Game Engineering
CS486/686-2016S-06
Determinants and Inverses

David Galles
Department of Computer Science
University of San Francisco
The Determinant of a $2\times2$ matrix is defined as:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$
06-1: **Determinant**

- We can define the determinant for a 3x3 matrix in terms of the determinants of 2x2 submatrices (minors)

\[
\begin{vmatrix}
  a & b & c \\ d & e & f \\ g & h & i \\ \end{vmatrix} = a \begin{vmatrix}
  e & f \\ h & i \\ \end{vmatrix} - b \begin{vmatrix}
  d & f \\ g & i \\ \end{vmatrix} + c \begin{vmatrix}
  d & e \\ g & h \\ \end{vmatrix}
\]

\[
= a(ei - hf) - b(di - gf) + c(dh - ge)
\]

\[
= aei + bfg + cdh - afh - bdi - ceg
\]
06-2: Determinant

\[
\begin{vmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i \\
\end{vmatrix}
= a \begin{vmatrix}
  e & f \\
  h & i \\
\end{vmatrix}
- b \begin{vmatrix}
  d & f \\
  g & i \\
\end{vmatrix}
+ c \begin{vmatrix}
  d & e \\
  g & h \\
\end{vmatrix}
\]

\[
= a(ei - hf) - b(di - gf) + c(dh - ge)
\]

\[
= aei + bfg + cdh - afh - bdi - ceg
\]
# Determinant

\[
\begin{vmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i \\
\end{vmatrix}
\]

The determinant is calculated as follows:

\[
\begin{vmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i \\
\end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}
\]

The signs for each term in the determinant are:

\[
\begin{vmatrix}
  + & - & + \\
  - & + & - \\
  + & - & + \\
\end{vmatrix}
\]

The determinant is calculated by expanding along the first row:

\[
\begin{vmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i \\
\end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}
\]

The determinant is the sum of the products of the elements along the main diagonal, minus the sum of the products of the elements along the secondary diagonal.
06-4: Determinant

\[
\begin{vmatrix}
a & b & c \\
d & e & f \\
g & h & i \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
a & b & c \\
d & e & f \\
g & h & i \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
a & b & c \\
d & e & f \\
g & h & i \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
a & b & c \\
d & e & f \\
g & h & i \\
\end{vmatrix}
\]
### Determinant

<table>
<thead>
<tr>
<th></th>
<th>+</th>
<th>-</th>
<th>+</th>
</tr>
</thead>
<tbody>
<tr>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

\[
\begin{vmatrix}
  \text{a} & \text{b} & \text{c} \\
  \text{d} & \text{e} & \text{f} \\
  \text{g} & \text{h} & \text{i}
\end{vmatrix}
\]
06-6: Determinant

- We can expand determinants to 4x4 matrices ...

\[
\begin{vmatrix}
  a & b & c & d \\
  e & f & g & h \\
  i & j & k & l \\
  m & n & o & p \\
\end{vmatrix} =
\begin{vmatrix}
  f & g & h \\
  j & k & l \\
  n & o & p \\
\end{vmatrix} -
\begin{vmatrix}
  e & g & h \\
  i & k & l \\
  m & o & p \\
\end{vmatrix} +
\begin{vmatrix}
  e & f & h \\
  i & j & l \\
  m & n & p \\
\end{vmatrix} -
\begin{vmatrix}
  e & f & g \\
  i & j & k \\
  m & n & o \\
\end{vmatrix}
\]
There are many ways to define / calculate determinants

- Add all possible permutations
- Sign is parity of number of inversions

\[
\begin{vmatrix}
    a_1 & a_2 & a_3 \\
    b_1 & b_2 & b_3 \\
    c_1 & c_2 & c_3
\end{vmatrix} = a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 c_3 b_1 + a_3 b_1 c_2 - a_3 b_2 c_1
\]
Determinant properties

