Department of Computer Science **University of San Francisco**

Computer Science 411
Homework 1: Sets
Due Wednesday, Feb 1st 2012

1. Enumerate each of the following sets (that is, write our all elements of each of the following sets):

   (a) $\{x, y, \{x, y\}, \{\{x, y\}\}\} - \{x, y\}$

   (b) $2^{\{1,2,3\}} - 2^{\{1,2\}}$

   (c) $2^{\{a,b,c\}} \cap 2^{\{b,c,d\}}$

   (d) $\bigcup\{\{a\}, \{a, b\}, \bigcap\{\{c, d\}, \{d, e, f\}\}\}$

   (e) $2^{\{\}}$

   (f) $2^{2^{\{\}}}$

   (g) $\{a, b, c\} \times \{b\} \times \{a, c\}$

   (h) $2^{\{a,b\}} \times \{a, b\}$

   (i) $\{\} \times \{1\} \times \{1, 2\}$

2. Recall that $\Pi \subset 2^S$ is a partition of S if

   (a) $\{\} \notin \Pi$

   (b) $\forall(X, Y \in \Pi), X \neq Y \Rightarrow X \cap Y = \{\}$

   (c) $\bigcup \Pi = S$

   Show that all of these condiditions are necessary for Π to be a partition. That is, give a set S, and three sets P_1, P_2, P_3, such that each of the subsets P_i satisfies all but one of the above criteria.

3. Let $R = \{(a, d), (a, e), (b, a), (b, b), (d, d), (e, a), (e, c)\}$

   Give a graphical representation of each of the following:

   (a) R

   (b) R^{-1}

   (c) $R \bigcup R^{-1}$

   (d) $R \bigcap R^{-1}$

4. Prove or disprove the following statements. Some of them are trickier than you might think ...

   (a) For any relation R, $R \cup R^{-1}$ is symmetic.

(b) The transitive closure of a symmetic set is necessarily reflexive.

(c) For any two partial orders R_1 and R_2 over the same set A, $R_1 \cup R_2$ is also a partial order.

(d) For any two partial orders R_1 and R_2 over the same set A, $R_1 \cap R_2$ is also a partial order.

(*HINT:* You prove that a relation is a partial order by showing that it is reflexive, transitive, and anti-symmetric. It is easiest to disprove something by giving a counter-example)

5. Let A, B and C be finite sets, and let $f : A \mapsto B$, $g : B \mapsto C$, and $h : A \mapsto C$ be functions, such that h is the composition of f and g. Give necessary and sufficient conditions on f, g for:

   (a) h to be Onto

   (b) h to be One-to-one

   (c) h to be a bijection

6. Let A and B be two disjoint sets, with $|A| = n$ and $|B| = m$. Be sure to show your work on this question for full credit!

   (a) How many different relations $R \subseteq A \times B$ are there, in terms of n and m?

   (b) How many different functions $f : A \mapsto B$ are there, in terms of n and m?

   (c) How many different bijections $f : A \mapsto B$ are there, in terms of n and m?

   (d) How many different one-to-one functions $f : A \mapsto B$ are there, assuming that $n \leq m$?

   (*HINT:* Parts 6a to 6d above are ordered (roughly) from easiest to hardest.)

   (Graduate Only)

7. Let S be any set, and let P be the set of all partitions of S. Let R be the binary relation on P such that $(\Pi_1, \Pi_2) \in R$ if and only if, for every $S_1 \in \Pi_1$, there is an $S_2 \in \Pi_2$ such that $S_1 \subseteq S_2$.

   For example, given the set $S = \{a, b, c, d\}$, the ordered pairs $(\{\{a\}, \{b, c\}, \{d\}\}, \{\{a, b, c\}, \{d\}\})$ and $(\{\{a\}, \{b, c\}, \{d\}\}, \{\{a, d\}, \{b, c\}\})$ are in R.

   (a) Give a directed graph representation of R for the set $S = \{a, b, c\}$.

   (b) Show that R is a partial order on P (for the general case, not just example (a) above)

   (c) What elements of P are maximal and minimal (for the general case, not just example (a) above)

   (d) Suppose that P was an arbitrary collection of subsets of 2^S, instead of partitions of S. Would R still necessarily be a partial order?