Data Structures and Algorithms

CS245-2017S-08

Priority Queues – Heaps

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08-0: Priority Queue ADT

Operations

- Add an element / key pair
- Return (and remove) element with smallest key

Keys are “priorities”, with smaller keys having a “better” priority

08-1: Priority Queue ADT

Operations

- Add an element / key pair
- Return (and remove) element with smallest key

Implementation:

- Sorted Array
  Add Element
  Remove Smallest Key

08-2: Priority Queue ADT

Operations

- Add an element / key pair
- Return (and remove) element with smallest key

Implementation:

- Sorted Array

Add Element	$O(n)$
Remove Smallest Key	$O(1)$
	(using circular array)

08-3: Priority Queue ADT

Operations

- Add an element / key pair
- Return (and remove) element with smallest key

Implementation:

- Binary Search Tree
  Add Element
  Remove Smallest Key

08-4: Priority Queue ADT

Operations

- Add an element / key pair
- Return (and remove) element with smallest key

Implementation:

- Binary Search Tree

Add Element	$O(\lg n)$
Remove Smallest Key	$O(\lg n)$

If the tree is balanced

08-5: Priority Queue ADT

Operations

- Add an element / key pair
- Return (and remove) element with smallest key

Implementation:

- Binary Search Tree

Add Element	$O(n)$
Remove Smallest Key	$O(n)$

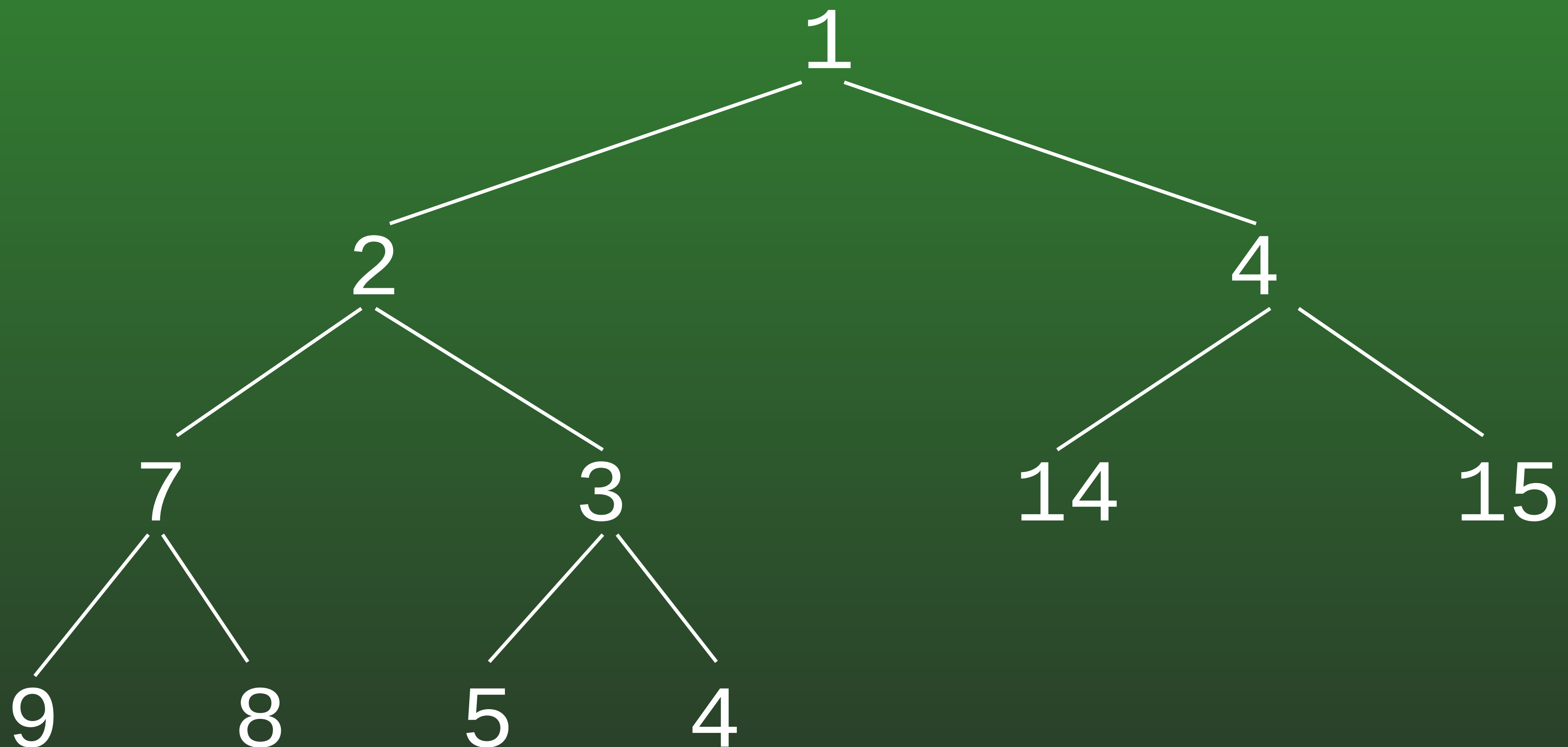
Computer Scientists are Pessimists

(Murphy was right)

08-6: Heap Definition

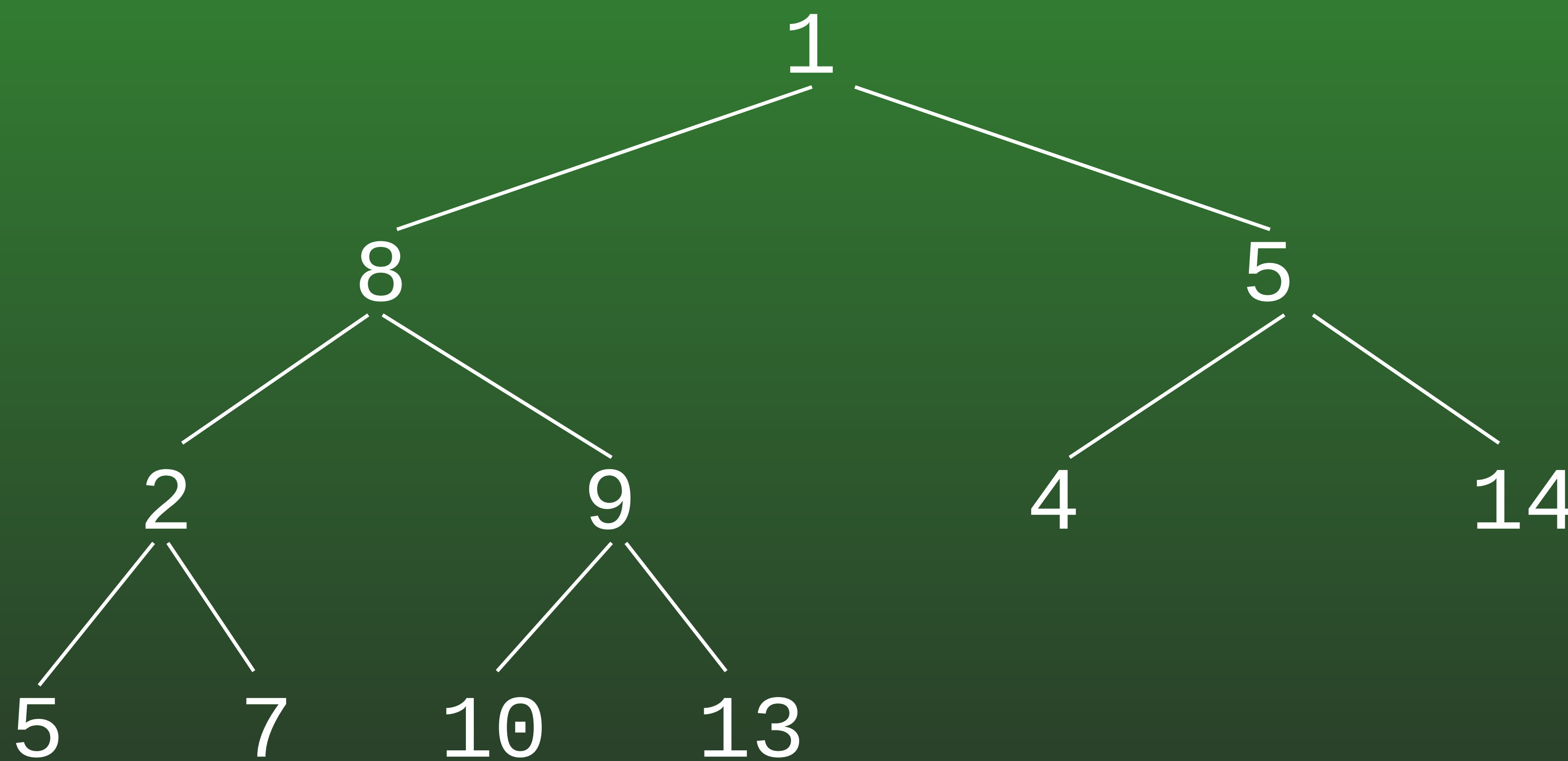
- Complete Binary Tree
- Heap Property
  - For every subtree in a tree, each value in the subtree is $\geq$ value stored at the root of the subtree

