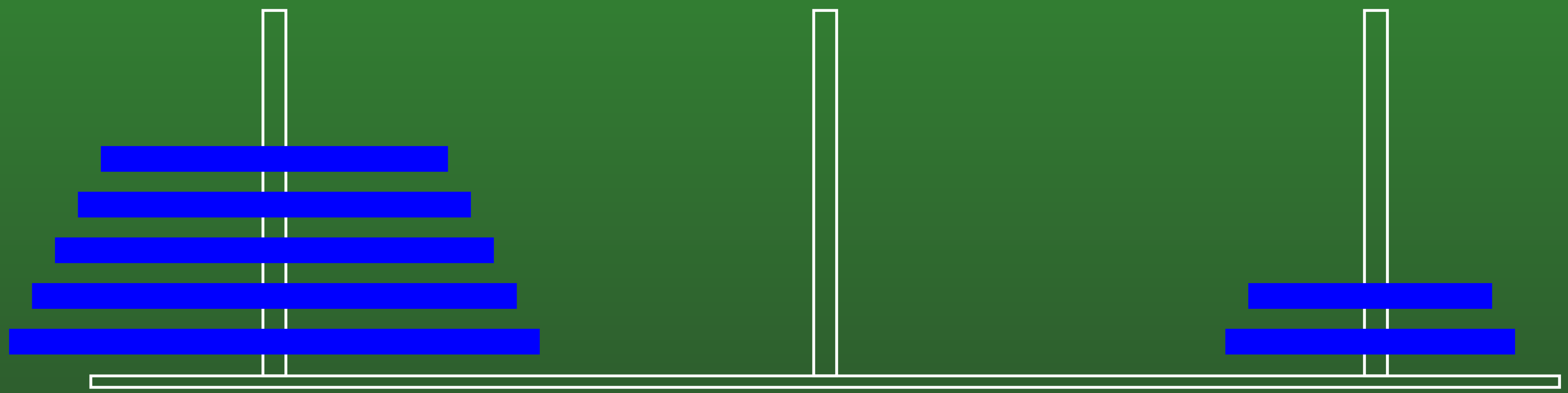
23-3: Towers of Hanoi



- Move one disk at a time
- Never place a larger disk on a smaller disk

Moves = 2

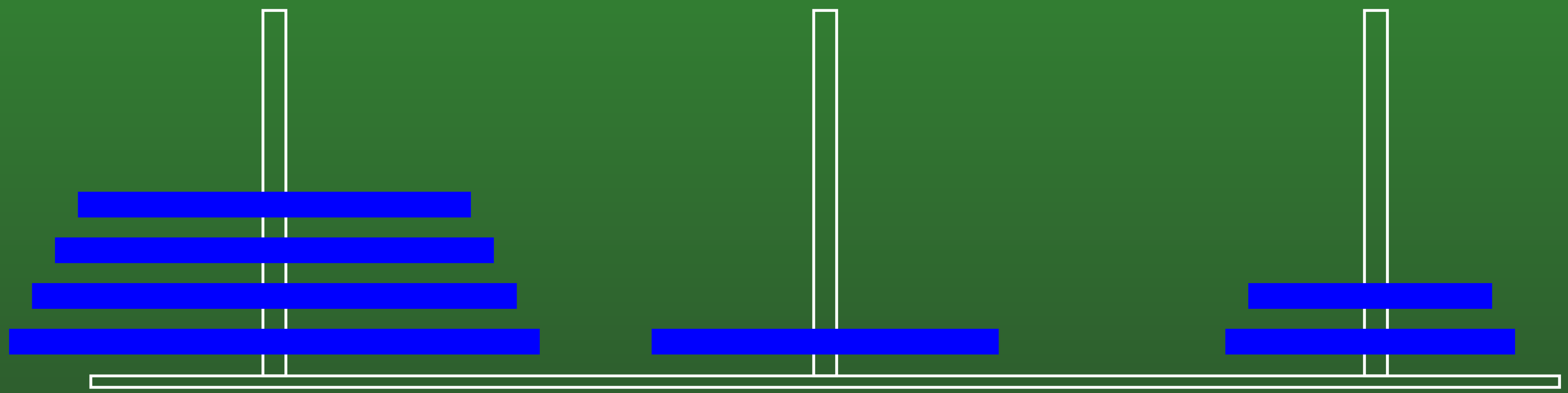
23-4: Towers of Hanoi



- Move one disk at a time
- Never place a larger disk on a smaller disk

Moves = 3

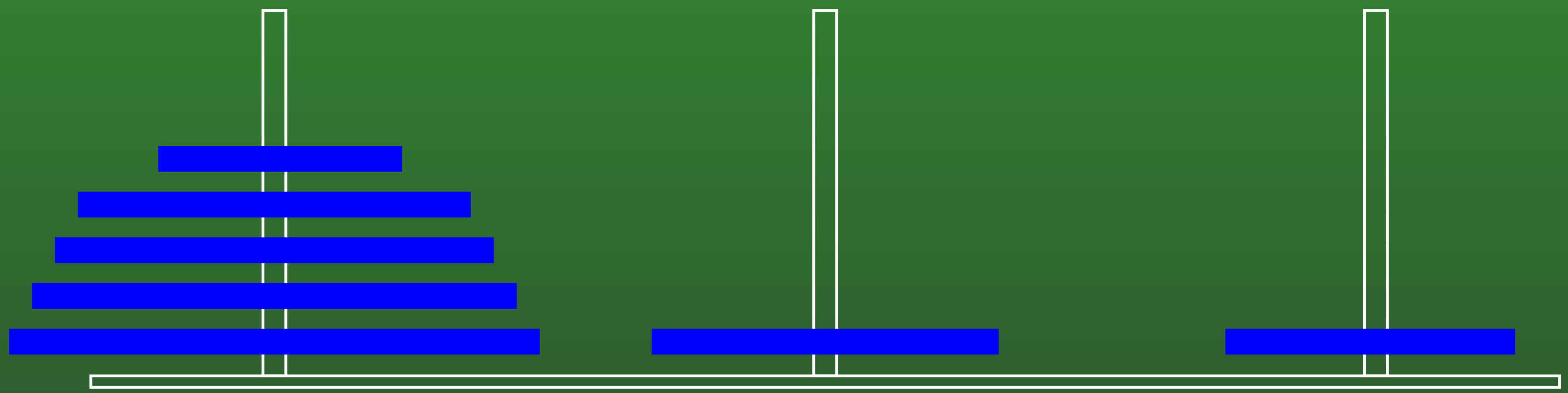
23-5: Towers of Hanoi



- Move one disk at a time
- Never place a larger disk on a smaller disk

Moves = 4

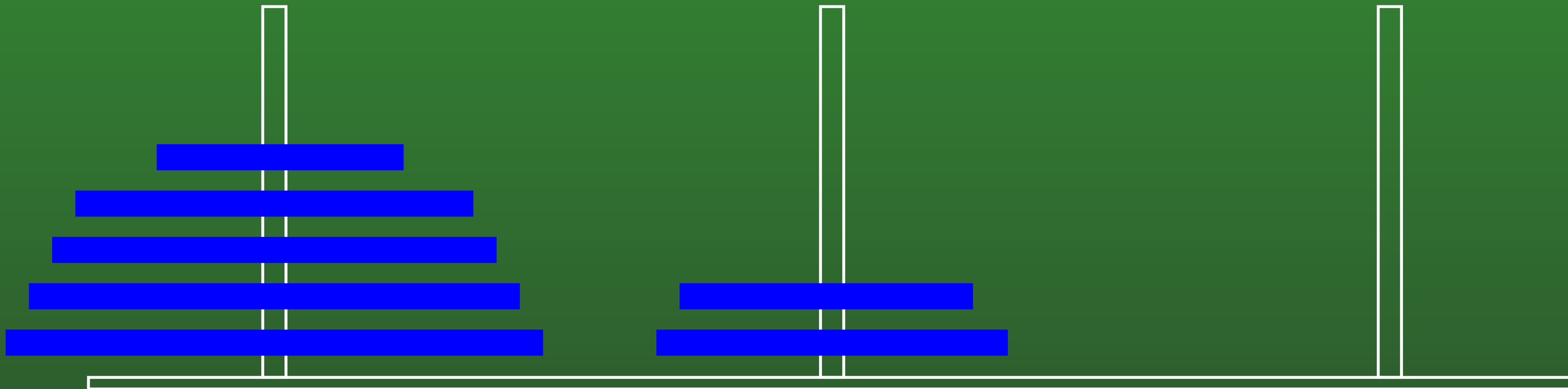
23-6: Towers of Hanoi



- Move one disk at a time
- Never place a larger disk on a smaller disk

Moves = 5

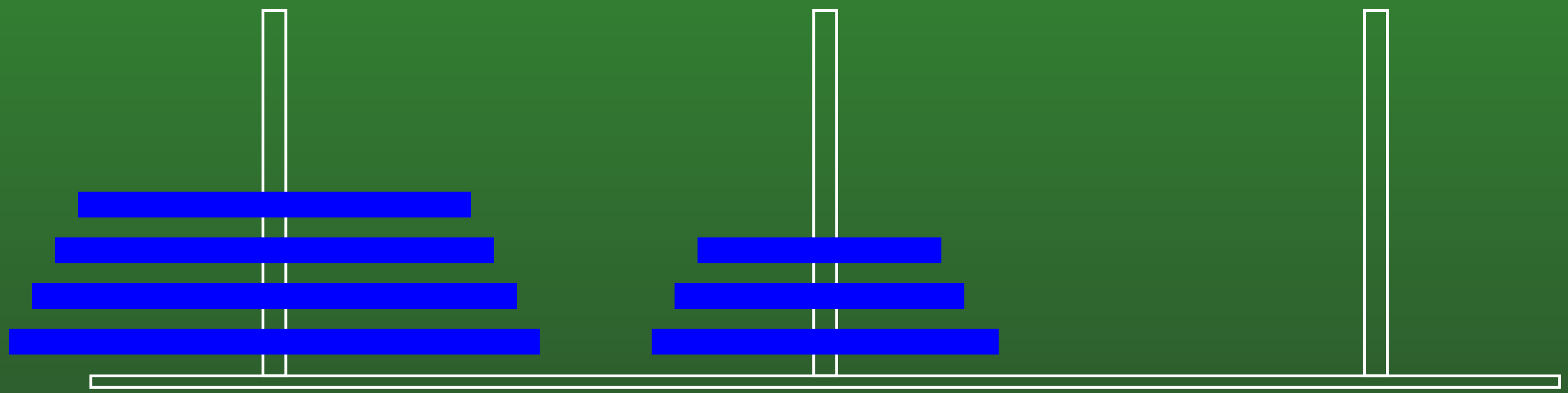
23-7: Towers of Hanoi



- Move one disk at a time
- Never place a larger disk on a smaller disk

Moves = 6

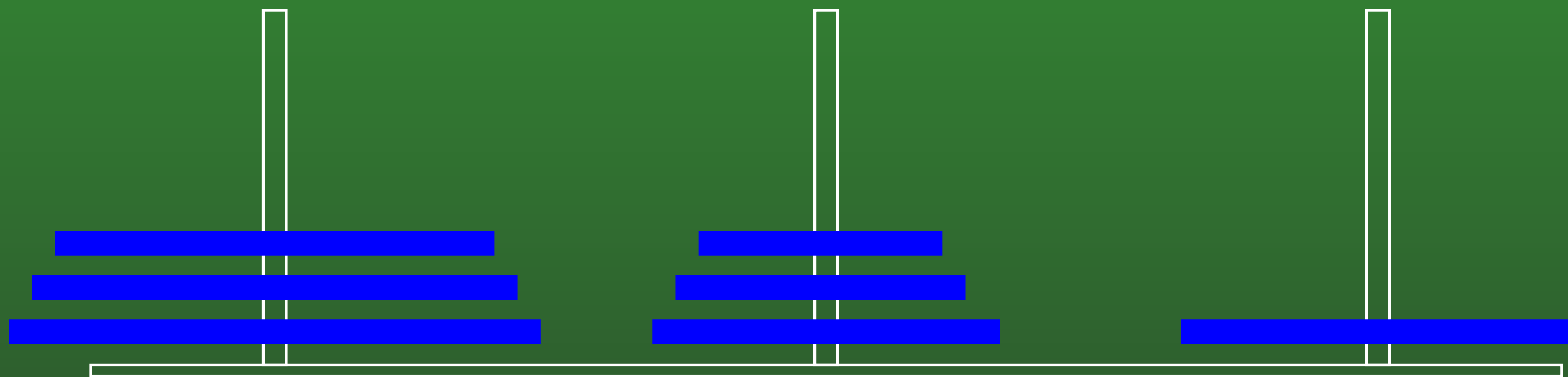
23-8: Towers of Hanoi



- Move one disk at a time
- Never place a larger disk on a smaller disk

Moves = 7

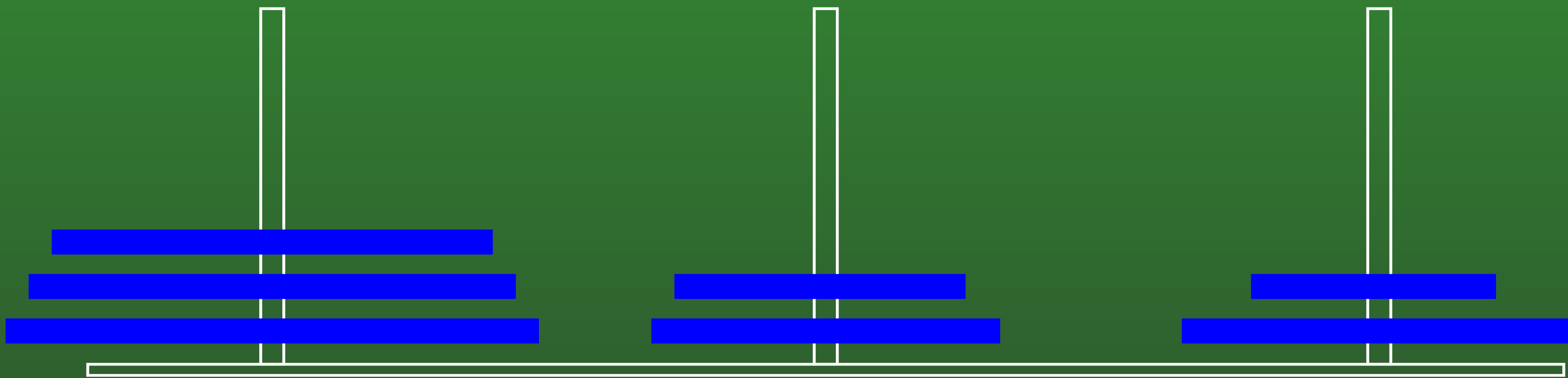
23-9: Towers of Hanoi



- Move one disk at a time
- Never place a larger disk on a smaller disk

Moves = 8

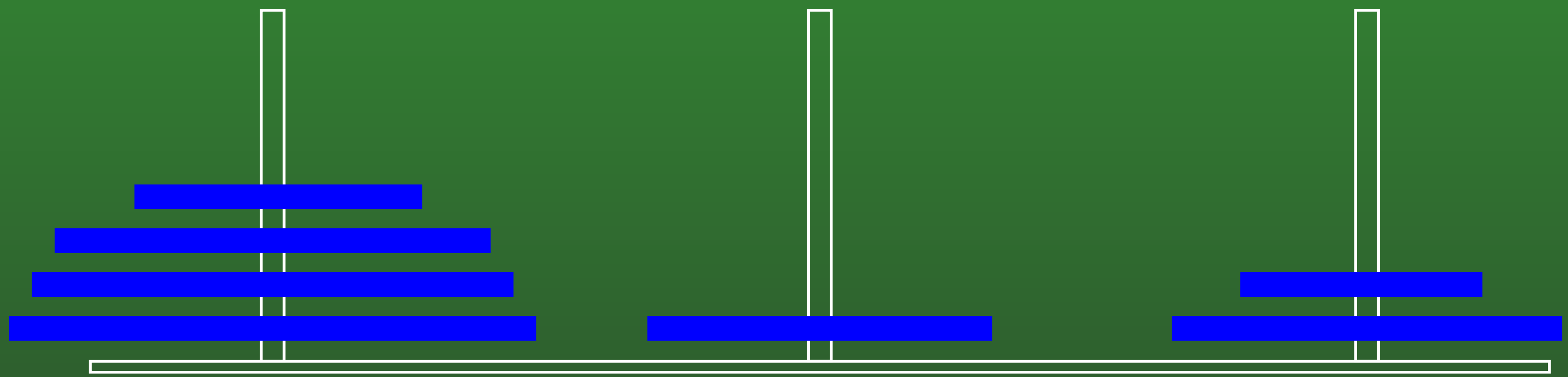
23-10: Towers of Hanoi



- Move one disk at a time
- Never place a larger disk on a smaller disk

Moves = 9

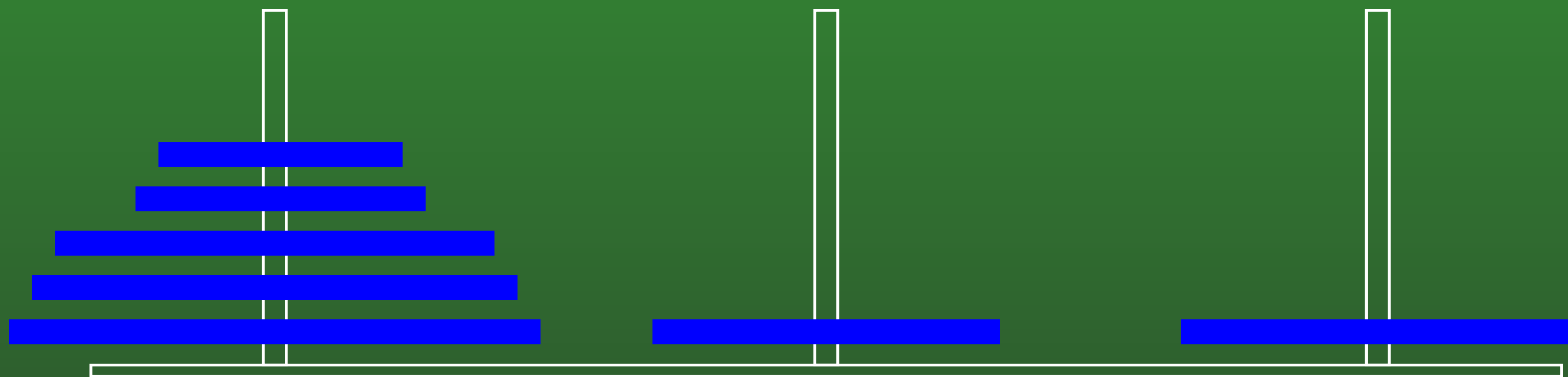
23-11: Towers of Hanoi



- Move one disk at a time
- Never place a larger disk on a smaller disk

Moves = 10

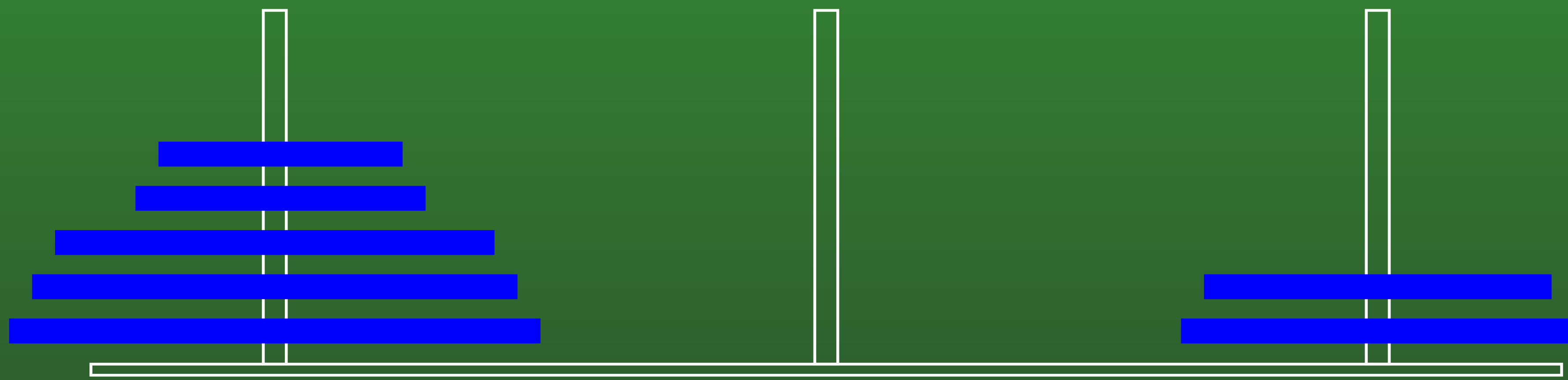
23-12: Towers of Hanoi



- Move one disk at a time
- Never place a larger disk on a smaller disk

Moves = 11

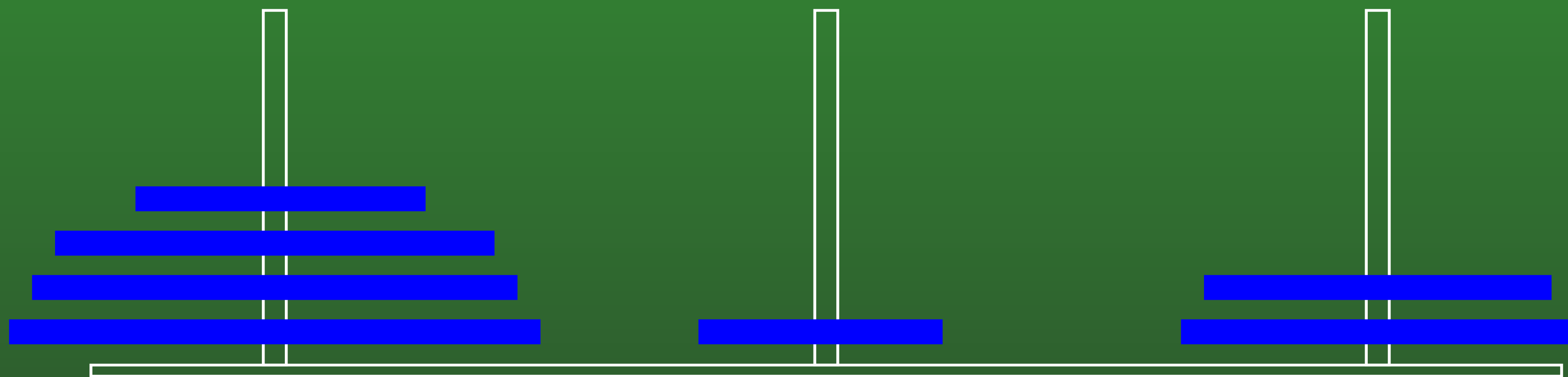
23-13: Towers of Hanoi



- Move one disk at a time
- Never place a larger disk on a smaller disk

Moves = 12

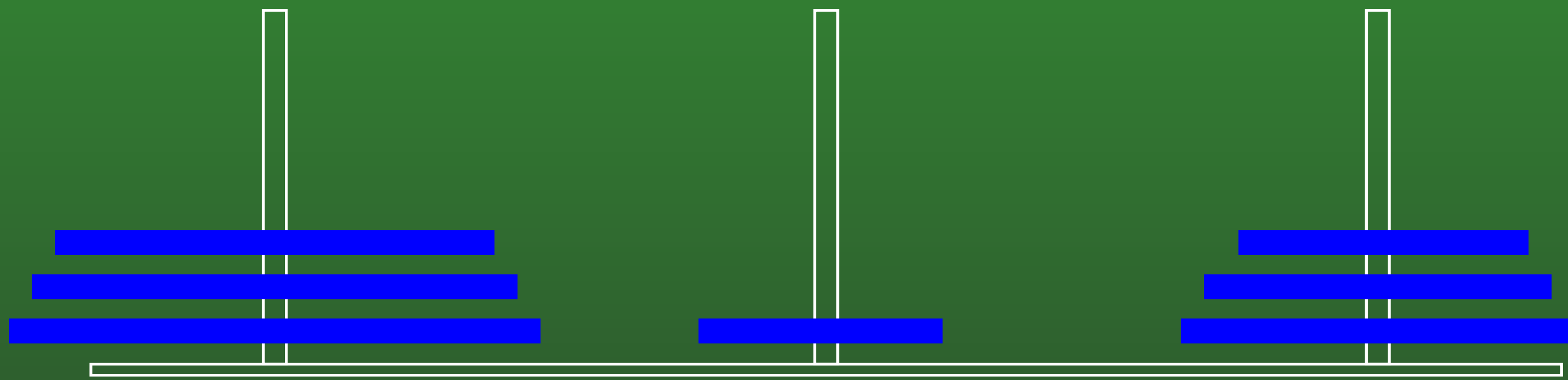
23-14: Towers of Hanoi



- Move one disk at a time
- Never place a larger disk on a smaller disk

Moves = 13

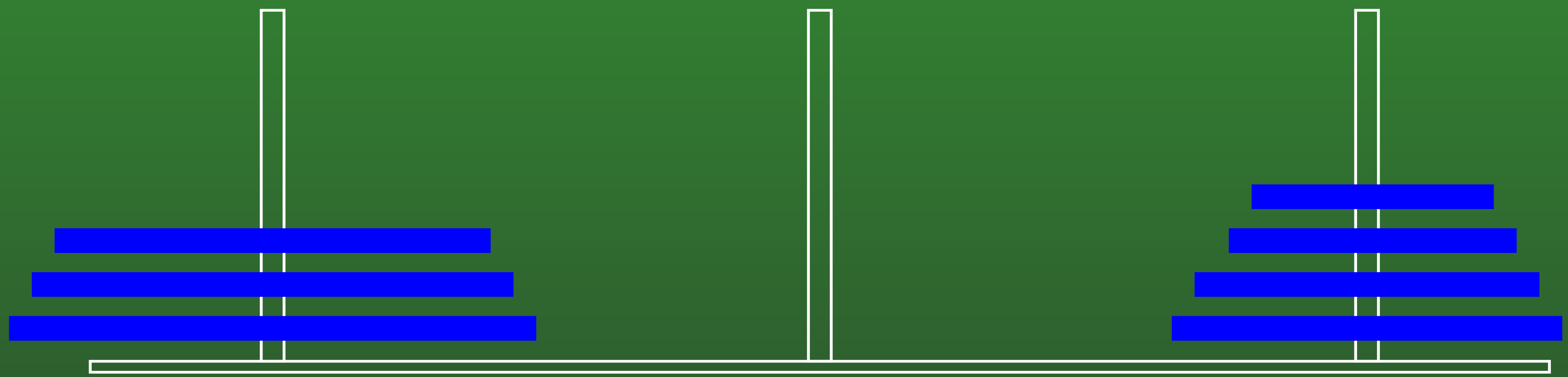
23-15: Towers of Hanoi



- Move one disk at a time
- Never place a larger disk on a smaller disk

Moves = 14

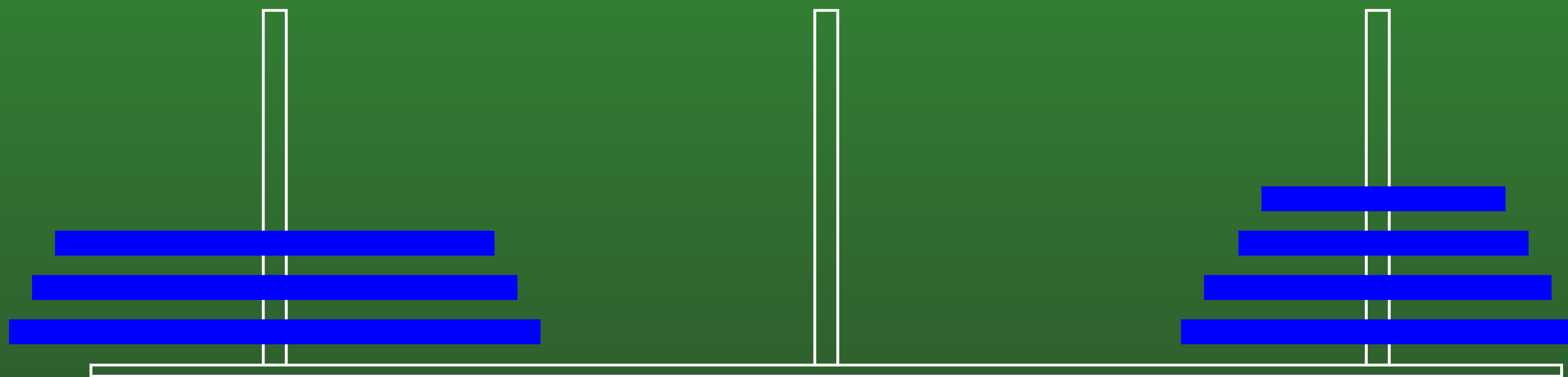
23-16: Towers of Hanoi



- Move one disk at a time
- Never place a larger disk on a smaller disk