- Multiplying a column (or row) by $k$ is the same as multiplying the determinant by $k$

\[
\begin{vmatrix}
ka & b \\
kc & d \\
\end{vmatrix} = \begin{vmatrix}
kad - kcb \\
\end{vmatrix} = k(ad - cb) = k \begin{vmatrix}
a & b \\
c & d \\
\end{vmatrix}
\]
06-9: Determinant

- Determinant properties
  - Multiplying a column (or row) by \( k \) is the same as multiplying the determinant by \( k \)

\[
\begin{vmatrix}
ka & b & c \\
kd & e & f \\
kg & h & i \\
\end{vmatrix}
= ka
\begin{vmatrix}
e & f \\
h & i \\
\end{vmatrix}
- b
\begin{vmatrix}
kd & f \\
kg & i \\
\end{vmatrix}
+ c
\begin{vmatrix}
kd & e \\
kg & h \\
\end{vmatrix}
\]

\[
= ka
\begin{vmatrix}
e & f \\
h & i \\
\end{vmatrix}
- kb
\begin{vmatrix}
d & f \\
g & i \\
\end{vmatrix}
+ kc
\begin{vmatrix}
d & e \\
g & h \\
\end{vmatrix}
\]

\[
= k
\begin{vmatrix}
a & b & c \\
d & e & f \\
g & h & i \\
\end{vmatrix}
\]
06-10: **Determinant**

- **Determinant properties**
  - Swapping two rows (or two columns) of a determinant reverse the sign (but preserves magnitude)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb = -(bc - ad) = -\begin{vmatrix} b & a \\ d & c \end{vmatrix}$$
Determinant properties

Swapping two rows (or two columns) of a determinant reverse the sign (but preserves magnitude)

\[
\begin{vmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i \\
\end{vmatrix}
= a \begin{vmatrix}
  e & f \\
  h & i \\
\end{vmatrix}
- b \begin{vmatrix}
  d & f \\
  g & i \\
\end{vmatrix}
+ c \begin{vmatrix}
  d & e \\
  g & h \\
\end{vmatrix}
\]

\[
= - \left( -a \begin{vmatrix}
  e & f \\
  h & i \\
\end{vmatrix}
+ b \begin{vmatrix}
  d & f \\
  g & i \\
\end{vmatrix}
+ c \begin{vmatrix}
  e & d \\
  h & g \\
\end{vmatrix} \right)
\]

\[
= - \left( b \begin{vmatrix}
  d & f \\
  g & i \\
\end{vmatrix}
- a \begin{vmatrix}
  e & f \\
  h & i \\
\end{vmatrix}
+ c \begin{vmatrix}
  e & d \\
  h & g \\
\end{vmatrix} \right) = - \begin{vmatrix}
  b & a & c \\
  e & d & f \\
  h & g & i \\
\end{vmatrix}
\]
06-12: Determinant

- Determinant properties
  - Other fun determinant properties
  - Take a “real” math class for more ...
Determinant: Geometry

- 2x2 determinant is signed area of parallelogram

\[
\begin{vmatrix}
 a & b \\
 c & d \\
\end{vmatrix}
\]

Parallelogram: \((0, 0), (a, b), (a + c, b + d), (c, d)\)
Determinant: Geometry

\[
\begin{vmatrix}
 a & b \\
 c & d \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
 2 & 1 \\
 1 & 3 \\
\end{vmatrix} = 6 - 1 = 5
\]

\[
(a, b) \quad (a+c, b+d)
\]

Area = 5
Signed area

- If the points $(0, 0), (a, b), (a + c, b + d), (c, d)$ go around the parallelogram *counter-clockwise*, the area is positive
- If the points $(0, 0), (a, b), (a + c, b + d), (c, d)$ go around the parallelogram *clockwise*, the area is negative
### Determinant: Geometry

\[
\begin{vmatrix}
  a & b \\
  c & d \\
\end{vmatrix}
= \begin{vmatrix}
  2 & 1 \\
  1 & 3 \\
\end{vmatrix}
= \begin{vmatrix}
  1 & 3 \\
  2 & 1 \\
\end{vmatrix}
\]

Counter-Clockwise - Positive Area

Clockwise - Negative Area
Compare determinant of a matrix to how matrix transforms objects

