06-0: **Ordered List ADT**

Operations:

- Insert an element in the list
- Check if an element is in the list
- Remove an element from the list
- Print out the contents of the list, in order

06-1: **Implementing Ordered List**

Using an Ordered Array – Running times:

Check
Insert
Remove
Print

06-2: **Implementing Ordered List**

Using an Ordered Array – Running times:

Check	$\Theta(\lg n)$
Insert	$\Theta(n)$
Remove	$\Theta(n)$
Print	$\Theta(n)$

06-3: **Implementing Ordered List**

Using an *Unordered* Array – Running times:

Check
Insert
Remove
Print

06-4: **Implementing Ordered List**

Using an *Unordered* Array – Running times:

Check	$\Theta(n)$
Insert	$\Theta(1)$
Remove	$\Theta(n)$ Need to find element first!
Print	$\Theta(n \lg n)$ (Given a fast sorting algorithm)

06-5: **Implementing Ordered List**

Using an Ordered Linked List – Running times:

Check
Insert
Remove
Print

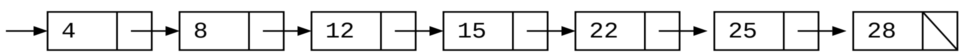
06-6: **Implementing Ordered List**

Using an Ordered Linked List – Running times:

Check	$\Theta(n)$
Insert	$\Theta(n)$
Remove	$\Theta(n)$
Print	$\Theta(n)$

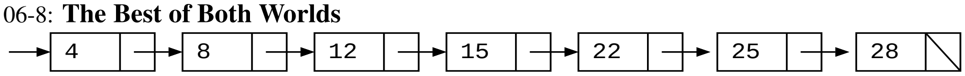
06-7: **The Best of Both Worlds**

- Linked Lists – Insert fast / Find slow
- Arrays – Find fast / Insert slow
- The only way to examine nth element in a linked list is to traverse (n-1) other elements

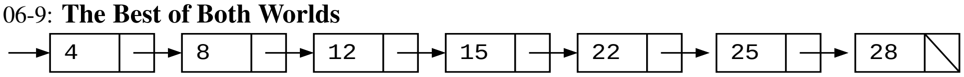


- If we could leap to the middle of the list ...

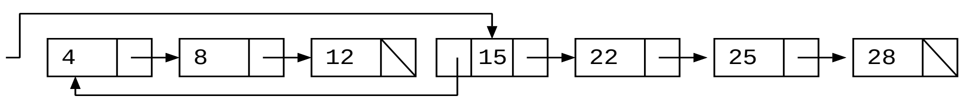
06-8: **The Best of Both Worlds**



06-9: **The Best of Both Worlds**

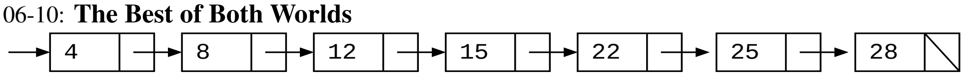


Move the initial pointer to the middle of the list:

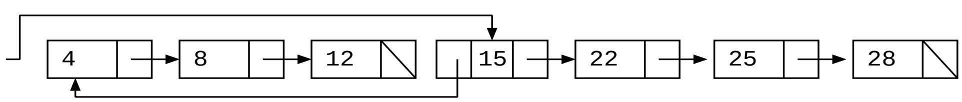


We've cut our search time in half! Have we changed the $\Theta()$ running time?

06-10: **The Best of Both Worlds**



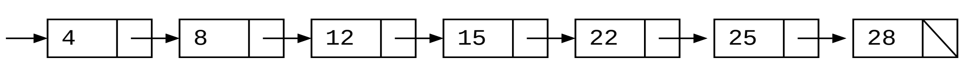
Move the initial pointer to the middle of the list:

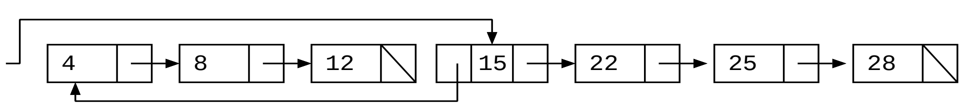


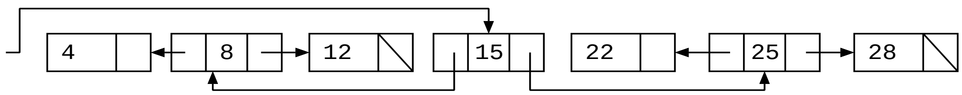
We've cut our search time in half! Have we changed the $\Theta()$ running time?

