14-0: **Counter Machines**

- Give a Non-Deterministic Finite Automata a counter
  - Increment the counter
  - Decrement the counter
  - Check to see if the counter is zero

14-1: **Counter Machines**

- A Counter Machine $M = (K, \Sigma, \Delta, s, F)$
  - K is a set of states
  - Σ is the input alphabet
  - $s \in K$ is the start state
  - $F \subset K$ are Final states
  - $\Delta \subseteq ((K \times (\Sigma \cup \epsilon) \times \{zero, \neg zero\}) \times (K \times \{-1, 0, +1\}))$
- Accept if you reach the end of the string, end in an accept state, and have an empty counter.

14-2: **Counter Machines**

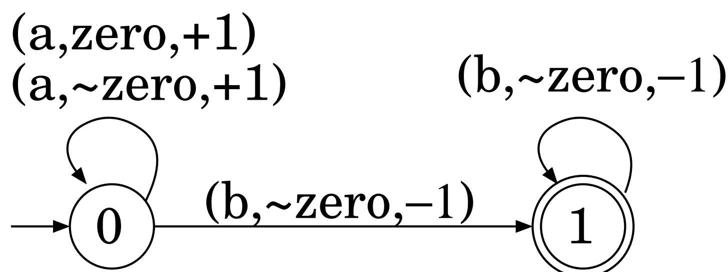
- Give a Non-Deterministic Finite Automata a counter
  - Increment the counter
  - Decrement the counter
  - Check to see if the counter is zero
- Do we have more power than a standard NFA?

14-3: **Counter Machines**

- Give a counter machine for the language $a^n b^n$

14-4: **Counter Machines**

- Give a counter machine for the language $a^n b^n$

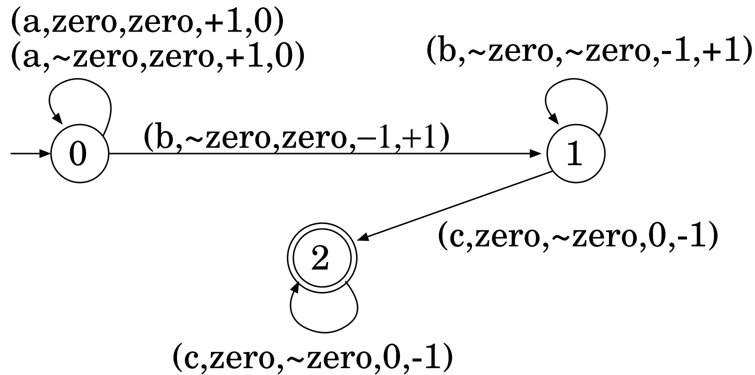


14-5: **Counter Machines**

- Give a 2-counter machine for the language $a^n b^n c^n$
  - Straightforward extension – examine (and change) two counters instead of one.

14-6: **Counter Machines**

- Give a 2-counter machine for the language $a^n b^n c^n$

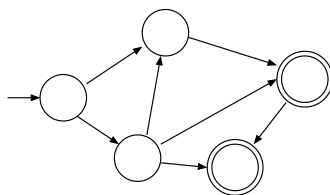


14-7: **Counter Machines**

- Our counter machines only accept if the counter is zero
  - Does this give us any more power than a counter machine that accepts whenever the end of the string is reached in an accept state?
  - That is, given a counter machine M that accepts only strings that both drive the machine to an accept state, and leave the counter empty, can we create a counter machine M' that accepts all strings that drive the machine to an accept state (regardless of the contents of the counter) so that $L[M] = L[M']$?

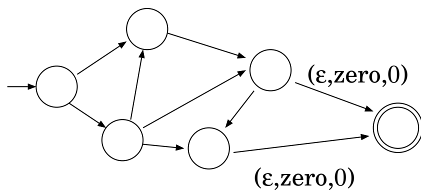
14-8: **Counter Machines**

- Our counter machines only accept if the counter is zero
  - Does this give us any more power than a counter machine that accepts whenever the end of the string is reached in an accept state?



14-9: **Counter Machines**

- Our counter machines only accept if the counter is zero
  - Does this give us any more power than a counter machine that accepts whenever the end of the string is reached in an accept state?

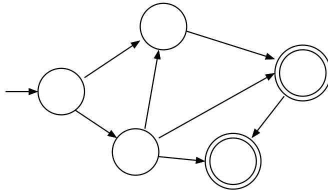


14-10: **Counter Machines**

- Our counter machines only accept if the counter is zero
  - Does this give us any less power than a counter machine that accepts whenever the end of the string is reached in an accept state?
  - That is, given a counter machine M that accepts all strings that drive the machine to an accept state (regardless of contents of counter), can we create a counter machine M' that accepts only strings that both drive the machine to an accept state and leave the counter empty, such that $L[M] = L[M']$?