Consider the transform matrix:

\[
\begin{vmatrix}
  a & b \\
  c & d \\
\end{vmatrix}
\]

What happens when we transform the unit square, using this matrix?
Cube Transformation

\[
\begin{bmatrix}
0 & 0 \\
1 & 0 \\
0 & 1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
= \begin{bmatrix}
a + c & b + d
\end{bmatrix}
\]
Cube Transformation

Transform with
\[
\begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix}
\]

(0,1)  (1,1)  (c,d)  (a+c, b+d)
(1,0)  

(a,b)
**Determinant as Scale**

- Determinant gives the volume of a unit square after transformation
  - Determinant gives the scaling factor for the transformation
  - Determinant $< 1 \Rightarrow$ “shrink” object
  - Determinant $= 1 \Rightarrow$ object size (area) unchanged
  - Determinant $> 1 \Rightarrow$ “grow” object
Sanity check:
Identity transform does not change an object – determinant should be 1
\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

Matrix that scales by factor $s$:
\[
\begin{bmatrix}
s & 0 \\
0 & 1
\end{bmatrix}
\]

This is just multiplying a column (or row!) by the scalar $s$ – multiplies the value of the determinant by $s$. 
06-22: Negative Determinant

- Determinant gives \textit{signed} area
- Pictures assume that \((a, b)\) is clockwise of \((c, d)\)
- What happens if \((a, b)\) is \textit{counterclockwise} of \((c, d)\)?
- What do you know about the transformation?
06-23: Negative Determinant

Transform with

\[
\begin{pmatrix}
    a & b \\
    c & d \\
\end{pmatrix}
\]

(a, b) → (c, d)
06-24: Negative Determinant

Transform with

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

\((a, b)\)  
\((c, d)\)

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]
If the determinant of a transformation matrix is negative
  - Transformation includes a reflection

Sanity check:
  - Flip the basis vectors in a transformation matrix, cause a reflection
  - Swapping rows (or columns) in a determinant changes the sign
If the determinant of a transformation matrix is negative
- Transformation includes a reflection

Sanity check II:
- What happens to a transformation matrix when we flip a dimension?
  - That is, multiply a column (assuming row vectors) by $-1$
- What does that do to the determinant?
Modify the transform by flipping the X axis
-- multiply the x column by -1
06-28: Negative Determinant

- Multiplying one column of the matrix flips that axis
- Multiplies the determinant by -1
  - Multiplying all elements of a row or column by $k$ changes the value of the determinant by a factor of $k$
3x3 determinants have the same properties as 2x2 determinants

Determinant of a 3x3 matrix is the scale factor for how the volume of the transformed object changes

-1 == Reflection, just like 2D case
06-30: **Determinant Geom 3D**

\[
\begin{vmatrix}
  r_{1x} & r_{1y} & r_{1z} \\
  r_{2x} & r_{2y} & r_{2z} \\
  r_{3x} & r_{3y} & r_{3z}
\end{vmatrix}
\]
What does it mean (geometrically) when a 3x3 matrix has a zero determinant?

What kind of a transformation is it?
Transformation Matrix with Zero Determinant

Basis vectors transformed to same plane
Given a square matrix $M$, the inverse $M^{-1}$ is the matrix such that

- $MM^{-1} = I$
- $M^{-1}M = I$

Since matrix multiplication is associative, for any vector $v$:

- $(vM)M^{-1} = v$

Matrix Inverse undoes the transformation
06-34: Matrix Inverse

- Do all square matrices have an inverse?
06-35: Matrix Inverse

- Do all square matrices have an inverse?
  - Consider:
    \[
    M = \begin{bmatrix}
    0 & 1 \\
    0 & 0 \\
    \end{bmatrix}
    \]
  - What happens when we multiply \( M \) by a vector \( v \)? Why can’t we undo this operation?
Singular

- A matrix that has an inverse is *Non-Singular* or *Invertable*
- A matrix without an inverse is *Singular* or *Non-Invertable*
- A matrix is singular if and only if it has a determinant of 0
To calculate the inverse, we will need cofactor matrix