Repeat the process!

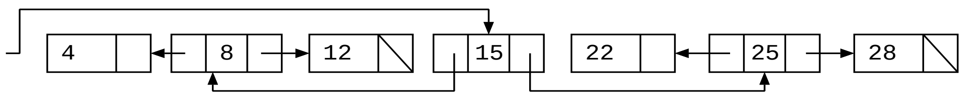
06-11: **The Best of Both Worlds**



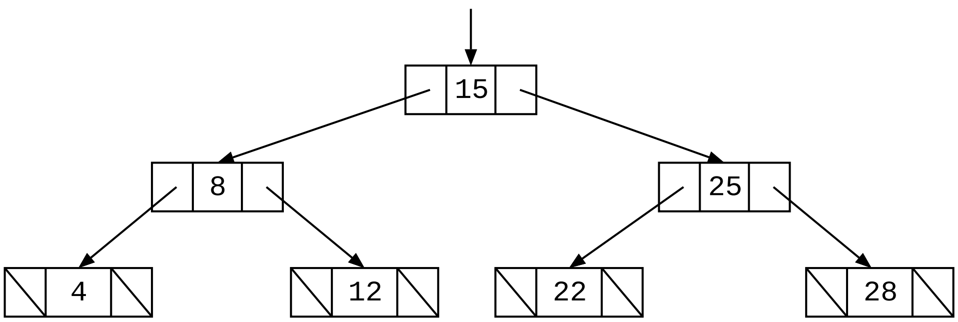




06-12: **The Best of Both Worlds** Grab the first element of the list:



Give it a good shake -



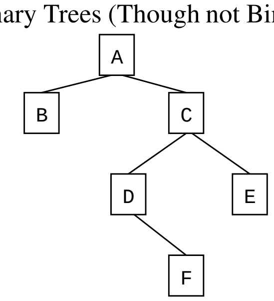
06-13: **Binary Trees**

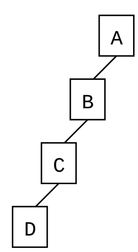
Binary Trees are Recursive Data Structures

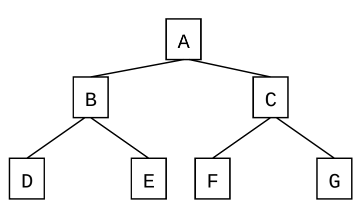
- Base Case: Empty Tree
- Recursive Case: Node, consiting of:
  - Left Child (Tree)
  - Right Child (Tree)
  - Data

06-14: **Binary Tree Examples**

The following are all Binary Trees (Though not Binary *Search* Trees)







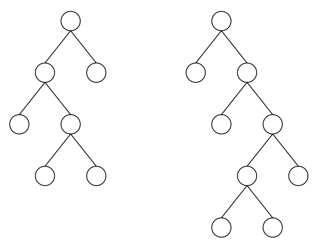
06-15: **Tree Terminology**

- Parent / Child

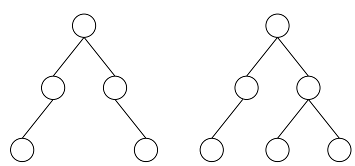
- Leaf node
- Root node
- Edge (between nodes)
- Path
- Ancestor / Descendant
- Depth of a node n
  - Length of path from root to n
- Height of a tree
  - (Depth of deepest node) + 1

