Automata Theory

CS411-2015S-FR

Final Review

David Galles

Department of Computer Science
University of San Francisco

FR-0: Sets & Functions

- Sets
  - Membership:
    - $a \in ? \{a, b, c\}$
    - $a \in ? \{b, c\}$
    - $a \in ? \{b, \{a, b, c\}, d\}$
    - $\{a, b, c\} \in ? \{b, \{a, b, c\}, d\}$

FR-1: Sets & Functions

- Sets
  - Membership:
    - $a \in \{a, b, c\}$
    - $a \notin \{b, c\}$
    - $a \notin \{b, \{a, b, c\}, d\}$
    - $\{a, b, c\} \in \{b, \{a, b, c\}, d\}$

FR-2: Sets & Functions

- Sets
  - Subset:
    - $\{a\} \subseteq ? \{a, b, c\}$
    - $\{a\} \subseteq ? \{b, c, \{a\}\}$
    - $\{a, b\} \subseteq ? \{a, b, c, d\}$
    - $\{a, b\} \subseteq ? \{a, b\}$
    - $\{\} \subseteq \{a, b, c, d\}$

FR-3: Sets & Functions

- Sets
  - Subset:
    - $\{a\} \subseteq \{a, b, c\}$
    - $\{a\} \not\subseteq \{b, c, \{a\}\}$
    - $\{a, b\} \subseteq \{a, b, c, d\}$
    - $\{a, b\} \subseteq \{a, b\}$
    - $\{\} \subseteq \{a, b, c, d\}$

FR-4: Sets & Functions

- Sets
  - Cross Product:
    - $A \times B = \{(a, b) : a \in A, b \in B\}$
    - $\{a, b\} \times \{a, b\} =$
    - $\{a, b\} \times \{\{a, b\}\} =$

FR-5: Sets & Functions

- Sets
  - Cross Product:
    - $A \times B = \{(a, b) : a \in A, b \in B\}$
    - $\{a, b\} \times \{a, b\} = \{(a, a), (a, b), (b, a), (b, b)\}$
    - $\{a, b\} \times \{\{a, b\}\} = \{(a, \{a, b\}), (b, \{a, b\})\}$

FR-6: Sets & Functions

- Sets
  - Power Set:
    - $2^A = \{S : S \subseteq A\}$
    - $2^{\{a,b\}} =$
    - $2^{\{a\}} =$
    - $2^{2^{\{a\}}} =$

Sets & Functions

- Sets
  - Power Set:
    - $2^A = \{S : S \subseteq A\}$
    - $2^{\{a,b\}} = \{\{\}, \{a\}, \{b\}, \{a, b\}\}$
    - $2^{\{a\}} = \{\{\}, \{a\}\}$
    - $2^{2^{\{a\}}} = \{\{\}, \{\{\}\}, \{\{a\}\}, \{\{\}, \{a\}\}\}$

FR-8: Sets – Partition

Π is a $\mathbf{partition}$ of S if:

- $\Pi \subset 2^S$
- $\{\} \notin \Pi$
- $\forall (X, Y \in \Pi), X \neq Y \implies X \cap Y = \{\}$
- $\bigcup \Pi = S$

{{a, c}, {b, d, e}, {f}} is a partition of {a,b,c,d,e,f}
{{a, b, c, d, e, f}} is a partition of {a,b,c,d,e,f}
{{a, b, c}, {d, e, f}} is a partition of {a,b,c,d,e,f}

FR-9: Sets – Partition

In other words, a $\mathbf{partition}$ of a set S is just a division of the elements of S into 1 or more groups.

- All the partitions of the set {a, b, c}?

FR-10: Sets – Partition

In other words, a $\mathbf{partition}$ of a set S is just a division of the elements of S into 1 or more groups.

- All the partitions of the set {a, b, c}?
  - {{a, b, c}}, {{a, b}, {c}}, {{a, c}, {b}}, {{a},{b, c}}, {{a}, {b}, {c}}

FR-11: Sets & Functions

- Relation
  - A relation R is a set of ordered pairs
  - That's all that a relation is
  - Relation Graphs

FR-12: Sets & Functions

- Properties of Relations
  - Reflexive
  - Symmetric
  - Transitive
  - Antisymmetric
- Equivalence Relation: Reflexive, Symmetric, Transitive
- Partial Order: Reflexive, Antisymmetric, Transitive
- Total Order: Partial order, for each $a, a' \in A$, either $(a, a') \in R$ or $(a', a) \in R$

FR-13: Sets & Functions

- What does a graph of an Equivalence relation look like?
- What does a graph of a Total Order look like
- What does a graph of a Partial Order look like?

FR-14: Closure

- A set $A \subseteq B$ is closed under a relation $R \subseteq ((B \times B) \times B)$ if:
  - $a_1, a_2 \in A \wedge ((a_1, a_2), c) \in R \implies c \in A$
  - That is, if a_1 and a_2 are both in A, and $((a_1, a_2), c)$ is in the relation, then c is also in A
- $\mathbb{N}$ is closed under addtion
- $\mathbb{N}$ is not closed under subtraction or division

FR-15: Closure

- Relations are also sets (of ordered pairs)
- We can talk about a relation R being closed over another relation R'
  - Each element of R' is an ordered triple of ordered pairs!

FR-16: Closure

- Relations are also sets (of ordered pairs)
- We can talk about a relation R being closed over another relation R'
  - Each element of R' is an ordered triple of ordered pairs!
- Example:
  - $R \subseteq A \times A$
  - $R' = \{(((a, b), (b, c)), (a, c)) : a, b, c \in A\}$
  - If R is closed under R', then . . .

FR-17: Closure

- Relations are also sets (of ordered pairs)
- We can talk about a relation R being closed over another relation R'
  - Each element of R' is an ordered triple of ordered pairs!
- Example:
  - $R \subseteq A \times A$
  - $R' = \{(((a, b), (b, c)), (a, c)) : a, b, c \in A\}$
  - If R is closed under R', then R is transitive!

FR-18: Closure

- **Reflexive closure** of a relation $R \subseteq A \times A$ is the smallest possible superset of R which is reflexive
  - Add self-loop to every node in relation
  - Add (a,a) to R for every $a \in A$
- **Transitive Closure** of a relation $R \subseteq A \times A$ is the smallest possible superset of R which is transitive
  - Add direct link for every path of length 2.
  - $\forall(a, b, c \in A)$ if $(a, b) \in R \wedge (b, c) \in R$ add (a, c) to R.

(examples on board)

FR-19: Sets & Functions

- Functions
  - Relation R over $A \times B$
  - For each $a \in A$:
    - Exactly one element $(x, y) \in R$ with $x = a$

FR-20: Sets & Functions

- For a function f over $(A \times A)$, what does the graph look like?
- For a function f over $(A \times B)$, what does the graph look like?

FR-21: Sets & Functions

- Functions
  - one-to-one: $f(a) \neq f(a')$ when $a \neq a'$ (nothing is mapped to twice)
  - onto: for each $b \in B$, $\exists a$ such that $f(a) = b$ (everything is mapped to)
  - bijection: Both one-to-one and onto

Sets & Functions

- For a function f over $(A \times B)$
  - What does the graph look like for a one-to-one function?
  - What does the graph look like for an onto function?
  - What does the graph look like for a bijection?