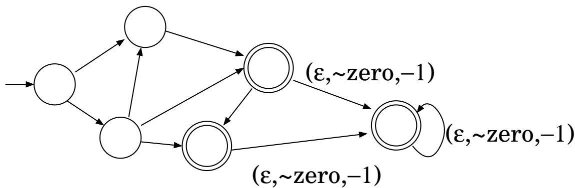
14-11: **Counter Machines**

- Our counter machines only accept if the counter is zero
  - Does this give us any less power than a counter machine that accepts whenever the end of the string is reached in an accept state?



14-12: **Counter Machines**

- Our counter machines only accept if the counter is zero
  - Does this give us any less power than a counter machine that accepts whenever the end of the string is reached in an accept state?



14-13: **Counter Machines**

- Give a Non-Deterministic Finite Automata two counters
- We can use two counters to simulate a stack
  - How?
  - *HINT:* We will simulate a stack that has two symbols, 0 and 1
  - *HINT2:* Think binary

14-14: **Counter Machines**

- We can use two counters to simulate a stack
  - One counter will store the contents of the stack
  - Other counter will be used as "scratch space"
- Stack will be represented as a binary number, with the top of the stack being the least significant bit

  - How can we push a 0?
  - How can we push a 1?

14-15: **Counter Machines**

- How can we push a 0?
  - Multiply the counter by 2
- How can we push a 1?
  - Multiply the counter by 2, and add 1

14-16: **Counter Machines**

- How can we multiply a counter by 2, if all we can do is increment
  - Remember, we have a “scratch counter”

14-17: **Counter Machines**

- How can we multiply a counter by 2, if all we can do is increment
  - Set the “Scratch Counter” to 0
  - While counter is not zero:
    - Decrement the counter
    - Increment the “Scratch Counter” twice

14-18: **Counter Machines**

- To Push a 0:
  - While Counter1 $\neq 0$
    - Increment Counter2
    - Increment Counter2
    - Decrement Counter1
  - Swap Counter1 and Counter2

14-19: **Counter Machines**

- To Push a 1:
  - While Counter1 $\neq 0$
    - Increment Counter2
    - Increment Counter2
    - Decrement Counter1
  - Increment Counter2
  - Swap Counter1 and Counter2

14-20: **Counter Machines**

- To Pop:

- While Counter1 $\neq 0$
  - Decrement Counter1
  - If Counter1 = 0, popped result is 1
  - Decrement Counter1
  - If Counter1 = 0, popped result is 0
  - Increment Counter2
- Swap Counter1 and Counter2

14-21: **Counter Machines**

- How do we check if the simulated stack is empty?
  - We need to use 1 (not zero) to represent an empty stack (why?)
  - Stack is empty if (counter1$-1 = 0$)

14-22: **Counter Machines**

- Example

Stack Counter	Scratch Counter
1	0

- Stack counter starts out as 1 (represents empty stack)
- Scratch counter starts out as 0

14-23: **Counter Machines**

- Example

Stack Counter	Scratch Counter
1	0

- Push 0

14-24: **Counter Machines**

- Example

Stack Counter	Scratch Counter
0	1

- Decrement Stack Counter, increment scratch counter

14-25: **Counter Machines**

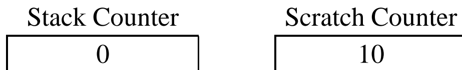
- Example

Stack Counter	Scratch Counter
0	10

- Decrement Stack Counter, increment scratch counter (twice)

14-26: **Counter Machines**

- Example



- Swap Scratch Counter and Stack Counter

While Scratch Counter ≠ Stack Counter
    Decrement Scratch Counter
    Increment Stack Counter

14-27: **Counter Machines**

- Example

- Swap Scratch Counter and Stack Counter

While Scratch Counter ≠ Stack Counter
    Decrement Scratch Counter
    Increment Stack Counter

14-28: **Counter Machines**

- Example

- Push 1

14-29: **Counter Machines**

- Example

- Decrement Stack Counter, increment scratch counter

14-30: **Counter Machines**

- Example

- Decrement Stack Counter, increment scratch counter (twice)

14-31: **Counter Machines**

- Example

Stack Counter	Scratch Counter
0	11

- Decrement Stack Counter, increment scratch counter

14-32: **Counter Machines**

- Example

Stack Counter	Scratch Counter
0	100

- Decrement Stack Counter, increment scratch counter (twice)

14-33: **Counter Machines**

- Example

Stack Counter	Scratch Counter
0	101

- Add one to scratch counter (since pushing 1, not 0)