08-7: Heap Examples



Valid Heap

08-8: Heap Examples



Invalid Heap

08-9: Heap Insert

- What is the only place we can insert an element in a heap, and maintain the complete binary tree property?

08-10: Heap Insert

- What is the only place we can insert an element in a heap, and maintain the complete binary tree property?
  - “End” of the tree – as a child of the shallowest leaf that is farthest to the left
  - Will the resulting tree still be a heap?

08-11: Heap Insert

- What is the only place we can insert an element in a heap, and maintain the complete binary tree property?
    - “End” of the tree – as a child of the shallowest leaf that is farthest to the left
- Inserting an element at the “end” of the heap may break the heap property
    - Swap the value up the tree (examples)

08-12: Heap Insert

- Running time for Insert?

08-13: Heap Insert

- Running time for Insert?
  - Place element at end of tree: $O(1)$ (We'll see a clever way to find the "end" of the tree in a bit)
  - Swap element up the tree: $O(\text{height of tree})$ (Worst case, swap all the way up to the root)
    - Height of a Complete Binary Tree with n nodes?

08-14: Heap Insert

- Running time for Insert?
  - Place element at end of tree: $O(1)$ (We'll see a clever way to find the "end" of the tree in a bit
  - Swap element up the tree: $O(\text{height of tree})$ (Worst case, swap all the way up to the root)
    - Height of a Complete Binary Tree with n nodes = $\Theta(\lg n)$
- Total running time: $\Theta(\lg n)$ in the worst case

08-15: Heap Remove Smallest

- Finding the smallest element is easy – at the root of the tree
- Removing the Root of the heap is hard
- What element is easy to remove? How could this help us?

08-16: Heap Remove Smallest

- Finding the smallest element is easy – at the root of the tree
- Removing the Root of the heap is hard
- Removing the element at the “end” of the heap is easy
  - Copy last element of heap into root
  - Remove the last element
    - Problem?

08-17: Heap Remove Smallest

- Finding the smallest element is easy – at the root of the tree
- Removing the Root of the heap is hard
- Removing the element at the "end" of the heap is easy
  - Copy last element of heap into root
  - Remove the last element
    - May break the heap property

08-18: Heap Remove Smallest

- Finding the smallest element is easy – at the root of the tree
- Removing the Root of the heap is hard
- Removing the element at the “end” of the heap is easy
  - Copy last element of heap into root
  - Remove the last element
    - Push the root down, until heap property is satisfied

08-19: Heap Remove Smallest

- Running time for remove smallest?