FR-23: Sets & Functions

- Infinite sets
  - Countable, Countably infinite
    - Bijection with the Natural Numbers
  - Uncountable, uncountable infinite
    - Infinite
    - No bijection with the Natural Numbers

FR-24: Infinite Sets

- We can show that a set is countable infinite by giving a bjiection between that set an the natural numbers
- Same thing as as imposing an ordering on an infinite set

FR-25: Countable Sets

- A set is **countable infinite** (or just **countable**) if it is equinumerous with $\mathbb{N}$.
  - Even elements of $\mathbb{N}$?

FR-26: Countable Sets

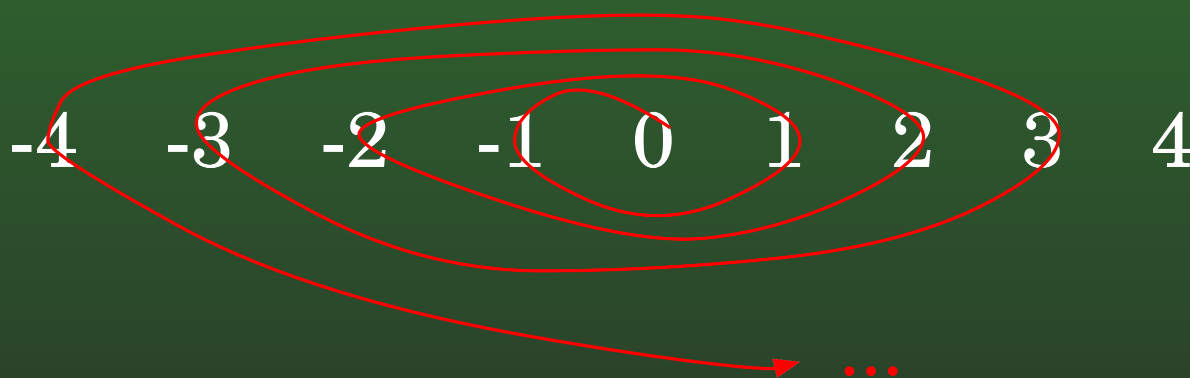
- A set is **countable infinite** (or just **countable**) if it is equinumerous with $\mathbb{N}$.
  - Even elements of $\mathbb{N}$?
  - $f(x) = 2x$

FR-27: Countable Sets

- A set is $\mathbf{countable\ infinite}$ (or just $\mathbf{countable}$) if it is equinumerous with $\mathbb{N}$.
  - Integers ($\mathbb{Z}$)?

FR-28: Countable Sets

- A set is **countable infinite** (or just **countable**) if it is equinumerous with $\mathbf{N}$.
  - Integers ($\mathbf{Z}$)?
  - $f(x) = \lceil \frac{x}{2} \rceil * (-1)^x$

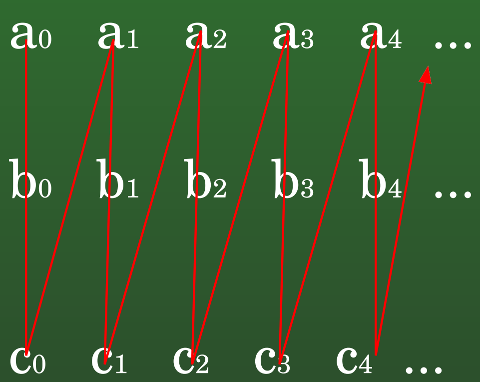


Countable Sets

- A set is **countable infinite** (or just **countable**) if it is equinumerous with $\mathbb{N}$.
  - Union of 3 (disjoint) countable sets A, B, C?

FR-30: Countable Sets

- A set is **countable infinite** (or just **countable**) if it is equinumerous with $\mathbb{N}$.
  - Union of 3 (disjoint) countable sets A, B, C?



- $$f(x) = \begin{cases} a_{\frac{x}{3}} & \text{if x mod 3 = 0} \\ b_{\frac{x-1}{3}} & \text{if x mod 3 = 1} \\ c_{\frac{x-2}{3}} & \text{if x mod 3 = 2} \end{cases}$$

FR-31: Countable Sets

- A set is **countable infinite** (or just **countable**) if it is equinumerous with $\mathbb{N}$.
  - $\mathbb{N} \times \mathbb{N}$?

$$\begin{array}{cccccc}
(0,0) & (0,1) & (0,2) & (0,3) & (0,4) & \ldots \\
(1,0) & (1,1) & (1,2) & (1,3) & (1,4) & \ldots \\
(2,0) & (2,1) & (2,2) & (2,3) & (2,4) & \ldots \\
(3,0) & (3,1) & (3,2) & (3,3) & (3,4) & \ldots \\
(4,0) & (4,1) & (4,2) & (4,3) & (4,4) & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

FR-32: Countable Sets

- A set is **countable infinite** (or just **countable**) if it is equinumerous with $\mathbf{N}$.
  - $\mathbf{N} \times \mathbf{N}$?

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	...
(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	...
(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	...
(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	...
(4,0)	(4,1)	(4,2)	(4,3)	(4,4)	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

  - $f((x, y)) = \frac{(x+y)*(x+y+1)}{2} + x$

Countable Sets

- A set is **countable infinite** (or just **countable**) if it is equinumerous with $\mathbb{N}$.
  - Real numbers between 0 and 1 (exclusive)?

FR-34: Uncountable R

- Proof by contradiction
  - Assume that R between 0 and 1 (exclusive) is countable
    - (that is, assume that there is some bijection from $\mathbf{N}$ to $\mathbf{R}$ between 0 and 1)
  - Show that this leads to a contradiction
    - Find some element of $\mathbf{R}$ between 0 and 1 that is not mapped to by any element in $\mathbf{N}$

FR-35: Uncountable R

- Assume that there is some bijection from $\mathbf{N}$ to $\mathbf{R}$ between 0 and 1

0	0.3412315569...
1	0.0123506541...
2	0.1143216751...
3	0.2839143215...
4	0.2311459412...
5	0.8381441234...
6	0.7415296413...
$\vdots$	$\vdots$

FR-36: Uncountable R

- Assume that there is some bijection from $\mathbf{N}$ to $\mathbf{R}$ between 0 and 1

0	0.3412315569...
1	0.0123506541...
2	0.1143216751...
3	0.2839143215...
4	0.2311459412...
5	0.8381441234...
6	0.7415296413...
$\vdots$	$\vdots$ $\ddots$

