The Method of Gaussian Reduction…

Carl Friedrich Gauss (1777-1855) devised a general method for obtaining the solutions for any system of linear equations. His method consists of making successive simplifications to the system, until eventually the equations themselves are simple enough that the solution to the system becomes apparent. His method is well-suited for use by modern computers.

Example 1

$$\begin{aligned} x + y + 2z &= 0 \\ 3x + y + 2z &= 2 \\ 2x - 2y + z &= -6 \end{aligned}$$

STEP 1: Subtract 3 times equation #1 from equation #2:

$$\begin{aligned} x + y + 2z &= 0 \\ -2y - 4z &= 2 \\ 2x - 2y + z &= -6 \end{aligned}$$

STEP 2: Subtract 2 times equation #1 from equation #3:

$$\begin{aligned} x + y + 2z &= 0 \\ -2y - 4z &= 2 \\ -4y - 3z &= -6 \end{aligned}$$

These two steps have simplified the system by "eliminating" the first variable x from all but the first equation. Next, we proceed to "eliminate" the second variable y from all but the second equation. It will be easier to accomplish this if we first multiply each term in the second equation by the fraction –1/2, so that y appears with a unit coefficient:

STEP 3: Multiply equation #2 by the (nonzero) constant -1/2:

$$\begin{aligned} x + y + 2z &= 0 \\ y + 2z &= -1 \\ -4y - 3z &= -6 \end{aligned}$$

STEP 4: Subtract 1 times equation #2 from equation #1:

$$\begin{aligned} x \qquad\qquad &= 1 \\ y + 2z &= -1 \\ -4y - 3z &= -6 \end{aligned}$$

STEP 5: Add 4 times equation #2 to equation #3:

$$\begin{aligned} x \qquad\qquad &= 1 \\ y + 2z &= -1 \\ 5z &= -10 \end{aligned}$$

At this stage, we have succeeded not only in "eliminating" the first variable x from all but the first equation, but also in "eliminating" the second variable y from all but the second equation. Now we turn our attention to the third variable z and proceed similarly.

STEP 6: Multiply equation #3 by the (nonzero) constant 1/5:

$$\begin{aligned} x \qquad\qquad &= 1 \\ y + 2z &= -1 \\ z &= -2 \end{aligned}$$

STEP 7: Subtract 2 times equation #3 from equation #2:

$$\begin{aligned} x \qquad\qquad &= 1 \\ y \qquad &= 3 \\ z &= -2 \end{aligned}$$

At this stage, since each variable has been "eliminated" from all except one equation, the system has become simple enough that it is possible to see directly what the solution is:

$$x = 1, \quad y = 3, \quad z = -2.$$

Thus, many tiny changes to the original system have produced in the end a system whose solution is quite obvious. We might well anticipate that this same approach could be used in other examples.

But sometimes things do not go quite so smoothly!

Example 2

$$\begin{aligned} 2x - 6y + 4z &= 14 \\ x - 3y + 5z &= 22 \\ 3x - 8y + 2z &= 3 \end{aligned}$$

STEP 1: Multiply equation #1 by the (nonzero) constant 1/2:

$$\begin{aligned} x - 3y + 2z &= 7 \\ x - 3y + 5z &= 22 \\ 3x - 8y + 2z &= 3 \end{aligned}$$

STEP 2: Add -1 times equation #1 to equation #2:

$$\begin{aligned} x - 3y + 2z &= 7 \\ 3z &= 15 \\ 3x - 8y + 2z &= 3 \end{aligned}$$

STEP 3: Add -3 times equation #1 to equation #3:

$$\begin{aligned} x - 3y + 2z &= 7 \\ 3z &= 15 \\ y - 4z &= -18 \end{aligned}$$

Here we have eliminated the first variable x from all but the first equation; but when we turn our attention to y we find that we cannot use the second equation to "eliminate" y from equations #1 and #3 because y doesn't occur in equation #2. In order to remedy this, we allow ourselves to "exchange" equation #2 with equation #3, in order that we can proceed past this annoying hurdle:

STEP 4: Exchange equation #2 with equation #3:

$$\begin{aligned} x - 3y + 2z &= 7 \\ y - 4z &= -18 \\ 3z &= 15 \end{aligned}$$

By swapping the positions of these two equations, we are now able to resume the reduction process and continue in the same manner as before.

STEP 5: Add 3 times equation #2 to equation #1:

$$\begin{aligned} x \qquad - 10z &= -47 \\ y - 4z &= -18 \\ 3z &= 15 \end{aligned}$$

STEP 6: Multiply equation #3 by the (nonzero) constant 1/3:

$$\begin{array}{rrrcr} x & & -\ 10z & = & -47 \\ & y & -\ \ 4z & = & -18 \\ & & z & = & 5 \end{array}$$

STEP 7: Add 10 times equation #3 to equation #1:

$$\begin{array}{rrrcr} x & & & = & 3 \\ & y & -\ \ 4z & = & -18 \\ & & z & = & 5 \end{array}$$

STEP 8: Add 4 times equation #3 to equation #2:

$$\begin{array}{rrrcr} x & & & = & 3 \\ & y & & = & 2 \\ & & z & = & 5 \end{array}$$

At this stage, the system of equations has become so simple that we can directly read off the solution:

$$x = 3, \quad y = 2, \quad z = 5.$$

Looking back over our work, we see that really only three kinds of operations were used:

1) Multiply every term in an equation by the same (nonzero) constant;

2) Add some multiple of one equation to another equation;

3) Exchange two of the equations.