06-16: **Full Binary Tree**

- Each node has 0 or 2 children
- Full Binary Trees

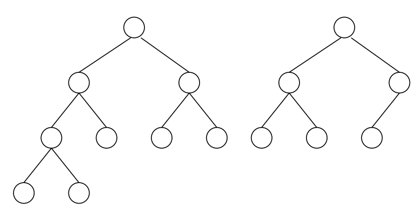


- *Not* Full Binary Trees

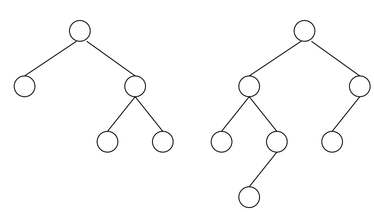


06-17: **Complete Binary Tree**

- Can be built by starting at the root, and filling the tree by levels from left to right



- Complete Binary Trees

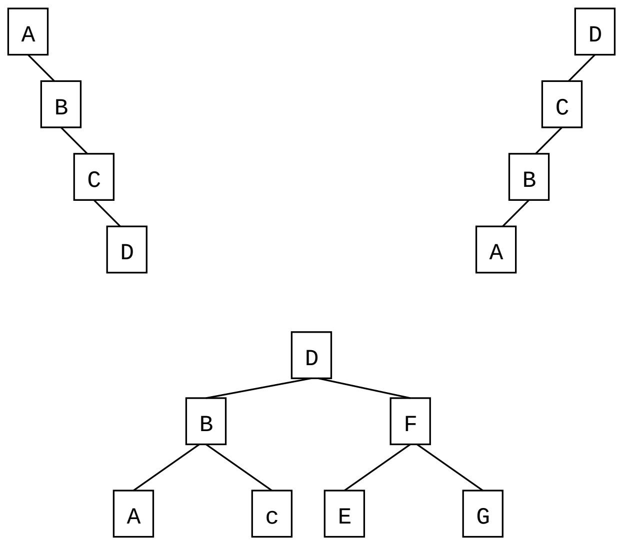


- *Not* Complete Binary Trees

06-18: **Binary Search Trees**

- Binary Trees
- For each node n, (value stored at node n) $\geq$ (value stored in left subtree)
- For each node n, (value stored at node n) $<$ (value stored in right subtree)

06-19: **Example Binary Search Trees**



06-20: **Implementing BSTs**

- Each Node in a BST is implemented as a class:

```
public class Node {
   public Comparable data;
   public Node left;
   public Node right;
}
```

06-21: **Implementing BSTs**

```
public class Node {

   public Node(Comparable data, Node left, Node right) {
      this.data = data;
      this.left = left;
      this.right = right;
   }

   public Node left() {
      return left;
   }
   public Node setLeft(Node newLeft) {
      left = newLeft
   }
   ... (etc)

   private Comparable data;
   private Node left;
   private Node right;
}
```

06-22: **Finding an Element in a BST**

- Binary Search Trees are recursive data structures, so most operations on them will be recursive as well

- Recall how to write a recursive algorithm ...

06-23: **Writing a Recursive Algorithm**

- Determine a small version of the problem, which can be solved immediately. This is the *base case*
- Determine how to make the problem smaller
- Once the problem has been made smaller, we can assume that the function that we are writing *will work correctly on the smaller problem* (Recursive Leap of Faith)
  - Determine how to use the solution to the smaller problem to solve the larger problem

06-24: **Finding an Element in a BST**

- First, the Base Case – when is it easy to determine if an element is stored in a Binary Search Tree?

06-25: **Finding an Element in a BST**

- First, the Base Case – when is it easy to determine if an element is stored in a Binary Search Tree?
  - If the tree is empty, then the element can't be there
  - If the element is stored at the root, then the element is there

06-26: **Finding an Element in a BST**

- Next, the Recursive Case – how do we make the problem smaller?

06-27: **Finding an Element in a BST**

- Next, the Recursive Case – how do we make the problem smaller?
  - Both the left and right subtrees are smaller versions of the problem. Which one do we use?

06-28: **Finding an Element in a BST**

- Next, the Recursive Case – how do we make the problem smaller?
  - Both the left and right subtrees are smaller versions of the problem. Which one do we use?
  - If the element we are trying to find is $<$ the element stored at the root, use the left subtree. Otherwise, use the right subtree.