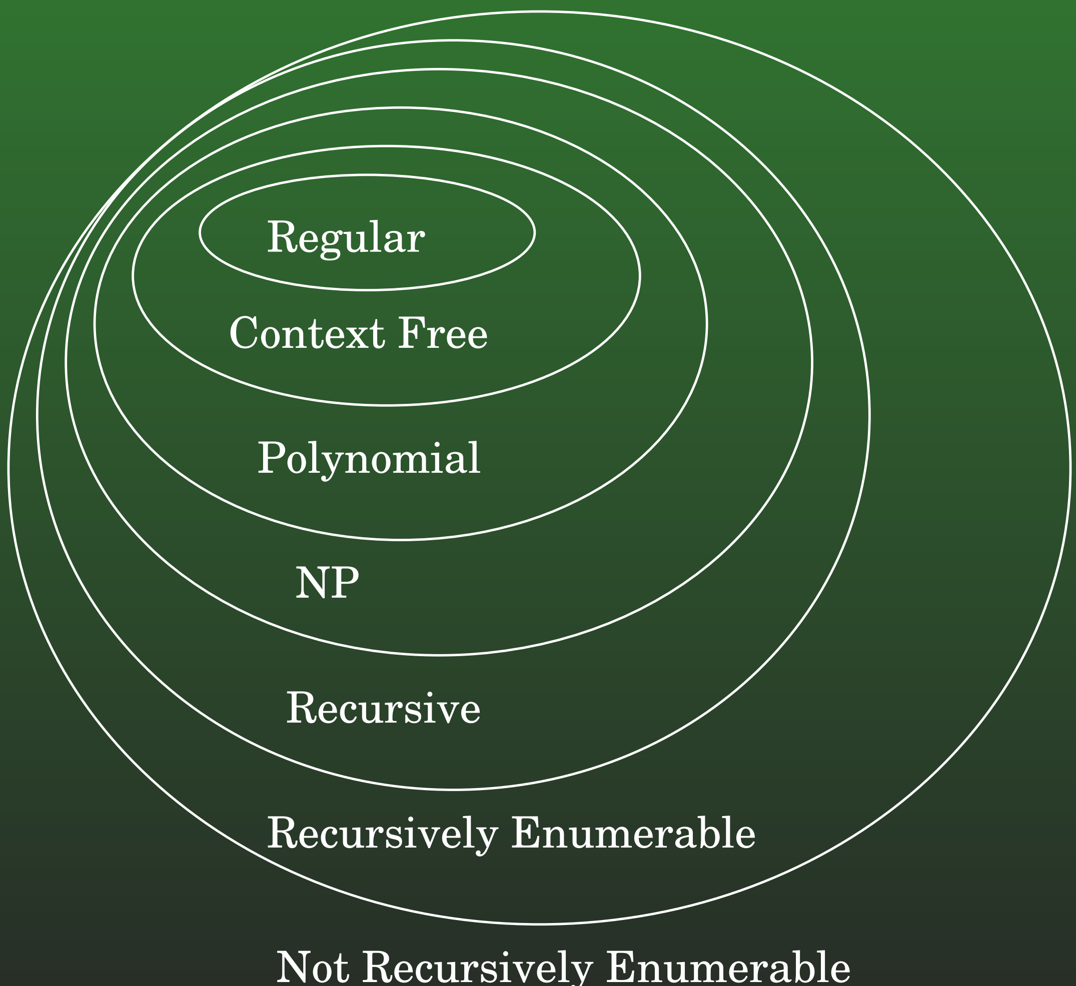
Consider: 0.425055...

FR-37: Formal Languages

- Alphabet Σ: Set of symbols
  - $\{0, 1\}, \{a, b, c\}$, etc
- String w: Sequence of symbols
  - $cat, dog, firehouse$ etc
- Language L: Set of strings
  - {cat, dog, firehouse}, {a, aa, aaa, $\ldots$}, etc
- Language class: Set of Languages
  - Regular languages, $\mathbf{P}$, $\mathbf{NP}$, etc.

FR-38: Formal Languages

- Language Hierarchy.



FR-39: # Regular Expressions

- Regular expressions are a way to describe formal languages
- Regular expressions are defined recursively
  - Base case – simple regular expressions
  - Recursive case – how to build more complex regular expressions from simple regular expressions

FR-40: Regular Expressions

- ϵ is a regular expression, representing $\{\epsilon\}$
- $\emptyset$ is a regular expression, representing {}
- $\forall$a $\in \Sigma$, a is a regular expression representing {a}
- if r_1 and r_2 are regular expressions, then $(r_1 r_2)$ is a regular expression
  - $L[(r_1 r_2)] = L[r_1] \circ L[r_2]$
- if r_1 and r_2 are regular expressions, then $(r_1 + r_2)$ is a regular expression
  - $L[(r_1 + r_2)] = L[r_1] \cup L[r_2]$
- if r is regular expressions, then (r^*) is a regular expression
  - $L[(r^*)] = (L[r])^*$

FR-41: r.e. Precedence

From highest to Lowest:

Kleene Closure *

Concatenation

Alternation +

ab*c+e = (a(b*)c) + e

(We will still need parentheses for some regular expres-
sions: (a+b)(a+b))

FR-42: Regular Expressions

- Intuitive Reading of Regular Expressions
  - Concatenation == “is followed by”
  - + == “or”
  - * == “zero or more occurances”
- (a+b)(a+b)(a+b)
- (a+b)*
- aab(aa)*

FR-43: Regular Languages

- A language L is regular if there exists a regular expression which generates it
- Give a regular expression for:
  - All strings over $\{a, b\}$ that have an odd # of a's

FR-44: Regular Languages

- A language L is regular if there exists a regular expression which generates it
- Give a regular expression for:
  - All strings over $\{a, b\}$ that have an odd # of a's
    $b^*a(b^*ab^*a)^*b^*$
  - All strings over $\{a, b\}$ that contain exactly two occurrences of bb (bbb counts as 2 occurrences!)

FR-45: Regular Languages

- A language L is regular if there exists a regular expression which generates it
- Give a regular expression for:
  - All strings over $\{a, b\}$ that have an odd # of a's
    $b^*a(b^*ab^*a)^*b^*$
  - All strings over $\{a, b\}$ that contain exactly two occurrences of bb (bbb counts as 2 occurrences!)
    $a^*(baa^*)^*bb(aa^*b)^*aa^*bb(aa^*b)^*a^* +$
    $a^*(baa^*)^*bbb(aa^*b)^*a^*$

FR-46: Regular Languages

- All strings over {0, 1} that begin (or end) with 11
- All strings over {0, 1} that begin (or end) with 11, but not both

FR-47: Regular Languages

- All strings over {0, 1} that begin (or end) with 11
  - 11 (0+1)* 11 + 11
- All strings over {0, 1} that begin (or end) with 11, but not both
  - 11(0+1)*0 + 11(0+1)*01 + 0(0+1)*11 + 10(0+1*)11

FR-48: Regular Languages

- Shortest string not described by following regular expressions?
  - a*b*a*b*
  - a*(ab)*(ba)*b*a*
  - a*b*(ab)*b*a*

FR-49: Regular Languages

- Shortest string not described by following regular expressions?
  - a*b*a*b*
    - baba
  - a*(ab)*(ba)*b*a*
    - baab
  - a*b*(ab)*b*a*
    - baab

FR-50: Regular Languages

- English descriptions of following regular expressions:
  - (aa+aaa)*
  - b(a+b)*b + a(a+b)*a + a + b
  - a*(baa*)*bb(aa*b)*a*

 # Regular Languages

- A language L is regular if there exists a DFA which accepts it
  - DFA for all strings with exactly 2 occurrences of bb