14-34: **Counter Machines**

- Example

Stack Counter	Scratch Counter
0	101

- Swap Scratch Counter and Stack Counter

While Scratch Counter $\neq$ Stack Counter
    Decrement Scratch Counter
    Increment Stack Counter

14-35: **Counter Machines**

- Example

Stack Counter	Scratch Counter
101	0

- Swap Scratch Counter and Stack Counter

While Scratch Counter $\neq$ Stack Counter
    Decrement Scratch Counter
    Increment Stack Counter

14-36: **Counter Machines**

- Example

Stack Counter	Scratch Counter
101	0

- Pop

14-37: **Counter Machines**

- Example

Stack Counter	Scratch Counter
100	0

- Decrement Stack counter

14-38: **Counter Machines**

- Example

Stack Counter	Scratch Counter
11	0

- Decrement Stack counter (twice)

14-39: **Counter Machines**

- Example

Stack Counter	Scratch Counter
11	1

- Increment Scratch counter

14-40: **Counter Machines**

- Example

Stack Counter	Scratch Counter
10	1

- Decrement Stack counter

14-41: **Counter Machines**

- Example

Stack Counter	Scratch Counter
1	1

- Decrement Stack counter (twice)

14-42: **Counter Machines**

- Example

- Increment Scratch counter

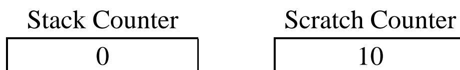
14-43: **Counter Machines**

- Example

- Decrement Stack counter

14-44: **Counter Machines**

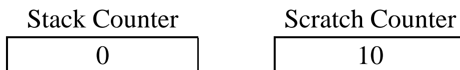
- Example



- Can't Decrement Stack counter a second time (empty), so popped value is 1

14-45: **Counter Machines**

- Example

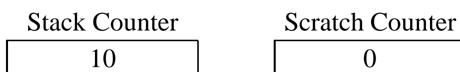


- Swap Scratch Counter and Stack Counter

While Scratch Counter $\neq$ Stack Counter
    Decrement Scratch Counter
    Increment Stack Counter

14-46: **Counter Machines**

- Example



- Swap Scratch Counter and Stack Counter

While Scratch Counter $\neq$ Stack Counter
    Decrement Scratch Counter
    Increment Stack Counter

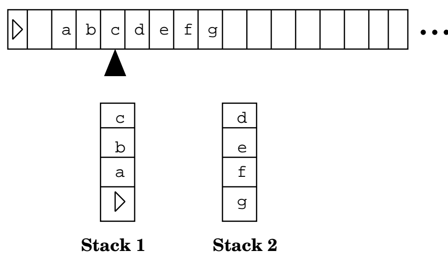
14-47: **Counter Machines**

- Two counters can simulate a stack

- Four counters can simulate two stacks
- What can we do with two stacks?

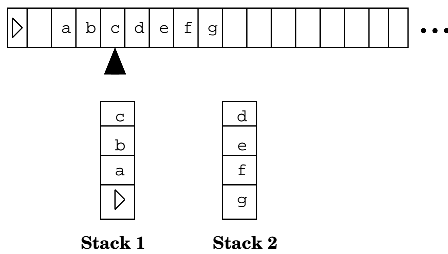
14-48: **Counter Machines**

- Two stacks can simulate a Turing Machine:
  - Stack 1: Everything to the left of the read/write head
  - Stack 2: Everything to the right of the read/write head
- Tape head points to top of Stack 1



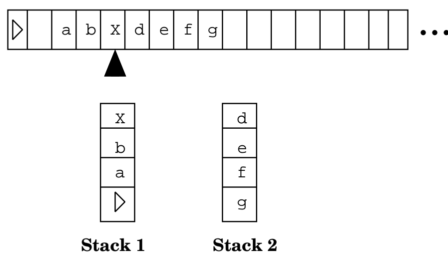
14-49: **Counter Machines**

- To write a new symbol at the Tape Head
  - Pop old value off the top of Stack 1
  - Push new value on the top of Stack 1



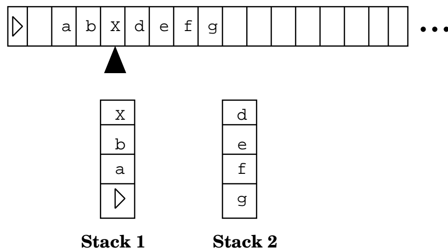
14-50: **Counter Machines**

- To write a new symbol at the Tape Head
  - Pop old value off the top of Stack 1
  - Push new value on the top of Stack 1