We’ve already seen cofactors (just not with that name), when calculating determinant
## 06-38: Cofactors

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>d</td>
<td>e</td>
<td>f</td>
<td>g</td>
</tr>
<tr>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>d</td>
<td>e</td>
<td>f</td>
<td>g</td>
</tr>
<tr>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>d</td>
<td>e</td>
<td>f</td>
<td>g</td>
</tr>
<tr>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
<th>i</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>d</td>
<td>e</td>
<td>f</td>
<td>g</td>
<td>h</td>
</tr>
<tr>
<td>-</td>
<td>d</td>
<td>e</td>
<td>f</td>
<td>g</td>
<td>h</td>
</tr>
<tr>
<td>+</td>
<td>d</td>
<td>e</td>
<td>f</td>
<td>g</td>
<td>h</td>
</tr>
</tbody>
</table>
06-39: Cofactors
06-40: Cofactor Matrix

\[
\begin{vmatrix}
    a & b & c \\
    d & e & f \\
    g & h & i
\end{vmatrix}
\]

Cofactor Matrix

\[
\begin{vmatrix}
    e & f & -d & -f \\
    h & i & g & i \\
    -b & c & a & c \\
    -b & c & a & c \\
    e & f & -d & -f \\
\end{vmatrix}
\]
Calculating Inverse

- Adjoint of a matrix $M$, $\text{adj}(M)$ is the transpose of the cofactor matrix.

- The inverse of a matrix is the adjoint divided by the determinant.
  - $M^{-1} = \frac{\text{adj}(M)}{|M|}$

- (can see how a matrix with a determinant of 0 would have a hard time having an inverse)

- Other ways of computing inverses (i.e. Gaussian elimination):
  - Fewer arithmetic operations
  - Algorithms require branches (expensive!)
  - Vector hardware makes adjoint method fast
First, Calculate determinant

\[
\begin{vmatrix}
1 & 2 & 1 \\
2 & 2 & 1 \\
1 & 1 & 2
\end{vmatrix}
\]

\[
\begin{vmatrix}
1 & 2 & 1 \\
2 & 2 & 1 \\
1 & 1 & 2
\end{vmatrix} = 1 \begin{vmatrix}
2 & 1 \\
1 & 2
\end{vmatrix} - 2 \begin{vmatrix}
2 & 1 \\
1 & 2
\end{vmatrix} + 1 \begin{vmatrix}
2 & 2 \\
1 & 1
\end{vmatrix} = -3
\]
Next, calculate cofactor matrix

\[
\begin{bmatrix}
1 & 2 & 1 \\
2 & 2 & 1 \\
1 & 1 & 2 \\
\end{bmatrix}
\]
Next, calculate cofactor matrix

\[
\begin{bmatrix}
1 & 2 & 1 \\
2 & 2 & 1 \\
1 & 1 & 2
\end{bmatrix}
\]

Cofactor matrix:

\[
\begin{bmatrix}
3 & -3 & 0 \\
-3 & 1 & 1 \\
0 & 1 & -2
\end{bmatrix}
\]
06-45: **Inverse Example**

Transpose the cofactor array to get the adjoint (in this example, the adjoint is equal to its transpose, but that doesn’t always happen)

\[
\begin{bmatrix}
1 & 2 & 1 \\
2 & 2 & 1 \\
1 & 1 & 2 \\
\end{bmatrix}
\]

Adjoint:

\[
\begin{bmatrix}
3 & -3 & 0 \\
-3 & 1 & 1 \\
0 & 1 & -2 \\
\end{bmatrix}
\]
Finally, divide adjoint by the determinant to get the inverse:

\[
\begin{bmatrix}
1 & 2 & 1 \\
2 & 2 & 1 \\
1 & 1 & 2
\end{bmatrix}
\]

Inverse:

\[
\begin{bmatrix}
-1 & 1 & 0 \\
1 & -\frac{1}{3} & -\frac{1}{3} \\
0 & -\frac{1}{3} & \frac{2}{3}
\end{bmatrix}
\]
06-47: Inverse Example

