Game Engineering

CS486/686-2016S-06

Determinants and Inverses

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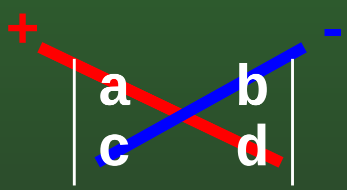
06-0: Determinant

- The Determinant of a 2x2 matrix is defined as:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$



06-1: Determinant

- We can define the determinant for a 3x3 matrix in terms of the determinants of 2x2 submatrices (minors)

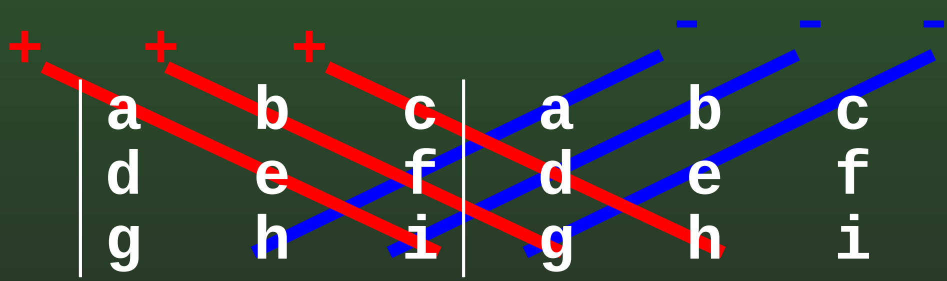
$$
\begin{aligned}
\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\
&= a(ei - hf) - b(di - gf) + c(dh - ge) \\
&= aei + bfg + cdh - afh - bdi - ceg
\end{aligned}
$$

06-2: Determinant

$$
\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}
= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}
$$

$$
= a(ei - hf) - b(di - gf) + c(dh - ge)
$$

$$
= aei + bfg + cdh - afh - bdi - ceg
$$



06-3: Determinant

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

06-4: Determinant

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$-\, d \begin{vmatrix} b & c \\ h & i \end{vmatrix} \; + \; e \begin{vmatrix} a & c \\ g & i \end{vmatrix} \; - \; f \begin{vmatrix} a & b \\ g & h \end{vmatrix}$$

06-5: Determinant

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$g\begin{vmatrix} b & c \\ e & f \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix} + i\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

06-6: Determinant

- We can expand determinants to 4x4 matricies ...

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

06-7: Determinant

- There are many ways to define / calculate determinants
  - Add all possible permutations
  - Sign is parity of number of inversions

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2c_3b_1 + a_3b_1c_2 - a_3b_2c_1$$

06-8: Determinant

- Determinant properties
  - Multiplying a column (or row) by k is the same as multiplying the determinant by k

$$
\begin{aligned}
\begin{vmatrix} ka & b \\ kc & d \end{vmatrix} &= kad - kcb \\
&= k(ad - cb) \\
&= k\begin{vmatrix} a & b \\ c & d \end{vmatrix}
\end{aligned}
$$

06-9: Determinant

- Determinant properties
  - Multiplying a column (or row) by k is the same as multiplying the determinant by k

$$
\begin{aligned}
\begin{vmatrix} ka & b & c \\ kd & e & f \\ kg & h & i \end{vmatrix}
&= ka \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} kd & f \\ kg & i \end{vmatrix} + c \begin{vmatrix} kd & e \\ kg & h \end{vmatrix} \\
&= ka \begin{vmatrix} e & f \\ h & i \end{vmatrix} - kb \begin{vmatrix} d & f \\ g & i \end{vmatrix} + kc \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\
&= k \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}
\end{aligned}
$$

06-10: Determinant

- Determinant properties
  - Swapping two rows (or two colums) of a determinant reverse the sign (but preserves magnitude)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb = -(bc - ad) = -\begin{vmatrix} b & a \\ d & c \end{vmatrix}$$

06-11: Determinant

- Determinant properties
  - Swapping two rows (or two colums) of a determinant reverse the sign (but preserves magnitude)

$$\begin{aligned}
\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\
&= -\left(-a\begin{vmatrix} e & f \\ h & i \end{vmatrix} + b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} e & d \\ h & g \end{vmatrix}\right) \\
&= -\left(b\begin{vmatrix} d & f \\ g & i \end{vmatrix} - a\begin{vmatrix} e & f \\ h & i \end{vmatrix} + c\begin{vmatrix} e & d \\ h & g \end{vmatrix}\right) = -\begin{vmatrix} b & a & c \\ e & d & f \\ h & g & i \end{vmatrix}
\end{aligned}$$

06-12: Determinant

- Determinant properties
  - Other fun determinant properties
  - Take a “real” math class for more ...