Moves = 15

Towers of Hanoi

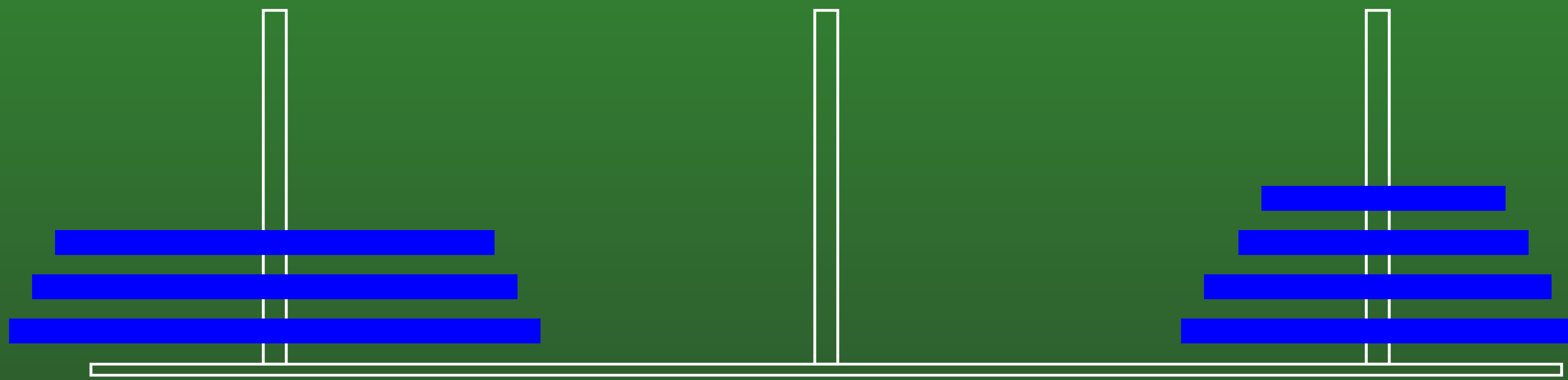


- Move one disk at a time
- Never place a larger disk on a smaller disk

Moves = 15

- Moving n disks requires $2^n - 1$ moves

23-18: Towers of Hanoi



- Move one disk at a time
- Never place a larger disk on a smaller disk

Moves = 15

- Moving n disks requires $2^n - 1$ moves
- Completely impractical for large values of n

23-19: Reductions

- A reduction from Problem 1 to Problem 2 allows us to solve Problem 1 in terms of Problem 2
  - Given an instance of Problem 1, create an instance of Problem 2
  - Solve the instance of Problem 2
  - Use the solution of Problem 2 to create a solution to Problem 1

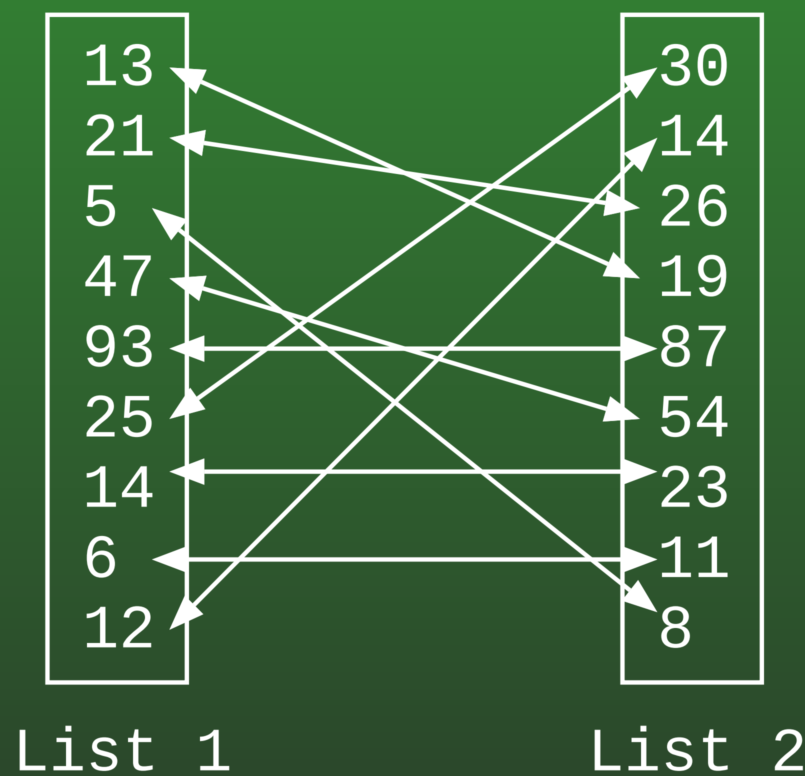
Reductions

- Example Problem: Pairing
  - Given two lists of integers of size n
  - Match the smallest element of each list together
  - Match the second smallest element of each list together
  - .. etc.

23-21: Reductions

List 1	List 2
13	30
21	14
5	26
47	19
93	87
25	54
14	23
6	11
12	8

23-22: Reductions



Reductions

- Reduction from Pairing to Sorting
  - Can we reduce the pairing problem to a sorting problem
  - That is, how can we use the sorting problem to solve the pairing problem?

23-24: Reductions

- Reduction from Pairing to Sorting
  - Lets us solve the Pairing problem by solving Sorting problem
  - Given any two lists L1 and L2 that we wish to pair:
    - Sort L1 and L2
    - Pair L1[i] with L2[i] for all i

Reductions

- Reduction from Pairing to Sorting
  - Reduction takes very little time
  - Time to solve Pairing (using this reduction) is the time to solve Sorting
  - We can solve Pairing in time $O(n \lg n)$ using sorting.

23-26: Reductions

- Reduction from Sorting to Pairing
  - Given an instance of Sorting, create an instance of pairing problem
  - Solve the paring problem
  - Use the solution of pairing problem to solve the sorting problem

Reductions

- Given an list L1:
  - Create a new list L2, such that L2[i] = i
  - Solve the paring problem, pairing L1 and L2
  - Use counting sort to sort L1, using the paired element from L2 as the key