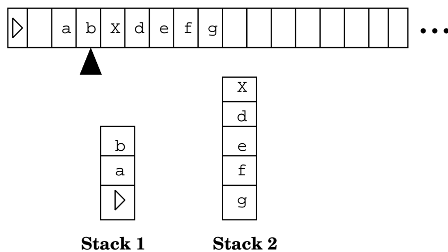
14-51: **Counter Machines**

- To move the tape head to the left
  - Pop symbol off Stack 1
  - Push it on Stack 2



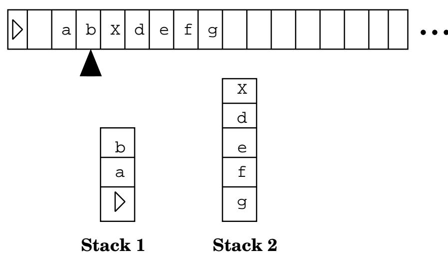
14-52: **Counter Machines**

- To move the tape head to the left
  - Pop symbol off Stack 1
  - Push it on Stack 2



14-53: **Counter Machines**

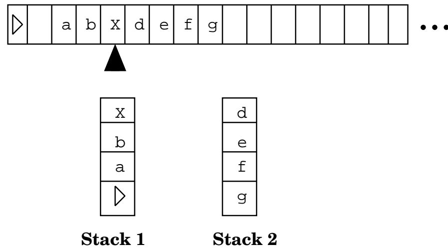
- To move the tape head to the right
  - Pop symbol off Stack 2
  - Push it on Stack 1



14-54: **Counter Machines**

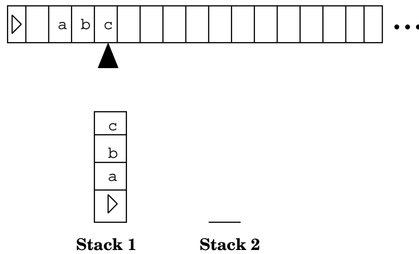
- To move the tape head to the right

- Pop symbol off Stack 2
- Push it on Stack 1



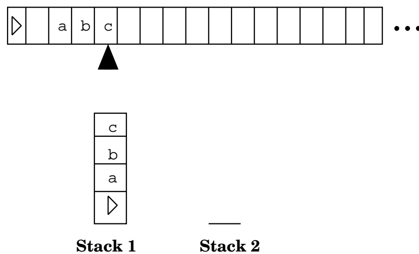
14-55: **Counter Machines**

- To move the tape head to the right, if Stack 2 is empty ...



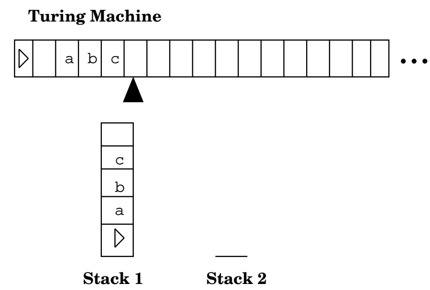
14-56: **Counter Machines**

- To move the tape head to the right, if Stack 2 is empty ...
  - Push a Blank Symbol on Stack 1



14-57: **Counter Machines**

- To move the tape head to the right, if Stack 2 is empty ...
  - Push a Blank Symbol on Stack 1



14-58: **Counter Machines**

- Four Counters ⇒ Two Stacks ⇒ Turing Machine
- If we can simulate a 4-counter machine with a 2-counter machine ...
- Two Counters ⇒ Four Counters ⇒ Two Stacks ⇒ Turing Machine

14-59: **2 Counter ⇒ 4 Counter**

- We can represent 4 counters using just one counter
- Counters have values i, j, k, l
- Master Counter: $2^i 3^j 5^k 7^l$
- When all counters have value 0, master counter has value 1

14-60: **2 Counter ⇒ 4 Counter**

- Master Counter: $2^i 3^j 5^k 7^l$
  - To increment counter j, multiply Master Counter by 3
    - Decrement Master Counter
    - Increment Scratch Counter 3 times
    - Repeat until Master Counter = 0
    - Move Scratch Counter to Master Counter

14-61: **2 Counter ⇒ 4 Counter**

- Master Counter: $2^i 3^j 5^k 7^l$
  - To decrement counter j, divide Master Counter by 3
    - Decrement Master Counter 3 times
    - Increment Scratch Counter
    - Repeat until Master Counter = 0
    - Copy Scratch Counter to Master Counter

14-62: **2 Counter ⇒ 4 Counter**

- Master Counter: $2^i 3^j 5^k 7^l$
  - To check of counter j is zero, see if MC mod 3 = 0