06-13: Determinant: Geometry

- 2x2 determinant is signed area of parallelogram

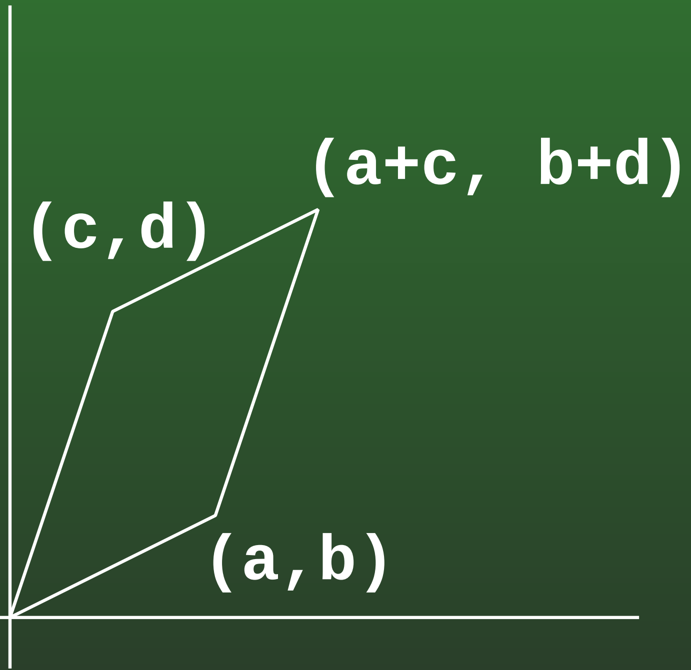
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

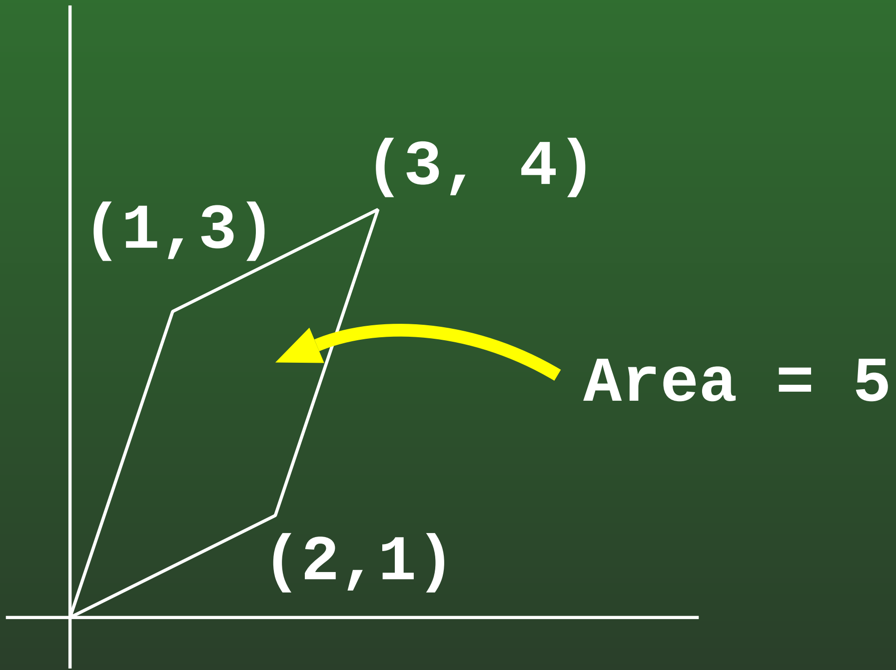
Parallelogram: $(0, 0), (a, b), (a + c, b + d), (c, d)$