23-28: Reductions

- Given an list L1:
  - Create a new list L2, such that L2[i] = i
  - Solve the paring problem, pairing L1 and L2
  - Use counting sort to sort L1, using the paired element from L2 as the key

How long does this take?

23-29: Reductions

- Given an list L1:
  - Create a new list L2, such that L2[i] = i
  - Solve the paring problem, pairing L1 and L2
  - Use counting sort to sort L1, using the paired element from L2 as the key

How long does this take?

- O(n + time to do pairing)

23-30: Reductions

- We can reduce Sorting to Pairing, such that:
  - Time to do Sorting takes O(n + time to do pairing)
- Sorting takes $\Omega(n \lg n)$ time
- Thus, the pairing problem must take at least $\Omega(n \lg n)$ time as well

23-31: Reductions

- We can use a Reduction to compare problems
- If there is a reduction from problem A to problem B that can be done quickly
- Problem A is known to be hard (cannot be solved quickly)
- Problem B cannot be solved quickly, either

23-32: NP Problems

- A problem is NP if a solution can be verified easily
  - Given a potential solution to the problem, verify that the solution does solve the problem
  - Verification takes polynomial (not exponential!) time
  - (Pretty low bar for “easily”)

NP Problems

- A problem is NP if a solution can be verified easily
  - Traveling Salesman Problem (TSP)
    - Given a graph with weighted vertices, and a cost bound k
    - Is there a cycle that contains all vertices in the graph, that has a total cost less than k?
  - Given any potential solution to the TSP, we can easily verify that the solution is correct

NP Problems

- A problem is NP if a solution can be verified easily
  - Graph Coloring
    - Given a graph and a number of colors k
    - Can we color every vertex using no more than k colors, such that all adjacent vertices have different colors?
  - Given any potential solution to the Graph Coloring problem, we can easily verify that the solution is correct

NP Problems

- A problem is NP if a solution can be verified easily
  - Satisfiability