- Decrement Master Counter 3 times (if we hit zero in the middle of this operation, MC mod 3 $\neq$ 0, if he hit zero at the end, MC mod 3 = 0)
- Increment Scratch Counter 3 times
- Repeat until Master Counter = 0
- Use Scratch Counter to restore Master Counter

14-63: **Counter Machines**

- Machine with:
  - Finite State Control
  - Two counters
    - Increment, Decrement, check for zero
- Has full power of a Turing machine – can compute anything

14-64: **Numerical Functions**

- New model of computation: Recursive Functions
  - Very simple functions
  - Method of combining functions
- End up with equivalent power of Turing Machines

14-65: **Numerical Functions**

- Basic Functions:
  - Zero function: $zero_k(n_1, \ldots, n_k) = 0$
  - Identity function: $id_{k,j}(n_1, \ldots, n_k) = n_j$
  - Successor function: $succ(n) = n + 1$ for all $n \in \mathbf{N}$

14-66: **Numerical Functions**

- Zero Function:
  - $zero_3(3, 11, 22) = 0$
  - $zero_2(9, 13) = 0$
  - $zero_0() = 0$

14-67: **Numerical Functions**

- Zero Function:
  - Why have $k-$ary zero function, instead of just defining a the constant 0, or a single $0-$ary function?
    - Notational convenience
    - “Historical Reasons”

14-68: **Numerical Functions**

- Identity function

- $id_{1,1}(4) = 4$
- $id_{4,2}(3, 7, 9, 5) = 7$
- $id_{5,5}(9, 11, 4, 5, 20) = 20$

14-69: **Numerical Functions**

- Successor Function
  - $succ(0) = 1$
  - $succ(1) = 2$
  - $succ(2) = 3$
  - $succ(57) = 58$

14-70: **Numerical Functions**

- Combining Functions:
  - Composition
    - $g : \mathbf{N}^k \mapsto \mathbf{N}$ any $k-$ary function
    - $h_1, \ldots h_k$ $l-$ary functions
    - Composition of g with $h_1, \ldots h_k$

$$f(n_1, \ldots n_l) = g(h_1(n_1, \ldots n_l), \ldots h_k(n_1, \ldots, n_l))$$

14-71: **Numerical Functions**

- Composition:
  - $plus2(x) = succ(succ(x))$
  - $plus3(x) = succ(succ(succ(x)))$

14-72: **Numerical Functions**

- Composition: Constant functions
  - $f() = 5 = succ(succ(succ(succ(succ(zero())))))$
  - $f(3, 2) = 2 = succ(succ(zero()))$

14-73: **Numerical Functions**

- Combining Functions:
  - Recursion
    - $k-$ary function g, $k + 2$-ary function h
    - Function f defined recursively by g and h:

$$\begin{aligned} f(n_1, \ldots n_k, 0) &= g(n_1, \ldots, n_k) \\ f(n_1, \ldots, n_k, m + 1) &= h(n_1, \ldots n_k, m, f(n_1, \ldots, n_k, m)) \end{aligned}$$

14-74: **Numerical Functions**

- Recursive functions:

$$
\begin{aligned}
plus(m, 0) &= m \\
plus(m, n+1) &= succ(plus(m, n))
\end{aligned}
$$

14-75: **Numerical Functions**

- Recursive functions:

$$
\begin{aligned}
plus(m, 0) &= m \\
plus(m, n+1) &= succ(plus(m, n))
\end{aligned}
$$

  - $g(n) = n$
  - $h(n_1, n_2, n_3) = succ(n_3)$

14-76: **Numerical Functions**

- Recursive functions:

$$
\begin{aligned}
mult(m, 0) &= zero(m) \\
mult(m, n+1) &= plus(m, mult(m, n))
\end{aligned}
$$

14-77: **Numerical Functions**

- Recursive functions:

$$
\begin{aligned}
mult(m, 0) &= zero(m) \\
mult(m, n+1) &= plus(m, mult(m, n))
\end{aligned}
$$

  - $g(n) = zero(n)$
  - $h(n_1, n_2, n_3) = plus(n_1, n_3)$

14-78: **Numerical Functions**

- Recursive functions:

$$
\begin{aligned}
exp(m, 0) &= suc(zero(m)) \\
exp(m, n+1) &= mult(m, exp(m, n))
\end{aligned}
$$

- $g(n) = succ(zero(n))$
- $h(n_1, n_2, n_3) = mult(n_1, n_3)$

14-79: **Numerical Functions**

- Recursive functions:

$$\begin{aligned} fact(0) &= suc(zero(m)) \\ fact(n+1) &= mult(n+1, fact(n)) \end{aligned}$$

- $g() = succ(zero())$
- $h(n_1, n_2) = mult(succ(n_1), n_2)$

14-80: **Numerical Functions**

- Recursive functions:

$$\begin{aligned} pred(0) &= 0 \\ pred(n+1) &= n \end{aligned}$$

14-81: **Numerical Functions**

- Recursive functions:

$$\begin{aligned} sub(m, 0) &= m \\ sub(m, n+1) &= pred(sub(m, n)) \end{aligned}$$

What is $sub(3, 5)$? Why?