DFA Definition

- A DFA is a 5-tuple $M = (K, \Sigma, \delta, s, F)$
  - K Set of states
  - Σ Alphabet
  - $\delta : (K \times \Sigma) \mapsto K$ is a Transition function
  - $s \in K$ Initial state
  - $F \subseteq K$ Final states

 # Regular Languages

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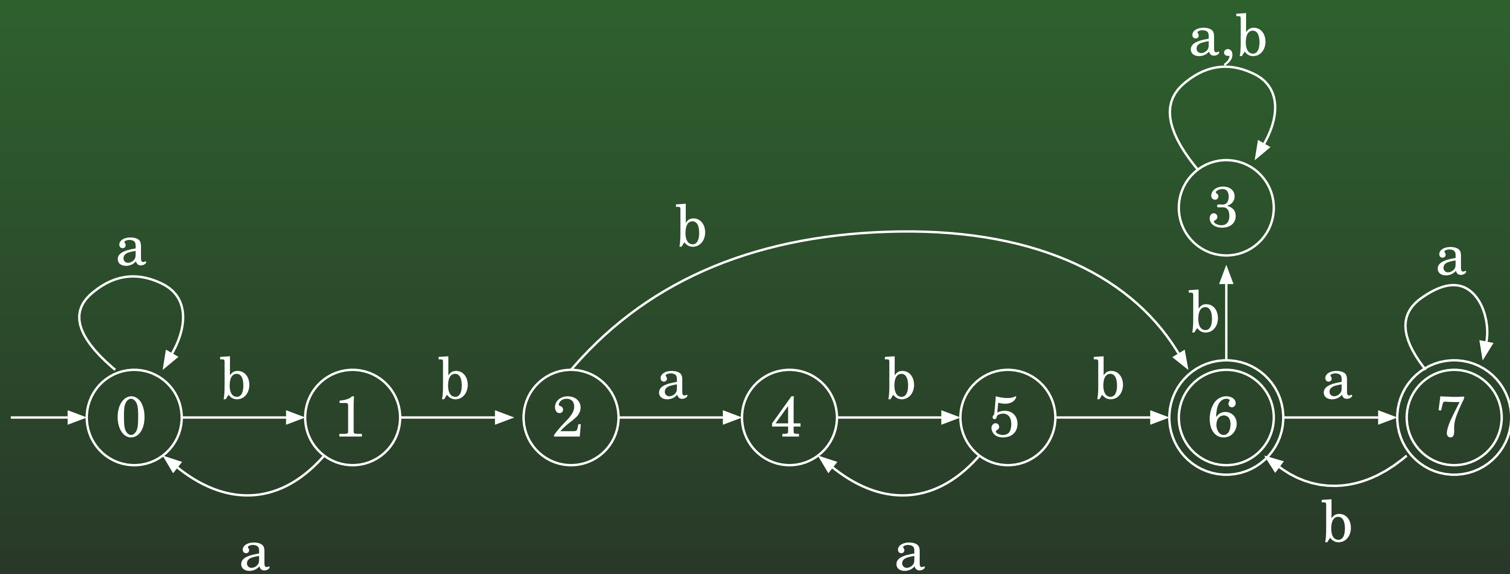

Regular Languages

- A language L is regular if there exists a DFA which accepts it
  - DFA for all strings over {0,1} that start and end with 111

Regular Languages

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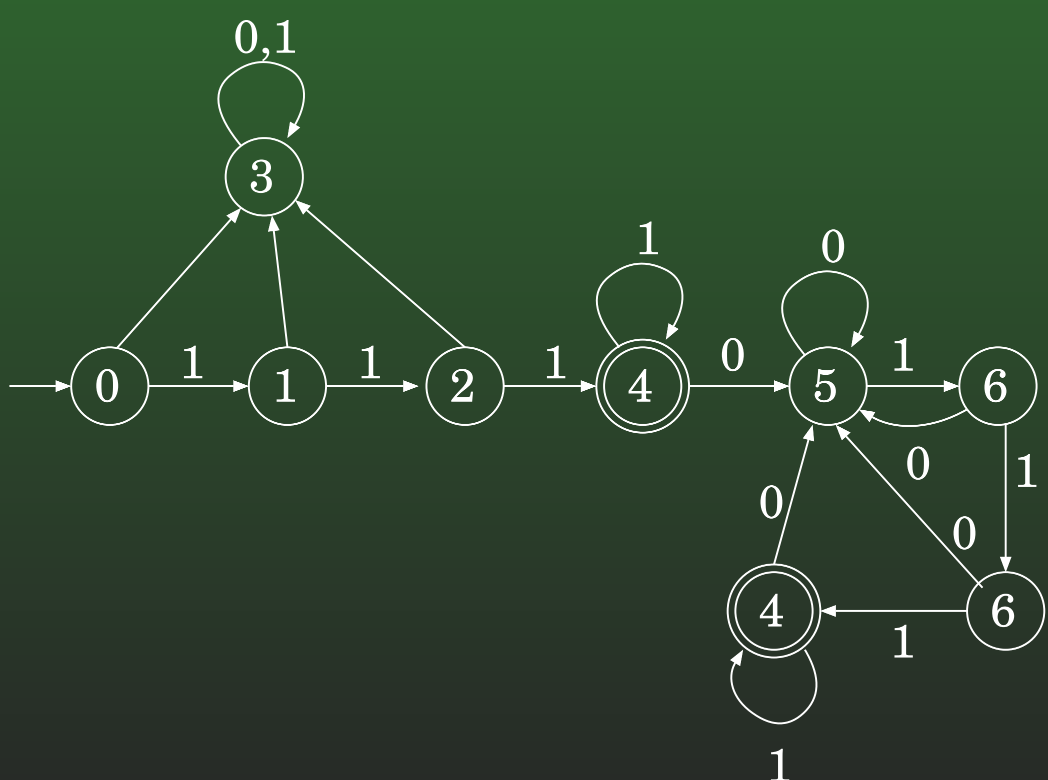
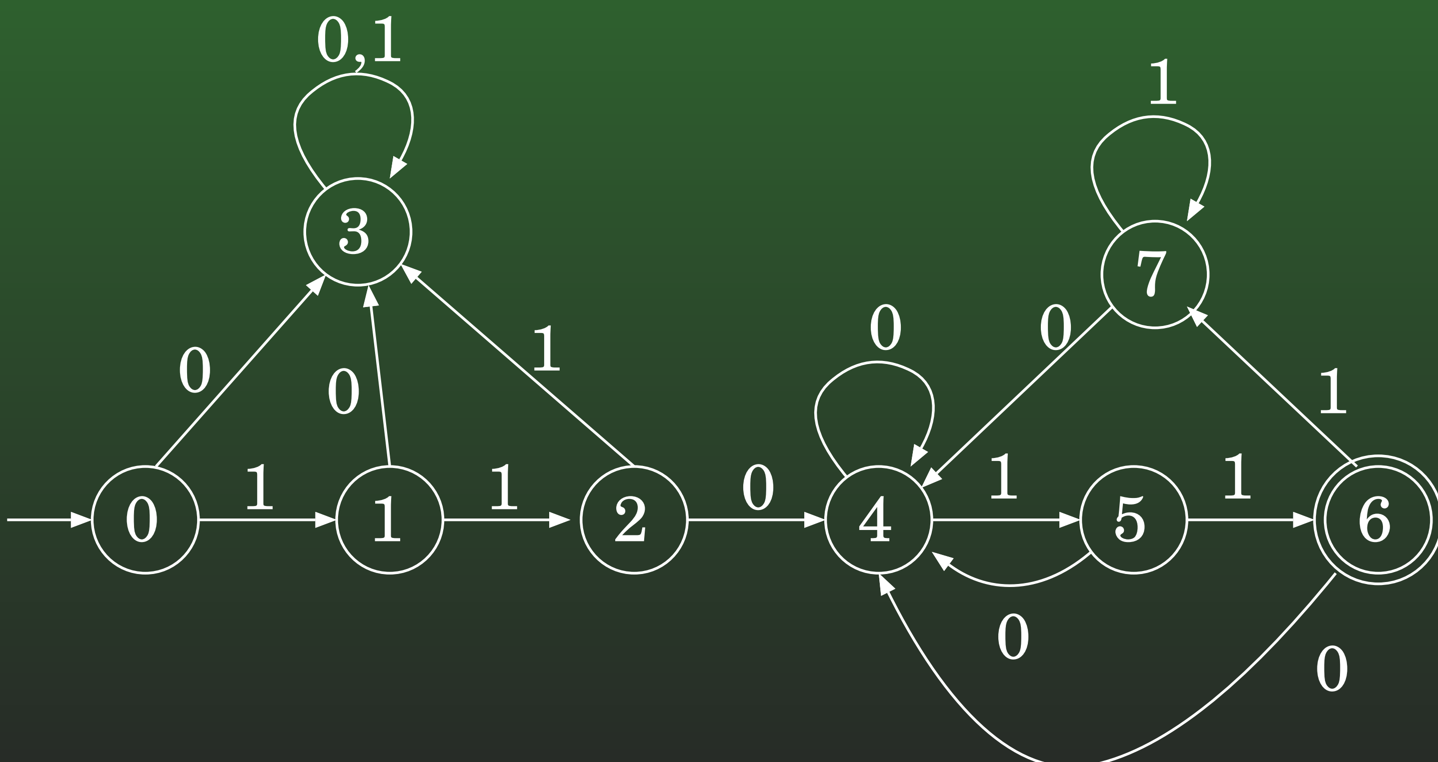