    - Given a boolean formula over a set of boolean variables $a_1 \ldots a_n$
      $(a_1 || !a_2) \&\& (a_2 || a_5 || !a1) \&\& \ldots$
    - Can we give a truth value to all variables $a_1 \ldots a_n$ so that the value of the formula is true?
  - Given any potential solution to the Satisfiability problem, we can easily verify that the solution is correct

23-36: NP Problems

- A problem is NP if a solution can be verified easily
  - Sorting
    - Given a list of elements L and an ordering of the elements $\leq$
    - Create a permutation of L such that $L[i] \leq L[i+1]$
  - Given any potential solution to the Sorting problem, we can easily verify that the solution is correct

23-37: NP Problems

- If we can guess an answer, we can verify it quickly
- NP stands for Non-Deterministic Polynomial
  - Non-Deterministic = we can guess
  - Polynomial = “quickly”
- NP problem: If we could guess an answer, we could verify it in polynomial (n, n^2, n^5 – not exponential) time

23-38: Non-Deterministic Machine

- Two Definitions of Non-Deterministic Machines:
  - “Oracle” – allows machine to magically make a correct guess
  - Massively parallel – simultaneously try to verify all possible solutions
    - Try all permutations of vertices in a graph, see if any form a cycle with cost < k
    - Try all colorings of a graph with up to k colors, see if any are legal
    - Try all permutations of a list, see if any are sorted

NP vs. P

- A problem is NP if a non-deterministic machine can solve it in polynomial time
  - Of course, we have no real non-deterministic machines
- A problem is in P (Polynomial), if a deterministic machine can solve it in polynomial time
  - Sorting is in P – can sort a list in polynomial time
  - All problems in P are also in NP
    - Ignore the oracle

NP-Complete

- An NP problem is “NP-Complete” if there is a reduction from *any* NP problem to that problem
- For example, Traveling Salesman (TSP) is NP-Complete
    - We can reduce *any* NP problem to TSP
    - If we could solve TSP in polynomial time, we could solve *all* NP problems in polynomial time
- Is TSP unique in this way?