14-82: **Numerical Functions**

- Predicate functions
  - $iszero(n) = 1$ if $n = 0$, and 0 otherwise

$$\begin{aligned} iszero(0) &= 1 \\ iszero(m+1) &= 0 \end{aligned}$$

14-83: **Numerical Functions**

- Predicate functions

- $geq(m, n) = 1$ if $m \geq n$, and 0 otherwise

$$geq(m, n) =$$

14-84: **Numerical Functions**

- Predicate functions
  - $geq(m, n) = 1$ if $m \geq n$, and 0 otherwise

$$geq(m, n) = iszero(sub(n, m))$$

14-85: **Numerical Functions**

- Predicate functions
  - $lt(m, n) = 1$ if $m < n$, and 0 otherwise

14-86: **Numerical Functions**

- Predicate functions
  - $lt(m, n) = 1$ if $m < n$, and 0 otherwise

$$lt(m, n) = sub(1, geq(m, n))$$

14-87: **Numerical Functions**

- Predicate functions
  - $and(m, n) = 1$ if $m = 1$ and $n = 1$, and 0 otherwise

14-88: **Numerical Functions**

- Predicate functions
  - $and(m, n) = 1$ if $m = 1$ and $n = 1$, and 0 otherwise

$$and(m, n) = mult(m, n)$$

14-89: **Numerical Functions**

- Predicate functions
  - $or(m, n) = 1$ if $m = 1$ or $n = 1$, and 0 otherwise

14-90: **Numerical Functions**

- Predicate functions
  - $or(m, n) = 1$ if $m = 1$ or $n = 1$, and 0 otherwise

$$or(m,n) = sub(1, iszero(plus(m,n)))$$

14-91: **Numerical Functions**

- Defining functions by cases:

$$f(n_1, \ldots, n_k) = \begin{cases} g(n_1, \ldots, n_k) & \text{if } p(n_1, \ldots, n_k) \\ h(n_1, \ldots, n_k) & \text{otherwise} \end{cases}$$

14-92: **Numerical Functions**

- Defining functions by cases:

$$\begin{aligned} rem(0,n) &= 0 \\ rem(m+1,n) &= \begin{cases} 0 & \text{if } equal(rem(m,n), pred(n)) \\ rem(m,n)+1 & \text{otherwise} \end{cases} \end{aligned}$$

(Using first parameter of function as recursion control)

14-93: **Numerical Functions**

- Defining functions by cases:

$$\begin{aligned} div(0,n) &= 0 \\ div(m+1,n) &= \begin{cases} div(m,n)+1 & \text{if } equal(rem(m,n), pred(n)) \\ div(m,n) & \text{otherwise} \end{cases} \end{aligned}$$

(Using first parameter of function as recursion control)

14-94: **Numerical Functions**

- Defining functions by cases:

$$f(n_1, n_2, \ldots, n_k) = \begin{cases} g(n_1, n_2, \ldots, n_k) & \text{if } P(n_1, n_2, \ldots, n_k) \\ h(n_1, n_2, \ldots, n_k) & \text{otherwise} \end{cases}$$

How can we get "functions by cases" using the tools we already have?

14-95: **Numerical Functions**

- Defining functions by cases:

$$f(n_1, n_2, \ldots, n_k) = \begin{cases} g(n_1, n_2, \ldots, n_k) & \text{if } P(n_1, n_2, \ldots, n_k) \\ h(n_1, n_2, \ldots, n_k) & \text{otherwise} \end{cases}$$

$$\begin{aligned} f(n_1, n_2, \ldots, n_k) = \; & P(n_1, n_2, \ldots, n_k) * g(n_1, n_2, \ldots, n_k) \\ & +((1 - P(n_1, n_2, \ldots, n_k)) * \\ & \;\; h(n_1, n_2, \ldots, n_k)) \end{aligned}$$

14-96: **Numerical Functions**

- Are there any functions which we can compute, that cannot be computed with primitive recursive functions?

14-97: **Numerical Functions**

- Are there any functions which we can compute, that cannot be computed with primitive recursive functions?
  - Yes!
  - Use a diagonalization argument
- To make life easier, we will only consider functions that take a single argument (unary functions)

14-98: **Numerical Functions**

- Unary Primitive Recursive Functions can be enumerated
  - That is, we can define an order over all unary primitive recursive functions, $f_1(n), f_2(n), f_3(n), \ldots$
  - How can we order them?