FR-56: Regular Languages

- A language L is regular if there exists a DFA which accepts it
  - DFA for all strings over {0,1} that start with 110, end with 011

FR-57: Regular Languages

- A language L is regular if there exists a DFA which accepts it
  - DFA for all strings over {0,1} that start with 110, end with 011



Regular Languages

- Give a DFA for all strings over {0,1} that begin or end with 11
- Give a DFA for all strings over {0,1} that begin or end with 11 (but not both)

FR-59: Regular Languages

- Give a DFA for all strings over {0,1} that contain 101010
- Give a DFA for all strings over {0,1} that contain 101 or 010
- Give a DFA for all strings over {0,1} that contain 010 and 101

FR-60: DFA Configuration & $\vdash_M$

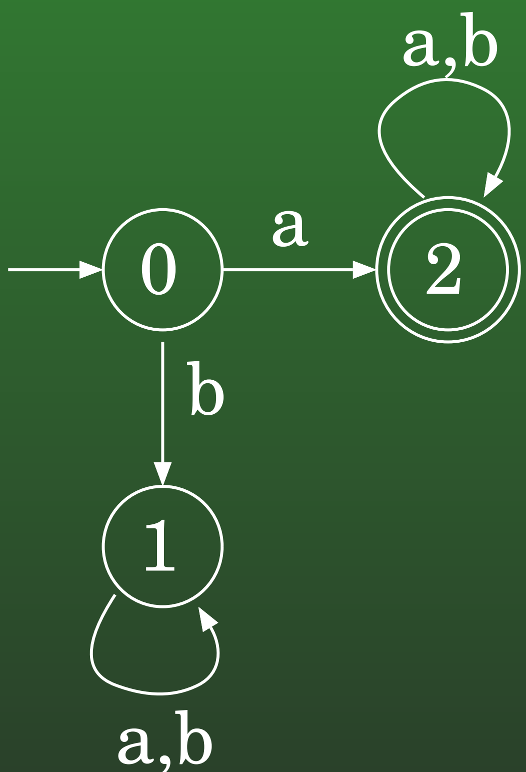
- Way to describe the computation of a DFA
- **Configuration**: What state the DFA is currently in, and what string is left to process
  - $\in K \times \Sigma^*$
  - $(q_2, abba)$ Machine is in state q_2, has $abba$ left to process
  - (q_8, bba) Machine is in state q_8, has bba left to process
  - (q_4, ϵ) Machine is in state q_4 at the end of the computation (accept iff $q_4 \in F$)

FR-61: DFA Configuration & $\vdash_M$

- Way to describe the computation of a DFA
- **Configuration**: What state the DFA is currently in, and what string is left to process
  - $\in K \times \Sigma^*$
- Binary relation $\vdash_M$: What machine M yields in one step
  - $\vdash_M \subseteq (K \times \Sigma^*) \times (K \times \Sigma^*)$
  - $\vdash_M = \{((q_1, aw), (q_2, w)) : q_1, q_2 \in K_M, w \in \Sigma_M^*, a \in \Sigma_M, ((q_1, a), q_2) \in \delta_M\}$

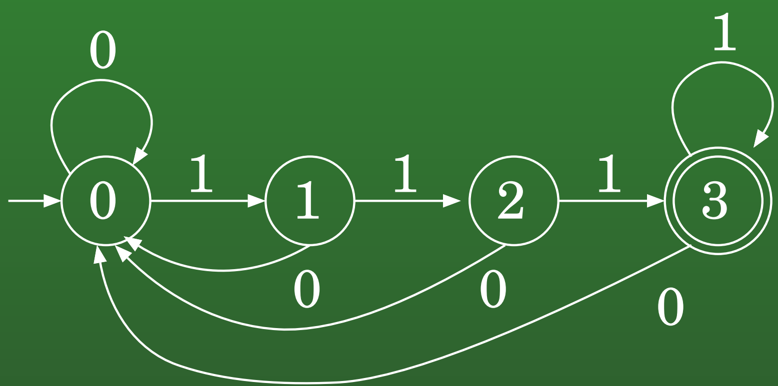
FR-62: DFA Configuration & $\vdash_M$

Given the following machine M:



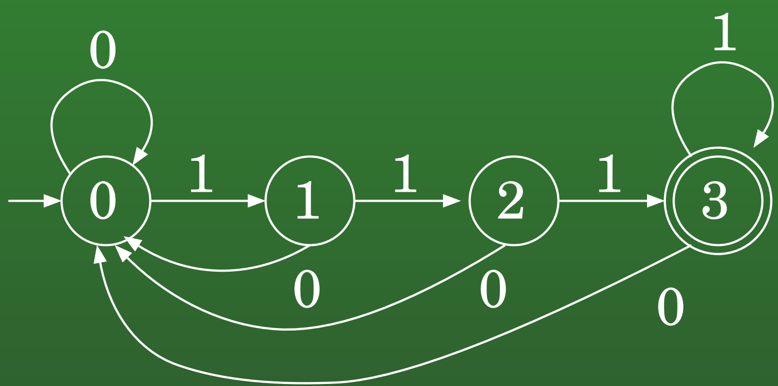
- $((q_0, \text{abba}), (q_2, \text{bba})) \in \vdash_M$
  - can also be written $(q_0, \text{abba}) \vdash_M (q_2, \text{bba})$