23-41: NP-Complete

- There are many NP-Complete problems
  - TSP
  - Graph Coloring
  - Satisfiability
  - .. many, many more
- If we could solve *any* of these problems quickly, we could solve *all* of them quickly
- All known solutions take exponential time

23-42: NP-Complete

- If a problem is NP-Complete, it almost certainly cannot be solved quickly (polynomial time)
  - If it could, then *all* NP problems could be solved quickly
  - Many people have tried for many years to find polynomial solutions for NP complete problems, all have failed
- However, no *proof* that NP-Complete problems require exponential time – open problem

23-43: NP =? P

- If we could solve any NP-Complete problem quickly (polynomial time), we could solve all NP problems quickly
- If that is the case, then NP=P
  - P is set of problems that can be solved by a standard machine in polynomial time
- Most everyone believes that NP $\neq$ P, and all NP-Complete problems require exponential time on standard computers – not yet been proven

NP-Completeness

- Why is NP-Completeness important?
  - If a problem is NP-Complete, no point in trying to come up with an algorithm to solve it
  - What can we do, if we need to solve a problem that is NP-Complete?

23-45: NP-Completeness

- What can we do, if we need to solve a problem that is NP-Complete?
  - If the problem we need to solve is very small (< 20), an exponential solution might be OK
  - We can solve an *approximation* of the problem
    - Color a graph using an non-optimal number of colors
    - Find a Traveling Salesman tour that is not optimal

Impossible Problems

- Some problems are “easy” – require a fairly small amount of time to solve
  - Sorting
- Some problems are “probably hard” – believed to require exponential time to solve
  - TSP, Graph Coloring, etc
- Some problems are “hard” – known to require an exponential amount of time to solve
  - Towers of Hanoi
- Some problems are impossible – *cannot* be solved

23-47: Halting Problem

- Program is running – seems to be taking a long time
- We'd like to know if the program will eventually finish, or if it is in an infinite loop
- Great debugging tool:
  - Takes as input the source code to a program p, and an input i
  - Determines if p will run forever when run on i

Halting Problem

- Program is running – seems to be taking a long time
- We'd like to know if the program will eventually finish, or if it is in an infinite loop
- Great debugging tool:
  - Takes as input the source code to a program p, and an input i
  - Determines if p will run forever when run on i
- No such tool can exist!

Halting Problem

- We will prove that the halting problem is unsolvable by contradiction
  - Assume that we have a solution to the halting problem
  - Derive a contradiction
  - Our original assumption (that the halting problem has a solution) must be false

23-50: Halting Problem

```
boolean halt(char [] program, char [] input) {

      /* code to determine if the program
         halts when run on the input */

      if (program halts on input)
         return true;
      else
         return false;
}
```

23-51: Halting Problem

```
boolean selfhalt(char [] program) {
    if (halt(program, program))
        return true;
    else
        return false;
}
```

23-52: Halting Problem

```
boolean selfhalt(char [] program) {
    if (halt(program, program))
        return true;
    else
        return false;
}

void contrary(char [] program) {
    if (selfhalt(program)
      while(true);   /* infinite loop */
}
```

23-53: Halting Problem

```
boolean selfhalt(char [] program) {
    if (halt(program, program))
         return true;
    else
         return false;
}

void contrary(char [] program) {
    if (selfhalt(program)
       while(true);   /* infinite loop */
}
```

- what happens when we call contrary, passing in its own source code as input?

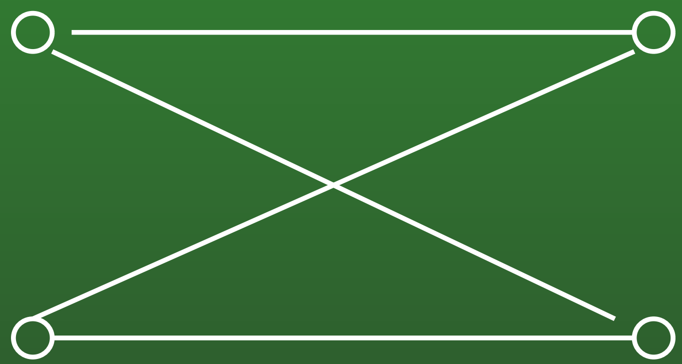
23-54: Reduction Example

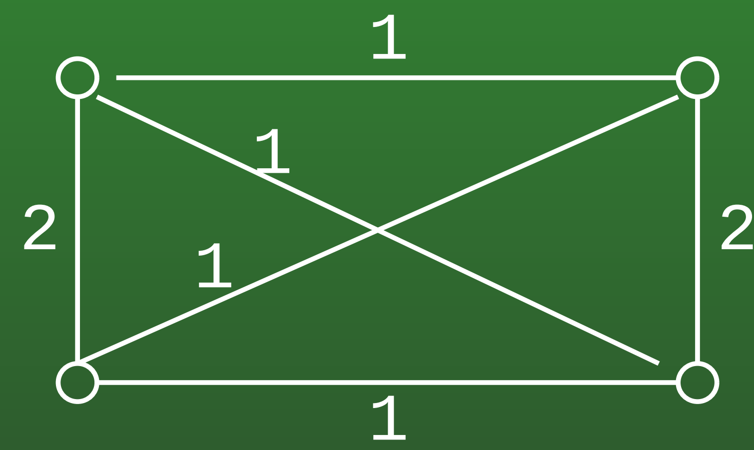
- Hamiltonian Cycle:
  - Given an unweighted, undirected graph G, is there a cycle that includes every vertex exactly once?
- Traveling Salesman Problem (TSP)
  - Given a complete, weighed, undirected graph G and a cost bound k, is there a cycle that incldes every vertex in G, with a cost $< k$?

Reduction Example

- If we could solve the Traveling Salesman problem in polynomial time, we could solve the Hamiltonian Cycle problem in polynomial time
  - Given any graph G, we can create a new graph G' and limit k, such that there is a Hamiltonian Circuit in G if and only if there is a Traveling Salesman tour in G' with cost less than k
  - Vertices in G' are the same as the vertices in G
  - For each pair of vertices x_i and x_j in G, if the edge (x_i, x_j) is in G, add the edge (x_i, x_j) to G' with the cost 1. Otherwise, add the edge (x_i, x_j) to G' with the cost 2.
  - Set the limit $k =$ # of vertices in G

23-56: Reduction Example





Limit = 4

23-57: Reduction Example

- If we could solve TSP in polynomial time, we could solve Hamiltonian Cycle problem in polynomial time
  - Start with an instance of Hamiltonian Cycle
  - Create instance of TSP
  - Feed instance of TSP into TSP solver
  - Use result to find solution to Hamiltonian Cycle

Reduction Example #2

- Given any instance of the Hamiltonian Cycle Problem:
  - We can (in polynomial time) create an instance of Satisfiability
  - That is, given any graph G, we can create a boolean formula f, such that f is satisfiable if and only if there is a Hamiltonian Cycle in G
- If we could solve Satisfiability in Polynomial Time, we could solve the Hamiltonian Cycle problem in Polynomial Time

23-59: Reduction Example #2

- Given a graph G with n vertices, we will create a formula with n^2 variables:
  - $x_{11}, x_{12}, x_{13}, \ldots x_{1n}$
    $x_{21}, x_{22}, x_{23}, \ldots x_{2n}$
    $\ldots$
    $x_{n1}, x_{n2}, x_{n3}, \ldots x_{nn}$
- Design our formula such that x_{ij} will be true if and only if the ith element in a Hamiltonian Circuit of G is vertex # j

23-60: Reduction Example #2

- For our set of n^2 variables x_{ij}, we need to write a formula that ensures that:
  - For each i, there is exactly one j such that x_{ij} = true
  - For each j, there is exactly one i such that x_{ij} = true
  - If x_{ij} and $x_{(i+1)k}$ are both true, then there must be a link from v_j to v_k in the graph G

23-61: Reduction Example #2

- For each i, there is exactly one j such that x_{ij} = true
  - For each i in $1 \ldots n$, add the rules:
    - $(x_{i1} || x_{i2} || \ldots || x_{in})$
- This ensures that for each i, there is at least one j such that x_{ij} = true
- (This adds n clauses to the formula)

Reduction Example #2

- For each i, there is exactly one j such that x_{ij} = true

  for each i in $1 \ldots n$
      for each j in $1 \ldots n$
          for each k in $1 \ldots n$ $\quad j \neq k$
              Add rule $(!x_{ij} || !x_{ik})$

- This ensures that for each i, there is at most one j such that x_{ij} = true
- (this adds a total of n^3 clauses to the formula)

Reduction Example #2

- If x_{ij} and $x_{(i+1)k}$ are both true, then there must be a link from v_i to v_k in the graph G

for each i in $1 \ldots (n-1)$
    for each j in $1 \ldots n$
        for each k in $1 \ldots n$
            if edge (v_j, v_k) is *not* in the graph:
                Add rule $(!x_{ij}||!x_{(i+1)k})$

- (This adds no more than n^3 clauses to the formula)

23-64: Reduction Example #2

- If x_{nj} and x_{0k} are both true, then there must be a link from v_i to v_k in the graph G (looping back to finish cycle)

  for each j in $1 \ldots n$
      for each k in $1 \ldots n$
          if edge (v_n, v_0) is *not* in the graph:
              Add rule $(!x_{nj} || !x_{0k})$

- (This adds no more than n^2 clauses to the formula)

Reduction Example #2

- In order for this formula to be satisfied:
  - For each i, there is exactly one j such that x_{ij} is true
  - For each j, there is exactly one i such that x_{ji} is true
  - if x_{ij} is true, and $x_{(i+1)k}$ is true, then there is an arc from v_j to v_k in the graph G
- Thus, the formula can only be satisfied if there is a Hamiltonian Cycle of the graph

More NP-Complete Problems

- Exact Cover Problem
  - Set of elements A
  - $F \subset 2^A$, family of subsets
  - Is there a subset of F such that each element of A appears exactly once?