14-99: **Numerical Functions**

- Enumerating Unary Primitive Recursive Functions
  - Each function is created by combining basic functions (succ, zero, select, etc) using composition and recursion
  - Can describe any function using a string
  - Order the strings in lexographic order (shortest to longest, using standard string compare for strings of the same length)

14-100: **Numerical Functions**

- Let the unary primitive recursive functions be:
  $f_0, f_1, f_2, f_3, \ldots$
- Define a new function $g(n) = f_n(n) + 1$
  - We can compute $g(n)$ by first finding the nth unary recursive function f_n, computing $f_n(n)$, and adding 1 to the result

14-101: **Numerical Functions**

- Let the unary primitive recursive functions be:
  $f_0, f_1, f_2, f_3, \ldots$
- Define a new function $g(n) = f_n(n) + 1$
  - We can compute $g(n)$ by first finding the nth unary recursive function f_n, computing $f_n(n)$, and adding 1 to the result
- $g(n)$ can be computed (we just showed how)
- $g(n)$ cannot be computed by a primitive recursive function! (why not?)

14-102: **Numerical Functions**

- $g(n)$ can be computed (we just showed how)

- $g(n)$ cannot be computed by a primitive recursive function! (why not?)
  - Not computed by the 0th primitive recursive function
  - Not computed by the 1st primitive recursive function
  - Not computed by the 2nd primitive recursive function
  - ...

14-103: **Numerical Functions**

- There are some well defined functions, which we can compute, which cannot be computed by primitive recursive functions.
- Can we add anything to primitive recursive functions to give them more power, so that any well defined function that can be computed can be computed with recursive functions?

14-104: **Numerical Functions**

- Minimization
  - If g is a $(k+1)$-ary function. The minimialization of g is the k-ary function f defined as:

$$f(n_1, \ldots, n_k) = \begin{cases} \text{The least } m \text{ such that} \\ \qquad g(n_1, \ldots, n_k, m) = 1, \\ \qquad \text{if such an } m \text{ exists} \\ 0 \text{ otherwise} \end{cases}$$

Minimization of g is denoted $\mu m[g(n_1, \ldots n_k, m) = 1]$

14-105: **Numerical Functions**

- Minimization Examples

$$div(x, y) = \mu z[(y * (z + 1)) - z > 0]$$

"$-$" is "positive subtraction" (that is, if $y > x$, then $x - y = 0$)

$$\begin{aligned} div(x, y) &= z \\ y * z &\leq x \\ y * (z + 1) &> x \end{aligned}$$

14-106: **Numerical Functions**

- Minimization Examples

$$log(m, n) = \mu p[power(m + 2, p) \geq n + 1]$$

"$\geq$" is the "greater-than-or-equal" predicate

14-107: **Numerical Functions**

- Calculating minimalization:

$m \leftarrow 0$;
while ($g(n_1, \ldots, n_k, m) \neq 1$)
    $m \leftarrow m + 1$
return m

14-108: **Numerical Functions**

- Calculating minimalization:

$m \leftarrow 0$;
while ($g(n_1, \ldots, n_k, m) \neq 1$)
    $m \leftarrow m + 1$
return m

... of course, this may never terminate, if there is no value of m such that $g(n_1, \ldots, n_k, m) = 1$

14-109: **Numerical Functions**

- A function $g(n_1, \ldots, n_k, m)$ is minimalizable if
  - For each $n_1, \ldots, n_k \in \mathbf{N}$, there exists some m such that $g(n_1, \ldots, n_k, m) = 1$

  That is:

  $m \leftarrow 0$;
  while ($g(n_1, \ldots, n_k, m) \neq 1$)
      $m \leftarrow m + 1$
  return m

  always terminates, for all values $n_1, \ldots, n_k$

14-110: **Numerical Functions**

- μ-Recursive
  - A function is μ-recursive if it consists entirely of primitive-recursive functions, and minimalizations of minimalizable functions.
  - μ-recursive functions can calculate anything that can be decided by a Turing machine
    - (recall that "decide" means the TM halts on all inputs)

14-111: **Numerical Functions**

- μ-recursive functions can calculate anything that can be decided by a Turing machine
  - We can enumerate μ-recursive functions just like we enumerated primitive recursive functions $f_0, f_1, f_2, \ldots$
  - We can define the function $g(n) = f_n(n) + 1$
  - How can I assert that μ-recursive functions can compute anything that a Turing Machine can compute, when μ-recursive functions can't compute g?