08-20: Heap Remove Smallest

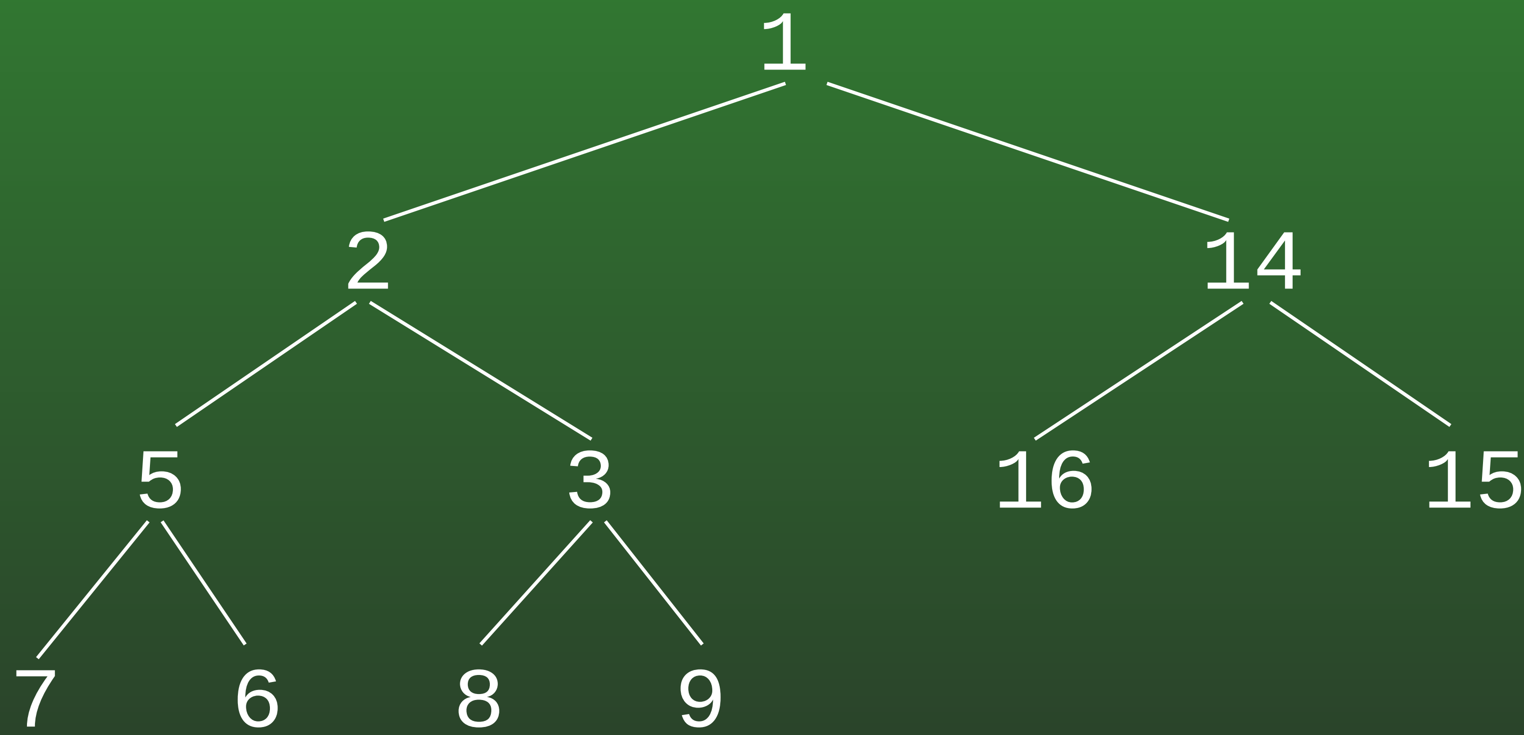
- Running time for remove smallest?
  - Copy last element into root, remove last element: $O(1)$, given a $O(1)$ time method to find the last element
  - Push the root down: $O(\text{height of the tree})$ (Worst case, push element all the way down)
    - As before, Complete Binary Tree with n elements has height $\Theta(\lg n)$
- Total time: $\Theta(\lg n)$ in the worst case

08-21: Representing Heaps

- Represent heaps using pointers, much like BSTs
  - Need to add parent pointers for insert to work correctly
  - Need to maintain a pointer to the location to insert the next element (this could be hard to update & maintain)
  - Space needed to store pointers – 3 per node – could be greater than the space need to store the data in the heap!
  - Memory allocation and deallocation is slow
- There is a better way!