More NP-Complete Problems

- Exact Cover Problem
  - $A = \{a, b, c, d, e, f, g\}$
  - $F = \{\{a, b, c\}, \{d, e, f\}, \{b, f, g\}, \{g\}\}$
  - Exact cover exists: $\{a, b, c\}, \{d, e, f\}, \{g\}$

23-68: More NP-Complete Problems

- Exact Cover Problem
  - $A = \{a, b, c, d, e, f, g\}$
  - $F = \{\{a, b, c\}, \{c, d, e, f\}, \{a, f, g\}, \{c\}\}$
  - No exact cover exists

23-69: More NP-Complete Problems

- Exact Cover is NP-Complete
  - Reduction from Satisfiability
  - Given any instance of Satisfiability, create (in polynomial time) an instance of Exact Cover
  - Solution to Exact Cover problem tells us solution to Satisfiability problem
  - Satisfiability is NP-Complete => Exact Cover is NP-Complete

Exact Cover is NP-Complete

- Given an instance of SAT:
  - $C_1 = (x_1 \vee \overline{x_2})$
  - $C_2 = (\overline{x_1} \vee x_2 \vee x_3)$
  - $C_3 = (x_2)$
  - $C_4 = (\overline{x_2} \vee \overline{x_3})$
- Formula: $C_1 \wedge C_2 \wedge C_3 \wedge C_4$
- Create an instance of Exact Cover
  - Define a set A and family of subsets F such that there is an exact cover of A in F if and only if the formula is satisfiable

23-71: Exact Cover is NP-Complete

$C_1 = (x_1 \vee \overline{x_2})$ $C_2 = (\overline{x_1} \vee x_2 \vee x_3)$ $C_3 = (x_2)$ $C_4 = (\overline{x_2} \vee \overline{x_3})$

$A = \{x_1, x_2, x_3, C_1, C_2, C_3, C_4, p_{11}, p_{12}, p_{21}, p_{22}, p_{23}, p_{31}, p_{41}, p_{42}\}$

$F = \{\{p_{11}\}, \{p_{12}\}, \{p_{21}\}, \{p_{22}\}, \{p_{23}\}, \{p_{31}\}, \{p_{41}\}, \{p_{42}\},$

$X_1, f = \{x_1, p_{11}\}$

$X_1, t = \{x_1, p_{21}\}$

$X_2, f = \{x_2, p_{22}, p_{31}\}$

$X_2, t = \{x_2, p_{12}, p_{41}\}$

$X_3, f = \{x_3, p_{23}\}$

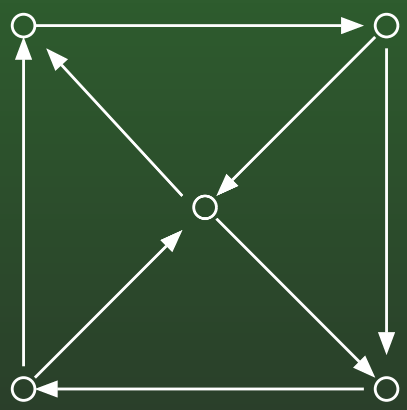
$X_3, t = \{x_3, p_{42}\}$

$\{C_1, p_{11}\}, \ \{C_1, p_{12}\}, \ \{C_2, p_{21}\}, \ \{C_2, p_{22}\}, \ \{C_2, p_{23}\}, \ \{C_3, p_{31}\},$

$\{C_4, p_{41}\}, \{C_4, p_{422}\}\}$

Directed Hamiltonian Cycle

- Given any directed graph G, determine if G has a a Hamiltonian Cycle
  - Cycle that includes every node in the graph exactly once, following the direction of the arrows



Directed Hamiltonian Cycle

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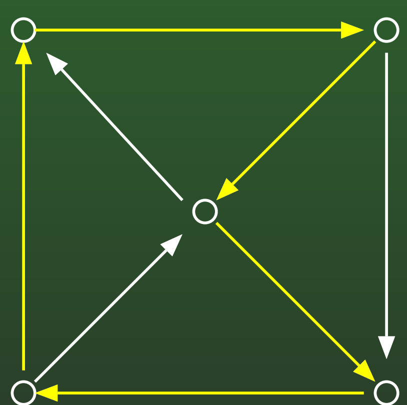
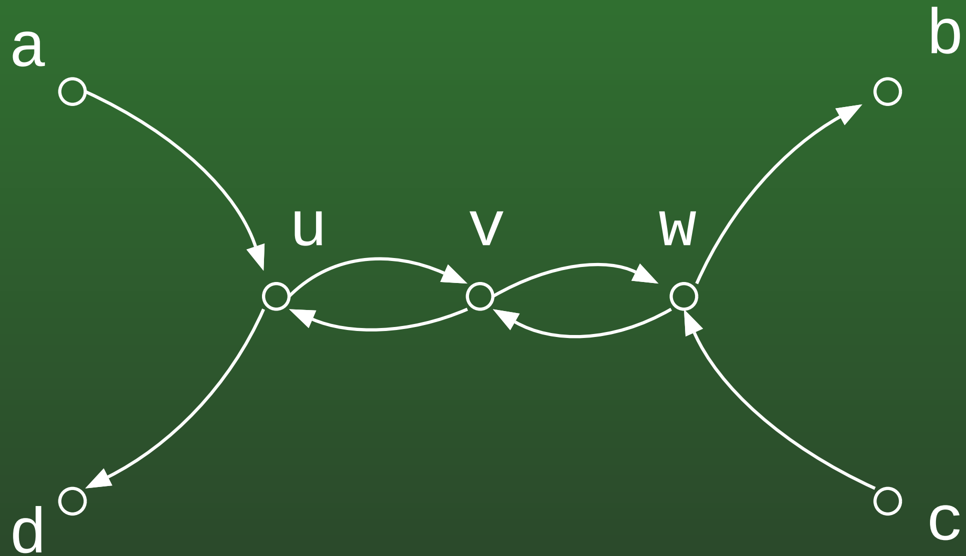


Directed Hamiltonian Cycle

- The Directed Hamiltonian Cycle problem is $\mathbf{NP}$-Complete
- Reduce Exact Cover to Directed Hamiltonian Cycle
  - Given any set A, and family of subsets F:
  - Create a graph G that has a hamiltonian cycle if and only if there is an exact cover of A in F

Directed Hamiltonian Cycle

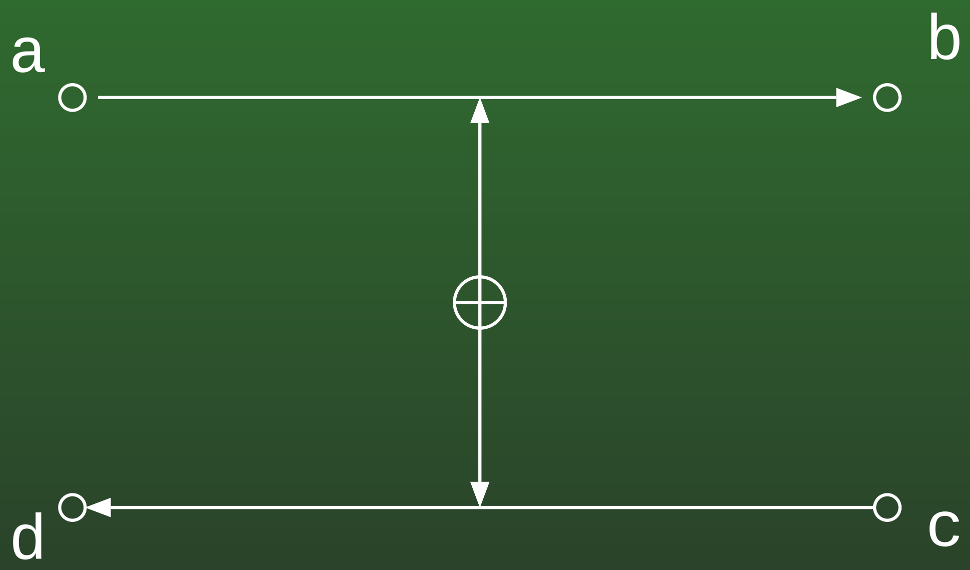
- Widgets:
  - Consider the following graph segment:



  - If a graph containing this subgraph has a Hamiltonian cycle, then the cycle must contain either $a \rightarrow u \rightarrow v \rightarrow w \rightarrow b$ or $c \rightarrow w \rightarrow v \rightarrow u \rightarrow d$ – but not both (why)?