14-112: **Numerical Functions**

- Method to compute $g(n)$ using a Turing machine:

- Enumerate first $n+1$ functions f
  - $f_0, f_1, \ldots, f_n$
- Compute $f_n(n)$
- Output $f_n(n)+1$

14-113: **Numerical Functions**

- Method to compute $g(n)$ using a Turing machine:
  - Enumerate first $n+1$ functions f
    - $f_0, f_1, \ldots, f_n$
  - **Compute** $f_n(n)$
  - Output $f_n(n)+1$
- Function f_n might not be minimalizable! If $f_n(n)$ is not minimalizable, then $f_n(n) = 0$, but we have no way of discovering this!

14-114: **Recursive Languages**

- μ-recursive functions can calculate anything that can be decided by a Turing machine.
- $\{L : L$ is decided by some TM $M\}$ is the recursive languages
- How can a function from the natural numbers to the natural numbers decide a language?

14-115: **Recursive Languages**

- How can a function from the natural numbers to the natural numbers decide a language?
  - Any string can be encoded as a number
    - ASCII-style encoding scheme to encode each symbol in string
    - Append codes of each symbol together to get a (really large) number

$\Sigma = \{a, \ldots, z\}$, $en(a) = 10$, $en(b) = 11$, ..., $en(z) = 35$
$en(abbz) = 10111135$

14-116: **Recursive Languages**

- How can a function from the natural numbers to the natural numbers decide a language?
  - Any string can be encoded as a number
  - Predicate function can be used to determine membership
    $L[f] = \{w : f(en(w)) = 1\}$

14-117: **μ-Recursive Functions & TMs**

- Any μ-recursive function can be decided by a Turing machine
  - Each of the primitive-recursive functions can easily be simulated by a Turing machine
  - Any minimalizable function can be computed by a Turing machine that tries all values in order

```
m ← 0;
while (g(n_1, ..., n_k, m) ≠ 1)
    m ← m + 1
return m
```

14-118: **μ-Recursive Functions & TMs**

- Any function that can be decided by a Turing machine can be computed with a μ-recursive function
  - We can encode a configuration as a number
  - We can encode a sequence of configurations with a (much larger) number

  $\text{config}_1\text{config}_2\text{config}_3 \ldots \text{config}_n$
- Each configuration encodes tape contents, head location, and current state of the Turing Machine

14-119: **μ-Recursive Functions & TMs**

- We have a large number, which represents a series of configurations for a Turning Machine

  $\text{config}_1\text{config}_2\text{config}_3 \ldots \text{config}_n$
- We can write a primitive-recursive predicate function $isvalid$ that examines this string of configurations, and determines if it is legal
  - if $\text{config}_i\text{config}_j$ appears in the sequence
  - Turing machine will move from config_i to config_j in a single step

14-120: **μ-Recursive Functions & TMs**

- $isvalid(n)$
  - Predicate function
  - True if n is a number which represents a valid sequence of configurations of a Turing Machine
  - Writing $isvalid$ for a particular Turing Machine is reasonably straightforward
    - Extract 1st and 2nd configurations from the number (using div and mod)
    - Make sure that the transition from 1st to 2nd configuration is valid
    - Recursively check the rest of the transitions

14-121: **μ-Recursive Functions & TMs**

- Given a number which represents a valid sequence of configurations for the Turning Machine M, if:
  - If the first configuration represents the initial state and the input n
  - The last configuration contains a halting state h
  - The tape contents of the last configuration represents the value y
- Then the Turing Machine M gives the output y for the input n

14-122: **μ-Recursive Functions & TMs**

- Given a number which represents a sequence of configurations, and an input n, we can:
  - Determine if the sequence of configurations is valid
  - Ensure that the first configuration encodes n
  - Ensure that the last configuration contains a halting state

14-123: **μ-Recursive Functions & TMs**

- Function $check_compute(n, x)$
  - Takes as input a string of configurations n, and an initial configuration x
  - Returns 1 (true) if n is a valid series of computations that starts with x
    - $isvalid(n) = 1$
    - $first(n) = x$

14-124: **μ-Recursive Functions & TMs**

- Function $compute(x)$
  - Calculates and returns the string of valid configurations that starts with x and ends in a halting state

14-125: **μ-Recursive Functions & TMs**

- Function $compute(x)$
  - Calculates and returns the string of valid configurations that starts with x and ends in a halting state

$$compute(x) = \mu n[check_compute(n, x)]$$

14-126: **μ-Recursive Functions & TMs**

- Function f_M, that calculates the same function as the Turing Machine M:

$$f_M(x) = last(compute(x))$$