Sanity check

\[
\begin{bmatrix}
1 & 2 & 1 \\
2 & 2 & 1 \\
1 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
-1 & 1 & 0 \\
1 & -\frac{1}{3} & -\frac{1}{3} \\
0 & -\frac{1}{3} & \frac{2}{3}
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
Orthogonal Matrices

- A matrix $\mathbf{M}$ is orthogonal if:
  - $\mathbf{M}\mathbf{M}^T = \mathbf{I}$
  - $\mathbf{M}^T = \mathbf{M}^{-1}$

- Orthogonal matrices are handy, because they are easy to invert

- Is there a geometric interpretation of orthogonality?
Orthogonal Matrices

\[
\begin{bmatrix}
m_{11} & m_{12} & m_{13} \\
m_{21} & m_{22} & m_{23} \\
m_{31} & m_{32} & m_{33}
\end{bmatrix}
\begin{bmatrix}
m_{11} & m_{21} & m_{31} \\
m_{12} & m_{22} & m_{32} \\
m_{13} & m_{23} & m_{33}
\end{bmatrix}
= \begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1\end{bmatrix}
\]

Do all the multiplications ...
Orthogonal Matrices

\[
\begin{align*}
    m_{11}m_{11} + m_{12}m_{12} + m_{13}m_{13} &= 1 \\
    m_{11}m_{21} + m_{12}m_{22} + m_{13}m_{23} &= 0 \\
    m_{11}m_{31} + m_{12}m_{32} + m_{13}m_{33} &= 0 \\
    m_{21}m_{11} + m_{22}m_{12} + m_{23}m_{13} &= 0 \\
    m_{21}m_{21} + m_{22}m_{22} + m_{23}m_{23} &= 1 \\
    m_{21}m_{31} + m_{22}m_{32} + m_{23}m_{33} &= 0 \\
    m_{31}m_{11} + m_{32}m_{12} + m_{33}m_{13} &= 0 \\
    m_{31}m_{21} + m_{32}m_{22} + m_{33}m_{23} &= 0 \\
    m_{31}m_{31} + m_{32}m_{32} + m_{33}m_{33} &= 1
\end{align*}
\]

Hmmm... that doesn’t seem to help much
Recall that rows of matrix are basis after rotation

- \( \mathbf{v}_x = [m_{11}, m_{12}, m_{13}] \)
- \( \mathbf{v}_y = [m_{21}, m_{22}, m_{23}] \)
- \( \mathbf{v}_z = [m_{31}, m_{32}, m_{33}] \)

Let's rewrite the previous equations in terms of \( \mathbf{v}_x \), \( \mathbf{v}_y \), and \( \mathbf{v}_z \) ...
Orthogonal Matrices

\[
\begin{align*}
m_{11}m_{11} + m_{12}m_{12} + m_{13}m_{13} &= 1 & v_x \cdot v_x &= 1 \\
m_{11}m_{21} + m_{12}m_{22} + m_{13}m_{23} &= 0 & v_x \cdot v_y &= 0 \\
m_{11}m_{31} + m_{12}m_{32} + m_{13}m_{33} &= 0 & v_x \cdot v_z &= 0 \\
m_{21}m_{11} + m_{22}m_{12} + m_{23}m_{13} &= 0 & v_y \cdot v_x &= 0 \\
m_{21}m_{21} + m_{22}m_{22} + m_{23}m_{23} &= 1 & v_y \cdot v_y &= 1 \\
m_{21}m_{31} + m_{22}m_{32} + m_{23}m_{33} &= 0 & v_y \cdot v_z &= 0 \\
m_{31}m_{11} + m_{32}m_{12} + m_{33}m_{13} &= 0 & v_z \cdot v_x &= 0 \\
m_{31}m_{21} + m_{32}m_{22} + m_{33}m_{23} &= 0 & v_z \cdot v_y &= 0 \\
m_{31}m_{31} + m_{32}m_{32} + m_{33}m_{33} &= 1 & v_z \cdot v_z &= 1
\end{align*}
\]
What does it mean if $u \cdot v = 0$?
- (assuming both $u$ and $v$ are non-zero)