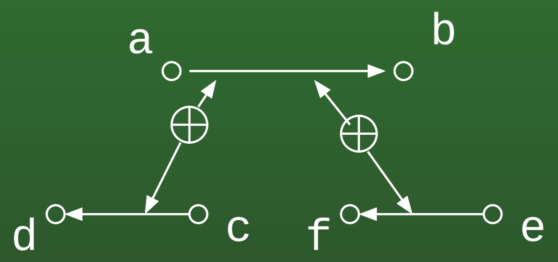
Directed Hamiltonian Cycle

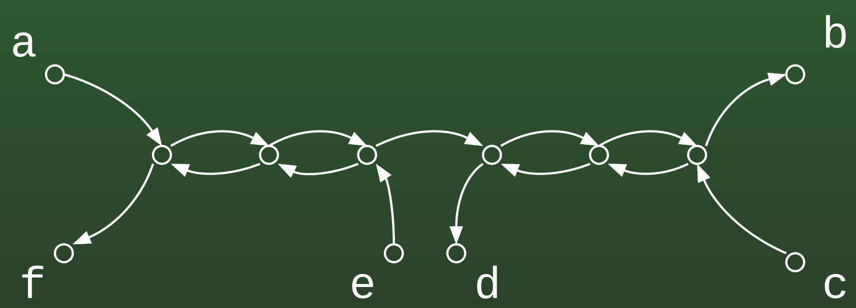
- Widgets:
  - XOR edges: Exactly one of the edges must be used in a Hamiltonian Cycle



Directed Hamiltonian Cycle

- Widgets:
  - XOR edges: Exactly one of the edges must be used in a Hamiltonian Cycle





Directed Hamiltonian Cycle

- Add a vertex for every variable in A (+ 1 extra)

$F_1 = \{a_1, a_2\}$
$F_2 = \{a_3\}$
$F_3 = \{a_2, a_3\}$

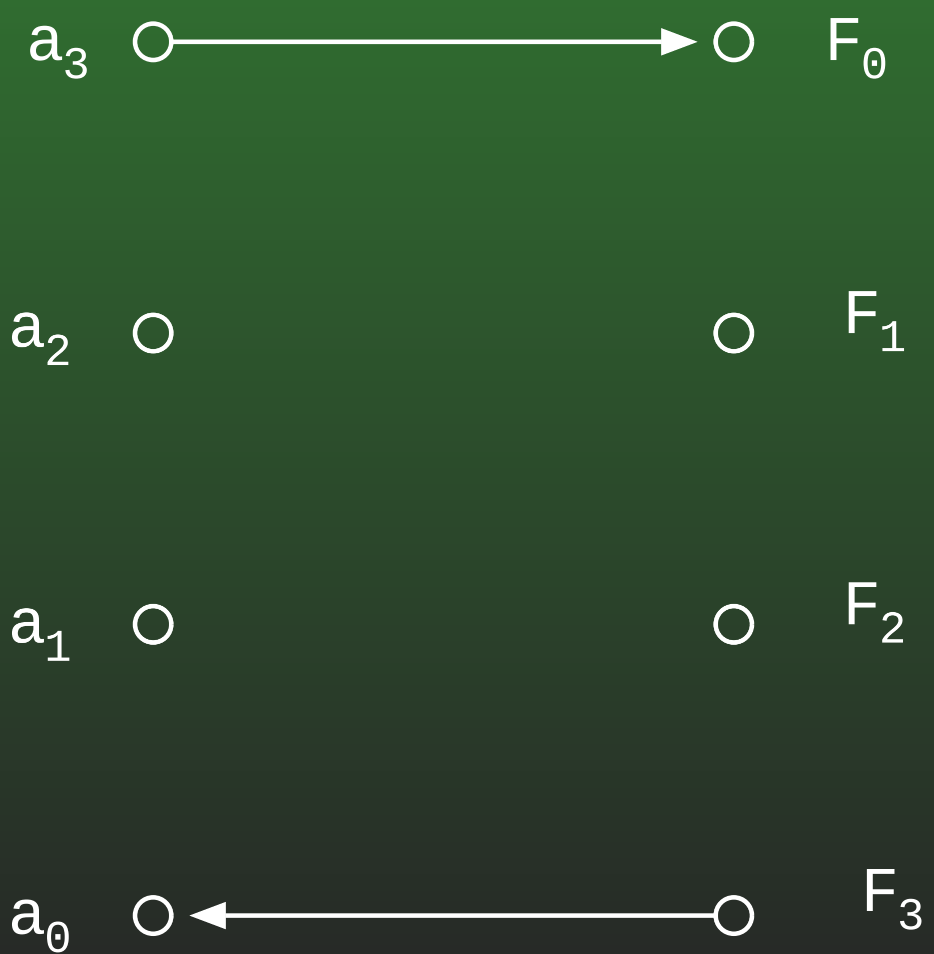
23-79: Directed Hamiltonian Cycle

- Add a vertex for every subset F (+ 1 extra)

$F_1 = \{a_1, a_2\}$
$F_2 = \{a_3\}$
$F_3 = \{a_2, a_3\}$

23-80: Directed Hamiltonian Cycle

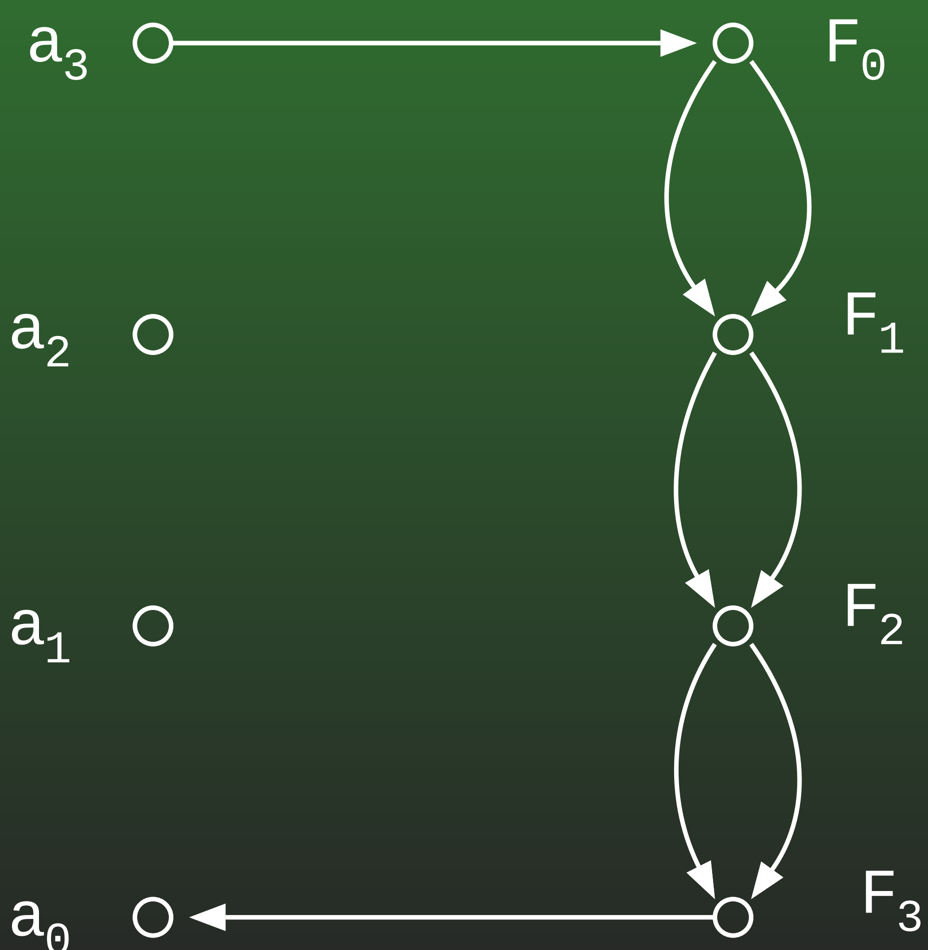
- Add an edge from the last variable to the 0th subset, and from the last subset to the 0th variable



$F_1 = \{a_1, a_2\}$
$F_2 = \{a_3\}$
$F_3 = \{a_2, a_3\}$

Directed Hamiltonian Cycle

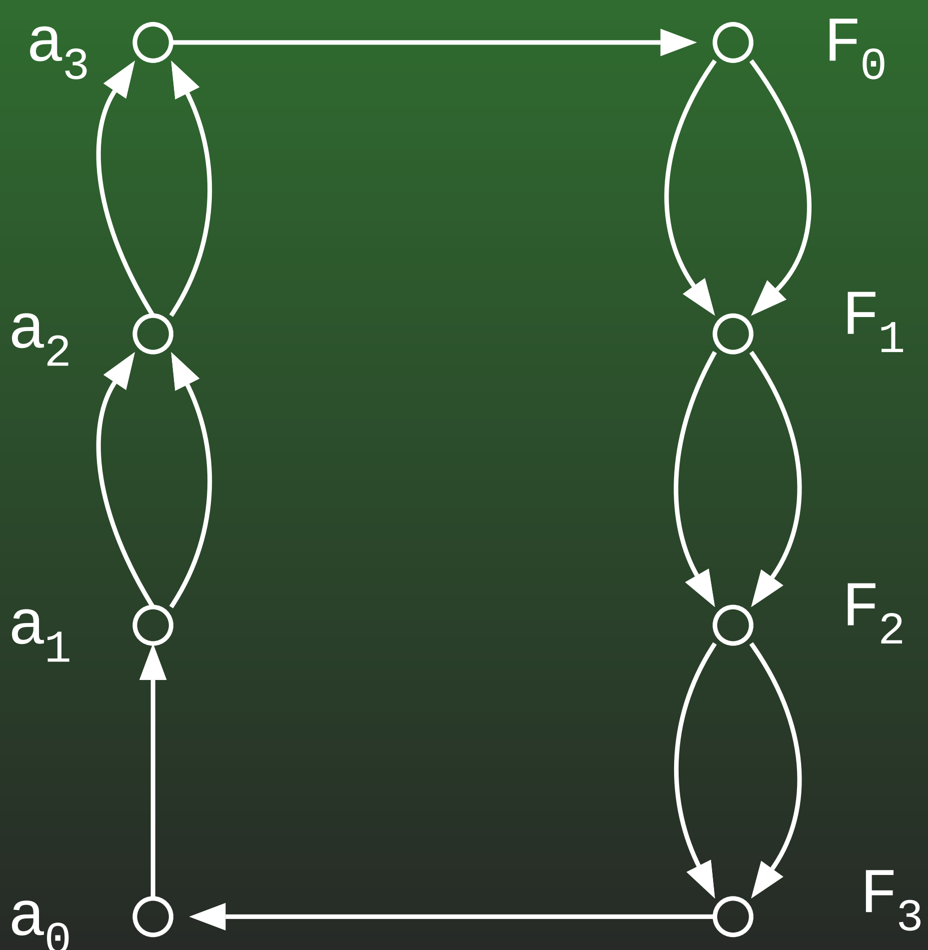
- Add 2 edges from F_i to F_{i+1}. One edge will be a "short edge", and one will be a "long edge".