FR-63: DFA Configuration & $\vdash_M$



$$
\begin{aligned}
(q_0, 11101) & \vdash_M (q_1, 1101) \\
& \vdash_M (q_2, 101) \\
& \vdash_M (q_3, 01) \\
& \vdash_M (q_0, 1) \\
& \vdash_M (q_1, \epsilon)
\end{aligned}
$$

FR-64: DFA Configuration & $\vdash_M$



$$
\begin{aligned}
(q_0, 10111) &\vdash_M (q_1, 0111) \\
&\vdash_M (q_0, 111) \\
&\vdash_M (q_1, 11) \\
&\vdash_M (q_2, 1) \\
&\vdash_M (q_3, \epsilon)
\end{aligned}
$$

FR-65: DFA Configuration & $\vdash_M^*$

- $\vdash_M^*$ is the reflexive, transitive closure of $\vdash_M$
  - Smallest superset of $\vdash_M$ that is both reflexive and transitive
  - "yields in 0 or more steps"
- Machine M accepts string w if:
  $(s_M, w) \vdash_M^* (f, \epsilon)$ for some $f \in F_M$

FR-66: DFA & Languages

- Language accepted by a machine $M = L[M]$
  - $\{w : (s_M, w) \vdash_M^* (f, \epsilon)$ for some $f \in F_M\}$
- DFA Languages, L_{DFA}
  - Set of all languages that can be defined by a DFA
  - $L_{DFA} = \{L : \exists M, L[M] = L\}$
- To think about: How does $L_{DFA} = L_{REG}$

FR-67: NFA Definition

- Difference between a DFA and an NFA
  - DFA has exactly only transition for each state/symbol pair
    - Transition function: $\delta : (K \times \Sigma) \mapsto K$
  - NFA has 0, 1 or more transitions for each state/symbol pair
    - Transition relation: $\Delta \subseteq ((K \times \Sigma) \times K)$

FR-68: NFA Definition

- A NFA is a 5-tuple $M = (K, \Sigma, \Delta, s, F)$
  - K Set of states
  - Σ Alphabet
  - $\Delta : (K \times \Sigma) \times K$ is a Transition relation
  - $s \in K$ Initial state
  - $F \subseteq K$ Final states

Fun with NFA

Create an NFA for:

- All strings over {a, b} that start with a and end with b

(also create a DFA, and regular expression)

FR-70: Fun with NFA

Create an NFA for:

- All strings over {a, b} that contain 010 or 101

FR-71: Regular Languages

- A language L is regular if there exists an NFA which accepts it
  - NFA for all strings over $\{a, b\}$ that contain $abba$

FR-72: Regular Languages

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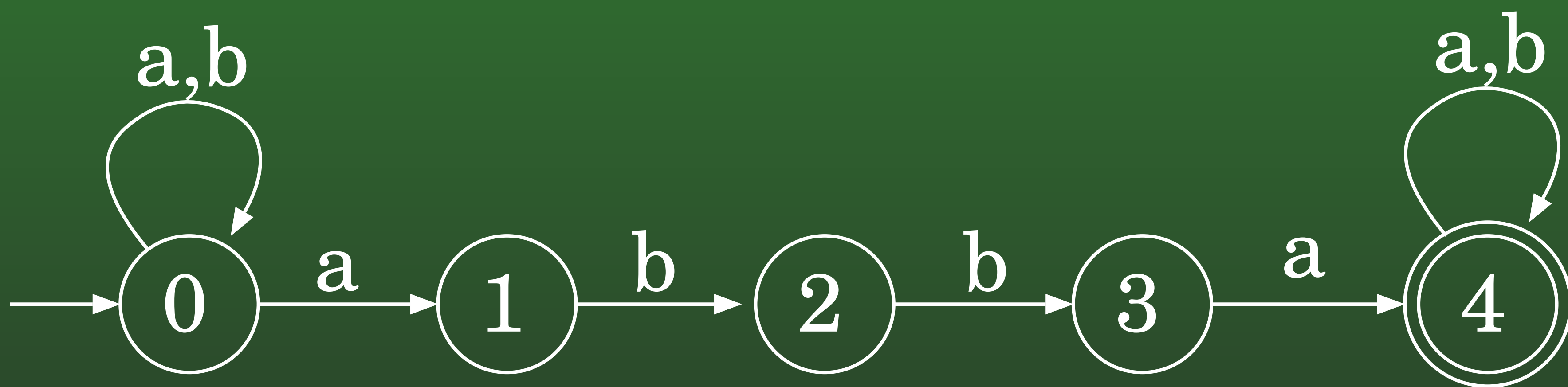
  - NFA for all strings over $\{a, b\}$ that contain $abba$

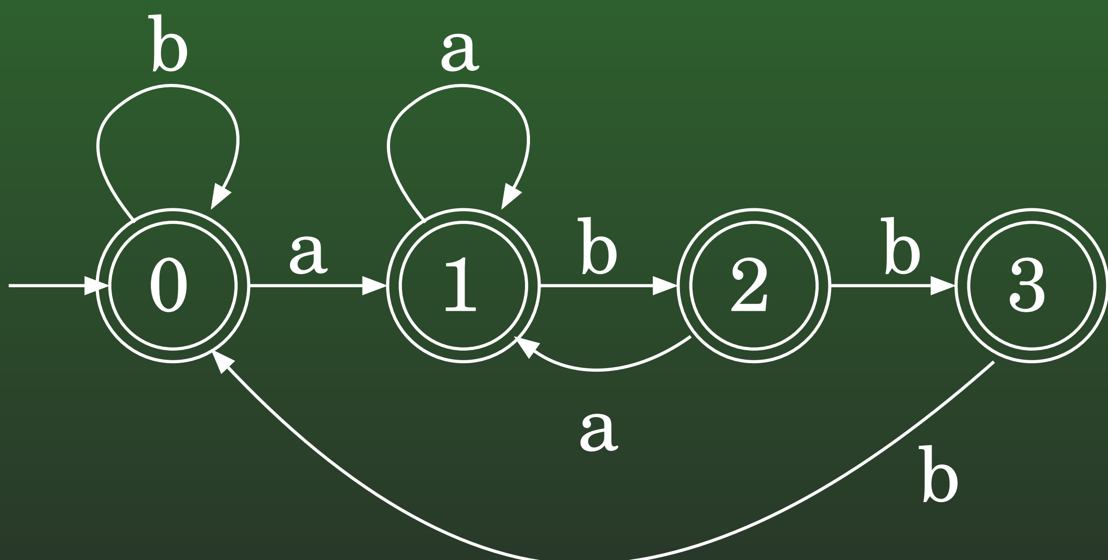


FR-73: Regular Languages

- A language L is regular if there exists an NFA which accepts it
  - NFA for all strings over $\{a, b\}$ that do not contain $abba$