06-29: **Finding an Element in a BST**

- Next, the Recursive Case – how do we make the problem smaller?
  - Both the left and right subtrees are smaller versions of the problem. Which one do we use?
  - If the element we are trying to find is $<$ the element stored at the root, use the left subtree. Otherwise, use the right subtree.
- How do we use the solution to the subproblem to solve the original problem?

06-30: **Finding an Element in a BST**

- Next, the Recursive Case – how do we make the problem smaller?

  - Both the left and right subtrees are smaller versions of the problem. Which one do we use?
  - If the element we are trying to find is $<$ the element stored at the root, use the left subtree. Otherwise, use the right subtree.
- How do we use the solution to the subproblem to solve the original problem?
  - The solution to the subproblem *is* the solution to the original problem (this is not always the case in recursive algorithms)

06-31: **Finding an Element in a BST**

To find an element e in a Binary Search Tree T:

- If T is empty, then e is not in T
- If the root of T contains e, then e is in T
- If $e <$ the element stored in the root of T:
  - Look for e in the left subtree of T

  Otherwise
  - Look for e in the right subtree of T

06-32: **Finding an Element in a BST**

```
boolean find(Node tree, Comparable elem) {
  if (tree == null)
    return false;
  if (elem.compareTo(tree.element()) == 0)
    return true;
  if (elem.compareTo(tree) < 0)
    return find(tree.left(), elem);
  else
    return find(tree.right(), elem);
}
```

06-33: **Printing out a BST**

To print out all element in a BST:

- Print all elements in the left subtree, in order
- Print out the element at the root of the tree
- Print all elements in the right subtree, in order

06-34: **Printing out a BST**

To print out all element in a BST:

- Print all elements in the left subtree, in order
- Print out the element at the root of the tree
- Print all elements in the right subtree, in order

- Each subproblem is a smaller version of the original problem – we can assume that a recursive call will work!

06-35: **Printing out a BST**

- What is the base case for printing out a Binary Search Tree – what is an easy tree to print out?

06-36: **Printing out a BST**

- What is the base case for printing out a Binary Search Tree – what is an easy tree to print out?
- An empty tree is extremely easy to print out – do nothing!
- Code for printing a BST ...

06-37: **Printing out a BST**

```
void print(Node tree) {
  if (tree != null) {
    print(tree.left());
    System.out.println(tree.element());
    print(tree.right());
  }
}
```

06-38: **Printing out a BST**
Examples

06-39: **Tree Traversals**

- PREORDER Traversal
  - Do operation on root of the tree
  - Traverse left subtree
  - Traverse right subtree
- INORDER Traversal
  - Traverse left subtree
  - Do operation on root of the tree
  - Traverse right subtree
- POSTORDER Traversal
  - Traverse left subtree
  - Traverse right subtree
  - Do operation on root of the tree

06-40: **PREORDER Examples**
06-41: **POSTORDER Examples**
06-42: **INORDER Examples**
06-43: **BST Minimal Element**
To find the minimal element in a BST:

- Base Case: When is it easy to find the smallest element in a BST?
- Recursive Case: How can we make the problem smaller?

  How can we use the solution to the smaller problem to solve the original problem?

06-44: **BST Minimal Element**

To find the minimal element in a BST:
Base Case:

- When is it easy to find the smallest element in a BST?

06-45: **BST Minimal Element**

To find the minimal element in a BST:
Base Case:

- When is it easy to find the smallest element in a BST?
  - When the left subtree is empty, then the element stored at the root is the smallest element in the tree.

06-46: **BST Minimal Element**

To find the minimal element in a BST:
Recursive Case:

- How can we make the problem smaller?

06-47: **BST Minimal Element**

To find the minimal element in a BST:
Recursive Case:

- How can we make the problem smaller?
  - Both the left and right subtrees are smaller versions of the same problem
- How can we use the solution to a smaller problem to solve the original problem?