What does it mean if $v \cdot v = 1$?
06-54: **Orthogonal Matrices**

- What does it mean if \( \mathbf{u} \cdot \mathbf{v} = 0 \)?
  - (assuming both \( \mathbf{u} \) and \( \mathbf{v} \) are non-zero)
  - \( \mathbf{u} \) and \( \mathbf{v} \) are perpendicular to each other (orthogonal)

- What does it mean if \( \mathbf{v} \cdot \mathbf{v} = 1 \)?
  - \( \| \mathbf{v} \| = 1 \)

- So, transformed basis vectors must be mutually perpendicular unit vectors
If a transformation matrix is orthogonal,
- Transformed basis vectors are mutually perpendicular unit vectors
- What kind of transformations are done by orthogonal matrices?
Orthogonal Matrices

- If a transformation matrix is orthogonal,
  - Transformed basis vectors are mutually perpendicular unit vectors
- What kind of transformations are done by orthogonal matrices?
  - Rotations & Reflections
Orthogonalizing a Matrix

- It is possible to have a matrix that *should* be orthogonal that is *not quite* orthogonal
  - Bad Data
  - Accumulated floating point error (matrix creep)
- If a matrix is just slightly non-orthogonal, we can modify it to be orthogonal
Orthogonalizing a Matrix

Not quite orthogonal

Move this vector down a bit
Orthogonalizing a Matrix

Just need to add this vector to \( v_2 \)

What is it?
Given two vectors $v_1$ and $v_2$ that are nearly orthogonal, we subtract from $v_2$ the component of $v_2$ that is parallel to $v_1$

$$v_2 = v_2 - \frac{v_1 \cdot v_2}{||v_1||^2} v_1$$

$$= v_2 - \frac{v_1 \cdot v_2}{v_1 \cdot v_1} v_1$$
We can easily extend this to 3 dimensions:

- Leave the first vector alone
- Tweak the second vector to be perpendicular to the first vector
- Tweak the third vector to be perpendicular to first two vectors
Orthogonalizing a Matrix

- Given a *nearly* orthogonal matrix with rows $r_1, r_2, \text{ and } r_3$:

\[
\begin{align*}
    r'_1 &= r_1 \\
    r'_2 &= r_2 - \frac{r_2 \cdot r'_1}{r'_1 \cdot r'_1} r'_1 \\
    r'_3 &= r_3 - \frac{r_3 \cdot r'_1}{r'_1 \cdot r'_1} r'_1 - \frac{r_3 \cdot r'_2}{r'_2 \cdot r'_2} r'_2
\end{align*}
\]

- Need to normalize $r_1, r_2, r_3$
- If we normalize $r'_1$ before calculating $r'_2$, then $r'_1 \cdot r'_1 = 1$, and we can remove a division
The problem with orthogonalizing a matrix this way is that there is a bias

- First row never changes
- Third row changes the most

What if we don’t want a bias?
The problem with orthogonalizing a matrix this way is that there is a bias.
- First row never changes
- Third row changes the most

What if we don’t want a bias?
- Change each vector a little bit in the correct direction
- Repeat until you get close enough
- Then run the “standard” method
Orthogonalizing a Matrix

\[ r'_1 = r_1 - k \frac{r_1 \cdot r_2}{r_2 \cdot r_2} r_2 - k \frac{r_1 \cdot r_3}{r_3 \cdot r_3} r_3 \]
\[ r'_2 = r_2 - k \frac{r_2 \cdot r_1}{r_1 \cdot r_1} r_1 - k \frac{r_2 \cdot r_3}{r_3 \cdot r_3} r_3 \]
\[ r'_3 = r_3 - k \frac{r_3 \cdot r_1}{r_1 \cdot r_1} r_1 - k \frac{r_3 \cdot r_2}{r_2 \cdot r_2} r_2 \]

- Do several iterations (smallish \( k \))
- Not guaranteed to get exact – run standard method when done