FR-74: Regular Languages

- A language L is regular if there exists an NFA which accepts it
  - NFA for all strings over $\{a, b\}$ that do not contain $abba$



FR-75: Regular Expression & NFA

- Give a regular expression for all strings over {a,b} that have an even number of a's, and a number of b's divisible by 3

FR-76: Pumping Lemma

- Not all languages are Regular
- L = all strings over $\{a, b, c\}$ that contain more a's than b's and c's combined

FR-77: Pumping Lemma

- To show that a language L is not regular, using the pumping lemma:
  - Let n be the constant of the pumping lemma
  - Create a string $w \in L$, such that $|w| > n$
  - For each way of breaking $w = xyz$ such that $|xy| \leq n$, $|y| > 0$:
    - Show that there is some i such that $xy^iz \notin L$
  - By the pumping lemma, L is not regular

Pumping Lemma

- Prove L = all strings over $\{a, b, c\}$ that contain more a's than b's and c's combined is not regular
- Let n be the constant of the pumping lemma
- Consider $w = b^n a^{n+1} \in L$
- If we break $w = xyz$ such that $|xy| \leq n$, then y must be all b's. Let $|y| = k$
- Consider $w' = xy^2x = b^{n+k}a^n$. $w' \notin L$ for any $k > 0$, thus by the pumping lemma, L is not regular

FR-79: Context-Free Languages

- A language is context-free if a CFG generates it
  - All strings over $\{a, b, c\}$ with same # of a's as b's

FR-80: Context-Free Languages

- A language is context-free if a CFG generates it
  - All strings over $\{a, b, c\}$ with same # of a's as b's

$$
\begin{aligned}
S &\rightarrow aSb \\
S &\rightarrow bSa \\
S &\rightarrow SS \\
S &\rightarrow cS \\
S &\rightarrow Sc \\
S &\rightarrow \epsilon
\end{aligned}
$$

FR-81: Context-Free Languages

- A language is context-free if a CFG generates it
  - All strings over $\{a, b, c\}$ with more a's than b's

FR-82: Context-Free Languages

- A language is context-free if a CFG generates it
  - All strings over $\{a, b, c\}$ with more a's than b's

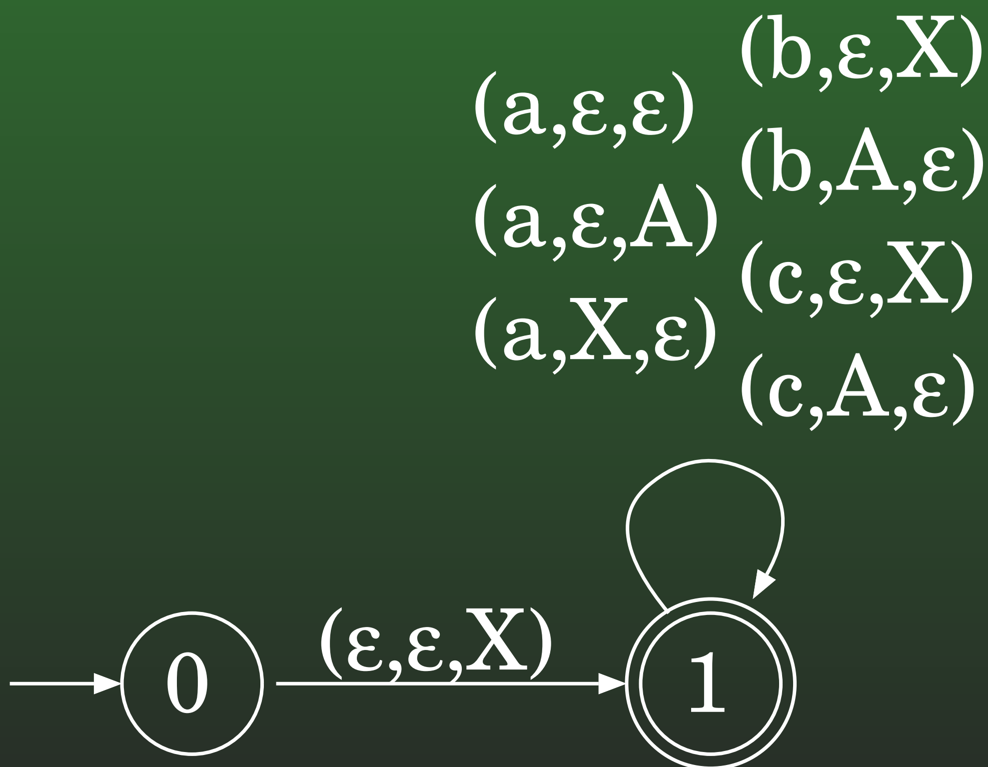
$$
\begin{aligned}
S &\rightarrow cS|Sc \\
S &\rightarrow aSb|bSa \\
S &\rightarrow aA|Aa \\
S &\rightarrow SA \\
A &\rightarrow aAb \\
A &\rightarrow bAa \\
A &\rightarrow AA \\
A &\rightarrow cA|Ac \\
A &\rightarrow aA|Aa \\
A &\rightarrow \epsilon
\end{aligned}
$$

FR-83: Context-Free Languages

- A language is context-free if a PDA accepts it
  - All strings over $\{a, b, c\}$ that contain more a's than b's and c's combined

Context-Free Languages

- A language is context-free if a PDA accepts it
  - All strings over $\{a, b, c\}$ that contain more a's than b's and c's combined



FR-85: Recursive Languages

- A language L is recursive if an always-halting Turing Machine accepts it
  - In other words, a Turing Machine decides L
- Create a Turing Machine for all strings over $\{a, b, c\}$ with an equal number of a's, b's and c's.