06-48: **BST Minimal Element**

To find the minimal element in a BST:
Recursive Case:

- How can we make the problem smaller?
  - Both the left and right subtrees are smaller versions of the same problem
- How can we use the solution to a smaller problem to solve the original problem?
  - The smallest element in the left subtree is the smallest element in the tree

06-49: **BST Minimal Element**

```
Comparable minimum(Node tree) {
  if (tree == null)
    return null;
  if (tree.left() == null)
    return tree.element();
  else
    return minimum(tree.left());
}
```

06-50: **BST Minimal Element**

Iterative Version

```
Comparable minimum(Node tree) {
  if (tree == null)
    return null;
  while (tree.left() != null)
    tree = tree.left();
  return tree.element();
}
```

06-51: **Inserting e into BST T**

- What is the base case – an easy tree to insert an element into?

06-52: **Inserting e into BST T**

- What is the base case – an easy tree to insert an element into?
  - An empty tree
  - Create a new tree, containing the element e

06-53: **Inserting e into BST T**

- Recursive Case: How do we make the problem smaller?

06-54: **Inserting e into BST T**

- Recursive Case: How do we make the problem smaller?
  - The left and right subtrees are smaller versions of the same problem.
  - How do we use these smaller versions of the problem?

06-55: **Inserting e into BST T**

- Recursive Case: How do we make the problem smaller?
  - The left and right subtrees are smaller versions of the same problem
  - Insert the element into the left subtree if $e \leq$ value stored at the root, and insert the element into the right subtree if $e >$ value stored at the root

06-56: **Inserting e into BST T**

- Base case – T is empty:
  - Create a new tree, containing the element e
- Recursive Case:
  - If e is less than the element at the root of T, insert e into left subtree
  - If e is greater than the element at the root of T, insert e into the right subtree

06-57: **Tree Manipulation in Java**

- Tree manipulation functions return trees
- Insert method takes as input the old tree and the element to insert, and returns the new tree, with the element inserted
  - Old value (pre-insertion) of tree will be destroyed
- To insert an element `e` into a tree `T`:
  - `T = insert(T, e);`

06-58: **Inserting e into BST T**

```
Node insert(Node tree, Comparable elem) {
  if (tree == null) {
    return new Node(elem);
  if (elem.compareTo(tree.element() <= 0)) {
    tree.setLeft(insert(tree.left(), elem));
    return tree;
  } else {
    tree.setRight(insert(tree.right(), elem));
    return tree;
  }
}
```

06-59: **Deleting From a BST**

- Removing a leaf:

06-60: **Deleting From a BST**

- Removing a leaf:
  - Remove element immediately

06-61: **Deleting From a BST**

- Removing a leaf:
  - Remove element immediately
- Removing a node with one child:

06-62: **Deleting From a BST**

- Removing a leaf:
  - Remove element immediately
- Removing a node with one child:
  - Just like removing from a linked list
  - Make parent point to child

06-63: **Deleting From a BST**

- Removing a leaf:
  - Remove element immediately
- Removing a node with one child:
  - Just like removing from a linked list
  - Make parent point to child
- Removing a node with two children:

06-64: **Deleting From a BST**

- Removing a leaf:
  - Remove element immediately
- Removing a node with one child:
  - Just like removing from a linked list
  - Make parent point to child
- Removing a node with two children:
  - Replace node with largest element in left subtree, or the smallest element in the right subtree

06-65: **Comparable vs. .key() method**

- We have been storing "Comparable" elements in BSTs
- Alternately, could use a "key()" method – elements stored in BSTs must implement a key() method, which returns an integer.
- We can combine the two methods
  - Each element stored in the tree has a key() method
  - key() method returns Comparable class

06-66: **BST Implementation Details**

- Use BSTs to implement Ordered List ADT
- Operations
  - Insert
  - Find
  - Remove
  - Print in Order
- The specification (interface) should not specify an implementation
  - Allow several different implementations of the same interface

06-67: **BST Implementation Details**

- BST functions require the root of the tree be sent in as a parameter

- Ordered list functions should *not* contain implementation details!
- What should we do?

06-68: **BST Implementation Details**

- BST functions require the root of the tree be sent in as a parameter
- Ordered list functions should *not* contain implementation details!
- What should we do?
  - Private variable, holds root of the tree
  - Private recursive methods, require root as an argument
  - Public methods call private methods, passing in private root