$F_1 = \{a_1, a_2\}$
$F_2 = \{a_3\}$
$F_3 = \{a_2, a_3\}$

23-82: Directed Hamiltonian Cycle

- Add an edge from a_{i-1} to a_i for $\mathbf{each}$ subset a_i appears in.



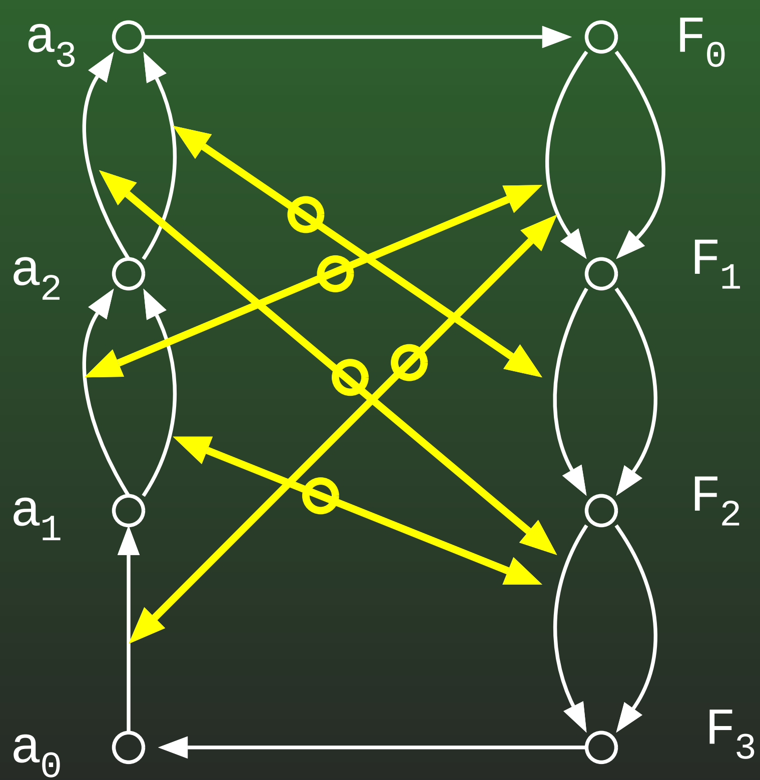
$F_1 = \{a_1, a_2\}$

$F_2 = \{a_3\}$

$F_3 = \{a_2, a_3\}$

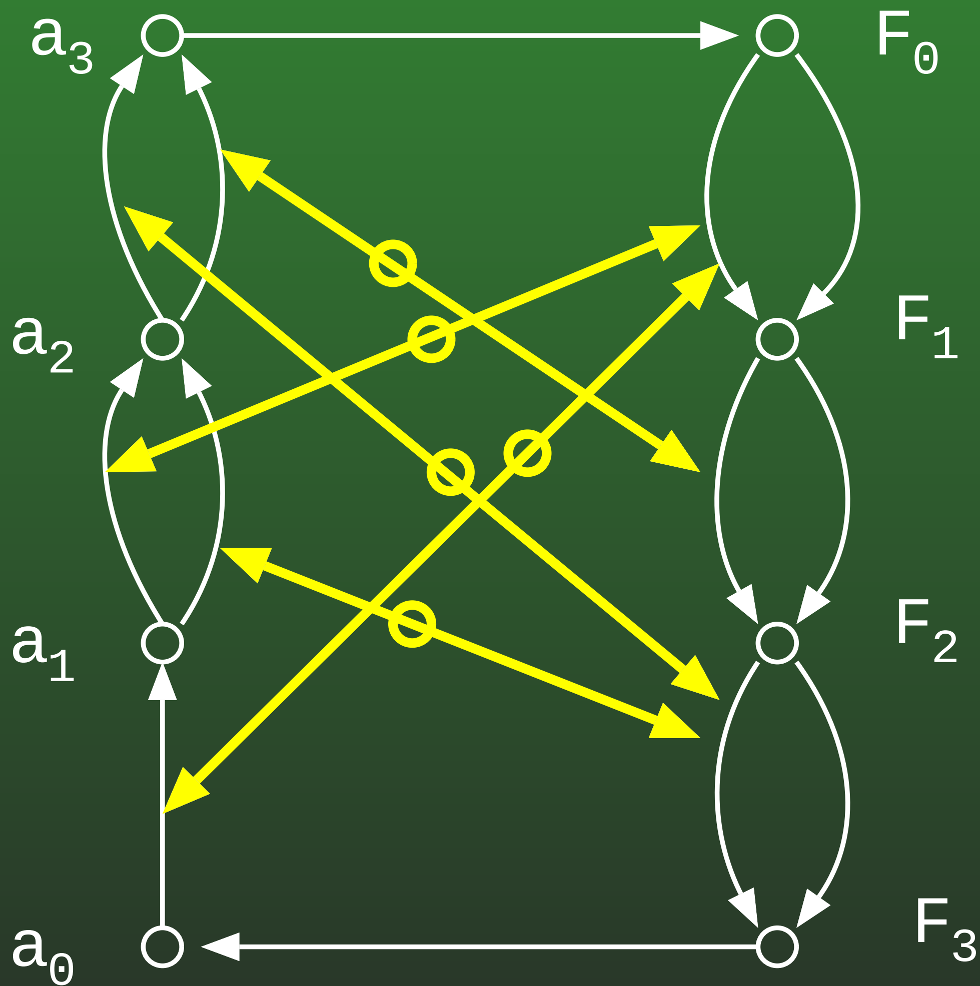
23-83: Directed Hamiltonian Cycle

- Each edge (a_{i-1}, a_i) corresponds to some subset that contains a_i. Add an XOR link between this edge and the long edge of the corresponding subset



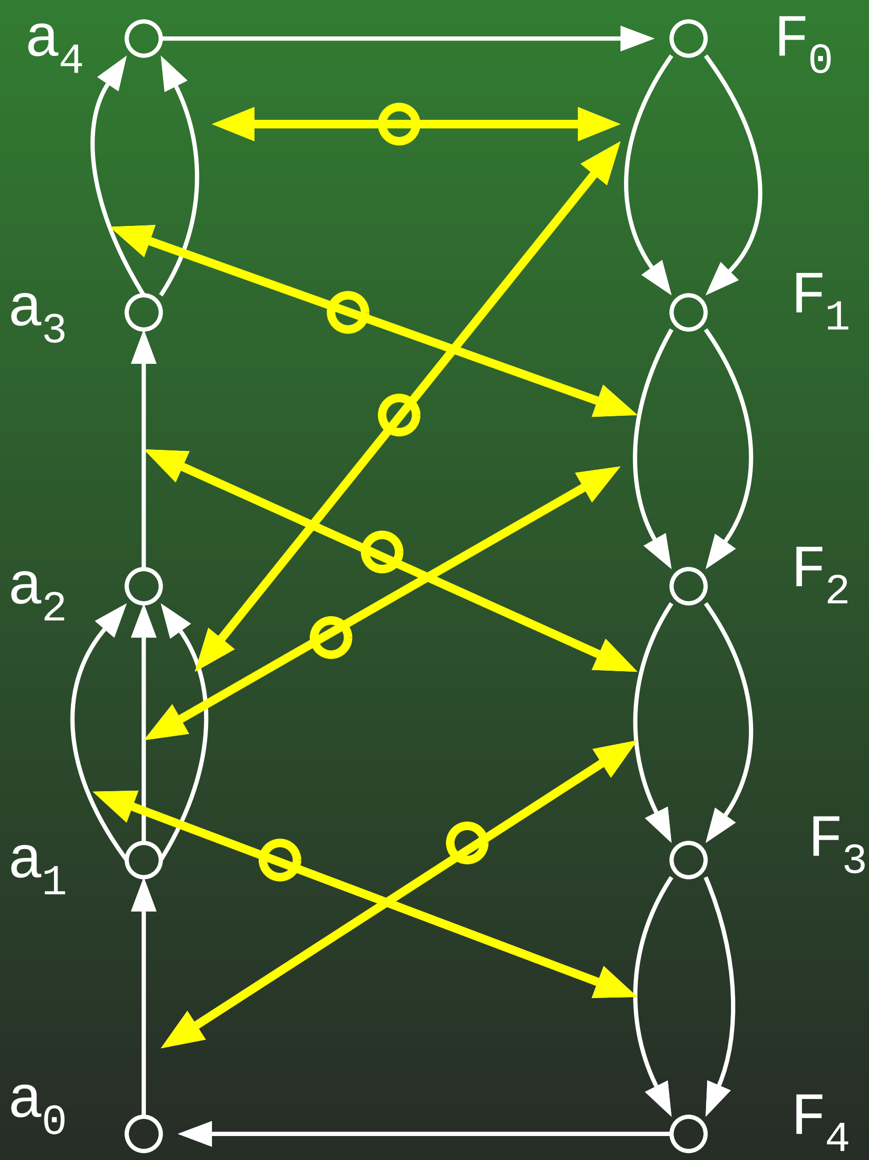
$F_1 = \{a_1, a_2\}$
$F_2 = \{a_3\}$
$F_3 = \{a_2, a_3\}$

23-84: Directed Hamiltonian Cycle



$F_1 = \{a_1, a_2\}$
$F_2 = \{a_3\}$
$F_3 = \{a_2, a_3\}$

Directed Hamiltonian Cycle



$F_1 = \{a_2, a_4\}$

$F_2 = \{a_2, a_4\}$

$F_3 = \{a_1, a_3\}$

$F_4 = \{a_2\}$

XOR edge