FR-86: Recursive Languages

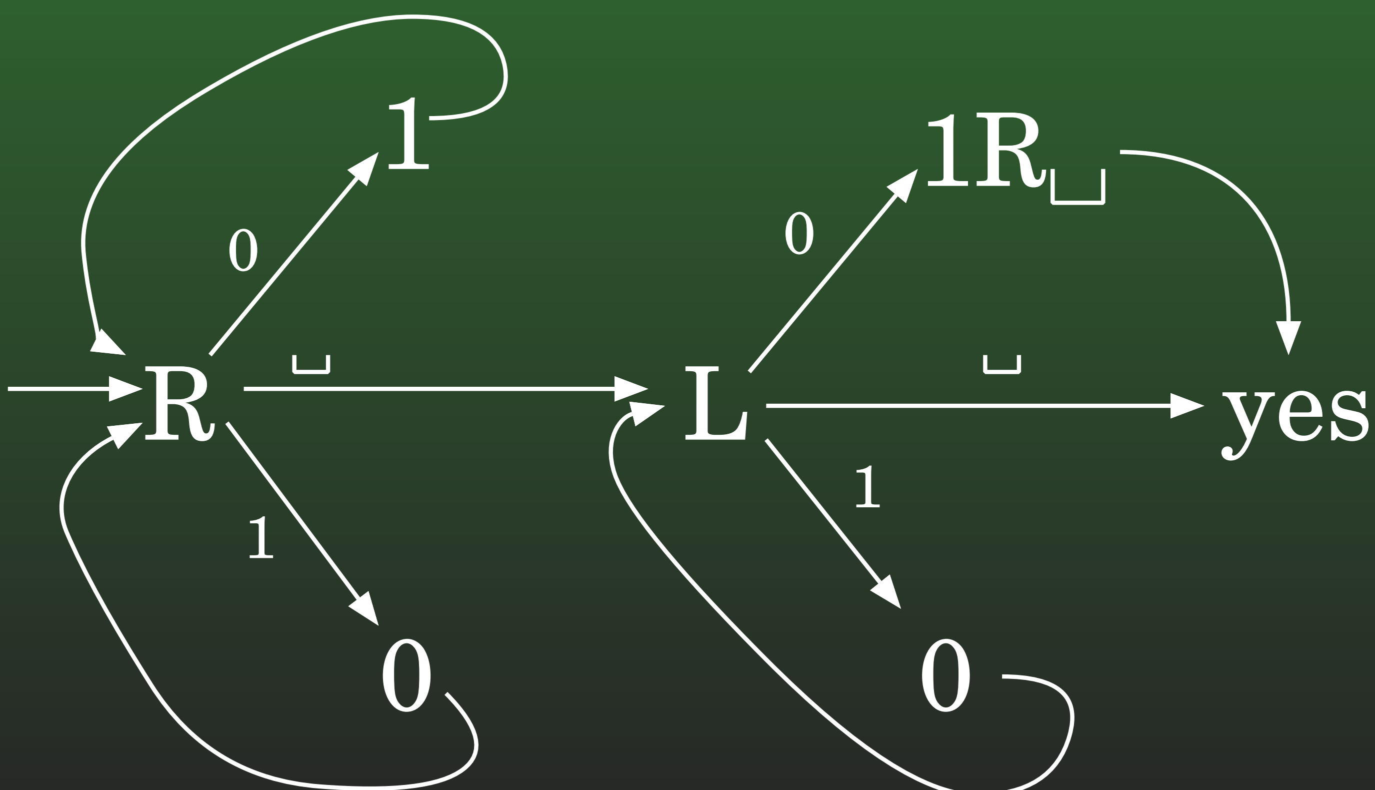
- Computing functions with TMs
  - Give a TM that computes negation, for a 2's complement binary number
  - (flip bits, add one, discard overflow)

FR-87: Recursive Languages

- Computing functions with TMs
  - Give a TM that computes negation, for a 2's complement binary number

FR-88: Recursive Languages

- Computing functions with TMs
  - Give a TM that computes negation, for a 2's complement binary number
  - (flip bits, add one, discard overflow)



FR-89: r.e. Languages

- A language L is recursively enumerable if there is some Turing Machine M that halts and accepts everything in L, and runs forever on everything not in L
- Give a TM that semi-decides $L = a^n b^n$
  - Note that this language is also context-free – context-free languages are a subset of the r.e. languages

 # r.e. Languages

- Enumeration Machines
  - Create a Turing Machine that enumerate the language:
    L = all strings of the form wcw, $w \in (a+b)^*$

FR-91: Counter Machines

- Finite automata with a counter (never negative)
- Add one, subtract 1, check for zero
- Create a 1-counter machine for all strings over {a,b} that contain the same number of a's as b's

FR-92: Unrestricted Grammars

$G = (V, \Sigma, R, S)$

- V = Set of symbols, both terminals & non-terminals
- $\Sigma \subset V$ set of terminals (alphabet for the language being described)
- $R \subset (V^*(V - \Sigma)V^* \times V^*)$ Set of rules
- $S \in (V - \Sigma)$ Start symbol

FR-93: Unrestricted Grammars

- $R \subset (V^*(V - \Sigma)V^* \times V^*)$ Set of rules
- In an Unrestricted Grammar, the left-hand side of a rule contains a string of terminals and non-terminals (at least one of which must be a non-terminal)
- Rules are applied just like CFGs:
  - Find a substring that matches the LHS of some rule
  - Replace with the RHS of the rule

FR-94: Unrestricted Grammars

- To generate a string with an Unrestricted Grammar:
  - Start with the initial symbol
  - While the string contains at least one non-terminal:
    - Find a substring that matches the LHS of some rule
    - Replace that substring with the RHS of the rule

FR-95: Unrestricted Grammars

- Example: Grammar for $L = \{a^n b^n c^n : n > 0\}$
  - First, generate $(ABC)^*$
  - Next, non-deterministically rearrange string
  - Finally, convert to terminals ($A \rightarrow a$, $B \rightarrow b$, etc.), ensuring that string was reordered to form $a^*b^*c^*$

FR-96: Unrestricted Grammars

- Example: Grammar for $L = \{a^n b^n c^n : n > 0\}$

$$
\begin{aligned}
S &\rightarrow ABCS \\
S &\rightarrow T_C \\
CA &\rightarrow AC \\
BA &\rightarrow AB \\
CB &\rightarrow BC \\
CT_C &\rightarrow T_C c \\
T_C &\rightarrow T_B \\
BT_B &\rightarrow T_B b \\
T_B &\rightarrow T_A \\
AT_A &\rightarrow T_A a \\
T_A &\rightarrow \epsilon
\end{aligned}
$$

FR-97: Unrestricted Grammars

$$\begin{aligned}
S &\Rightarrow ABCS \\
&\Rightarrow ABCABCS \\
&\Rightarrow ABACBCS \\
&\Rightarrow AABCBCS \\
&\Rightarrow AABBCCS \\
&\Rightarrow AABBCCT_C \\
&\Rightarrow AABBCT_Cc \\
&\Rightarrow AABBT_Ccc \\
&\Rightarrow AABBT_Bcc \\
&\Rightarrow AABT_Bbcc \\
&\Rightarrow AAT_Bbbcc \\
&\Rightarrow AAT_Abbcc \\
&\Rightarrow AT_Aabbcc \\
&\Rightarrow T_Aaabbcc \\
&\Rightarrow aabbcc
\end{aligned}$$

FR-98: Unrestricted Grammars

$$
\begin{aligned}
S &\Rightarrow ABCS & &\Rightarrow AAABBBBCCCT_C \\
&\Rightarrow ABCABCS & &\Rightarrow AAABBBCCT_Cc \\
&\Rightarrow ABCABCABCS & &\Rightarrow AAABBBCT_Ccc \\
&\Rightarrow ABACBCABCS & &\Rightarrow AAABBBT_Cccc \\
&\Rightarrow AABCBCABCS & &\Rightarrow AAABBBT_Bccc \\
&\Rightarrow AABCBACBCS & &\Rightarrow AAABBT_Bbccc \\
&\Rightarrow AABCABCBCS & &\Rightarrow AAABT_Bbbccc \\
&\Rightarrow AABACBCBCS & &\Rightarrow AAAT_Bbbbccc \\
&\Rightarrow AAABCBCBCS & &\Rightarrow AAAT_Abbbccc \\
&\Rightarrow AAABBCCBCS & &\Rightarrow AAT_Aabbbccc \\
&\Rightarrow AAABBCBCCS & &\Rightarrow AT_Aaabbbccc \\
&\Rightarrow AAABBBCCCS & &\Rightarrow T_Aaaabbbccc \Rightarrow aaabbbccc
\end{aligned}
$$