08-22: Representing Heaps

A Complete Binary Tree can be stored in an array:



	1	2	14	5	3	16	15	7	6	8	9		
0	1	2	3	4	5	6	7	8	9	10	11	12	13

08-23: CBTs as Arrays

- The root is stored at index 1
- For the node stored at index i:
  - Left child is stored at index $2 * i$
  - Right child is stored at index $2 * i + 1$
  - Parent is stored at index $\lfloor i/2 \rfloor$

08-24: CBTs as Arrays

Finding the parent of a node

```
int parent(int n) {
  return (n  / 2);
}
```

Finding the left child of a node

```
int leftchild(int n) {
  return 2 * n;
}
```

Finding the right child of a node

```
int rightchild(int n) {
  return 2 * n + 1;
}
```

08-25: Building a Heap

Build a heap out of n elements

08-26: Building a Heap

Build a heap out of n elements

- Start with an empty heap
- Do n insertions into the heap

```
MinHeap H = new MinHeap();
for(i=0 < i<A.size(); i++)
   H.insert(A[i]);
```

Running time?

08-27: Building a Heap

Build a heap out of n elements

- Start with an empty heap
- Do n insertions into the heap

```
MinHeap H = new MinHeap();
for(i=0 < i<A.size(); i++)
   H.insert(A[i]);
```

Running time? $O(n \lg n)$ – is this bound tight?

08-28: Building a Heap

Total time: $c_1 + \sum_{i=1}^{n} c_2 \lg i$

$$
\begin{aligned}
c_1 + \sum_{i=1}^{n} c_2 \lg i &\geq \sum_{i=n/2}^{n} c_2 \lg i \\
&\geq \sum_{i=n/2}^{n} c_2 \lg(n/2) \\
&= (n/2) c_2 \lg(n/2) \\
&= (n/2) c_2 ((\lg n) - 1) \\
&\in \Omega(n \lg n)
\end{aligned}
$$

Running Time: $\Theta(n \lg n)$

08-29: Building a Heap

Build a heap from the bottom up

- Place elements into a heap array
- Each leaf is a legal heap
- First potential problem is at location $\lfloor i/2 \rfloor$

08-30: Building a Heap

Build a heap from the bottom up

- Place elements into a heap array
- Each leaf is a legal heap
- First potential problem is at location $\lfloor i/2 \rfloor$

```
for(i=n/2; i>=0; i--)
  pushdown(i);
```

08-31: Building a Heap

How many swaps, worst case? If every `pushdown` has to swap all the way to a leaf:

$n/4$ elements	1 swap
$n/8$ elements	2 swaps
$n/16$ elements	3 swaps
$n/32$ elements	4 swaps
$\ldots$	

Total # of swaps:

$$n/4 + 2n/8 + 3n/16 + 4n/32 + \ldots + (\lg n)n/n$$

Decreasing a Key

- Given a specific element in a heap, how can we decrease the key of that element, and maintain the heap property?
    - Examples

08-33: Decreasing a Key

- Given a specific element in a heap, how can we decrease the key of that element, and maintain the heap property?
  - Examples
- Push the element up the tree, just like after an insert
  - Examples

08-34: Decreasing a Key

- Decrease the key of a specific element in a heap:
  - Decrease the key value
  - Push the element up the tree, just like after an insert
- Time required?

08-35: Decreasing a Key

- Decrease the key of a specific element in a heap:
  - Decrease the key value
  - Push the element up the tree, just like after an insert
- Time required: $\Theta(\lg n)$, in the worst case.

08-36: Removing an Element

- Given a specific element in a heap, how can we remove that element, and maintain the heap property?
  - Examples

08-37: Removing an Element

- Given a specific element in a heap, how can we remove that element, and maintain the heap property?
    - Examples
- Decrease key to a value $<$ root
- Remove smallest element

08-38: Removing an Element

- Given a specific element in a heap, how can we remove that element, and maintain the heap property?
  - Examples
- Decrease key to a value $<$ root. Time $\Theta(\lg n)$ worst case
- Remove smallest element. Time $\Theta(\lg n)$ worst case

08-39: Java Specifics

- When inserting an element, push value up until it reaches the root, or it's $\geq$ its parent
  - Our while statement will have two tests
- We can insert a *sentinel* value at index 0, guaranteed to be $\leq$ any element in the heap
  - Now our while loop only requires a single test