Determinant: Geometry

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6 - 1 = 5$$





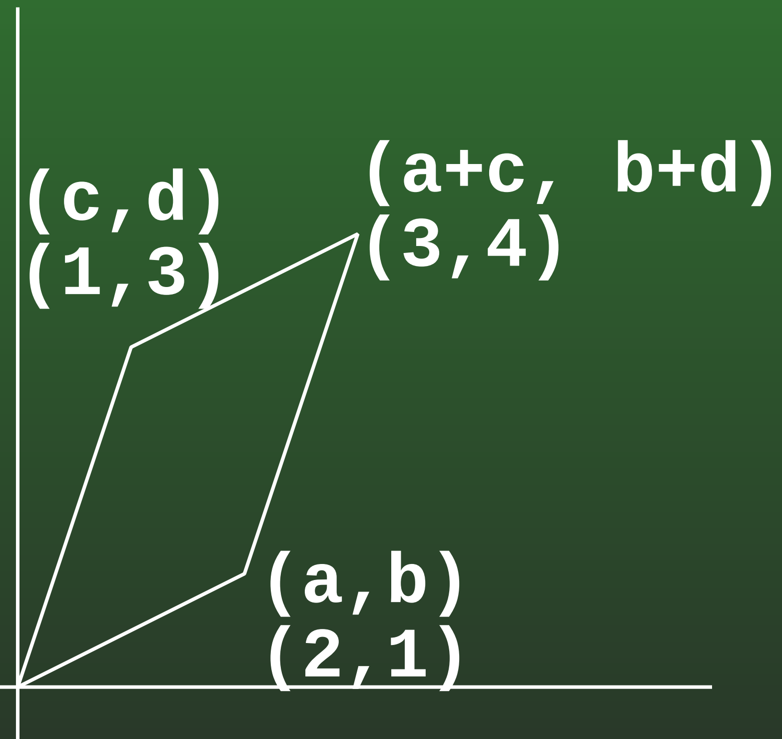
06-15: Determinant: Geometry

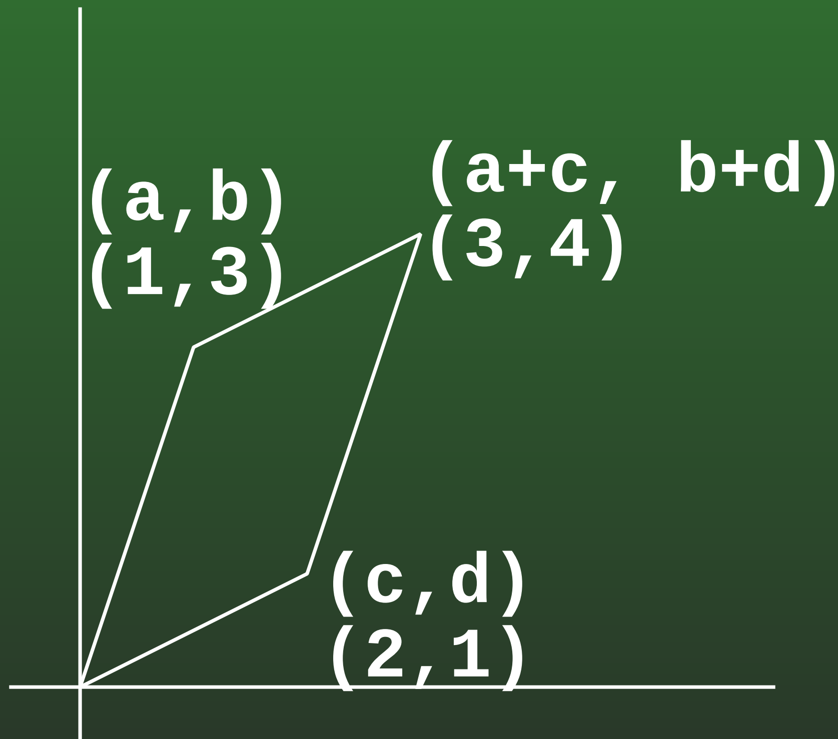
- *Signed* area
  - If the points $(0,0), (a,b), (a+c,b+d), (c,d)$ go around the parallelogram *counter-clockwise*, the area is positive
  - If the points $(0,0), (a,b), (a+c,b+d), (c,d)$ go around the parallelogram *clockwise*, the area is negative

06-16: Determinant: Geometry

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$$





Counter-Clockwise - Positive Area

Clockwise - Negative Area

06-17: Determinant: Geometry

- Compare determinant of a matrix to how matrix transforms objects
- Consider the transform matrix:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

- What happens when we transform the unit square, using this matrix?

06-18: Cube Transformation

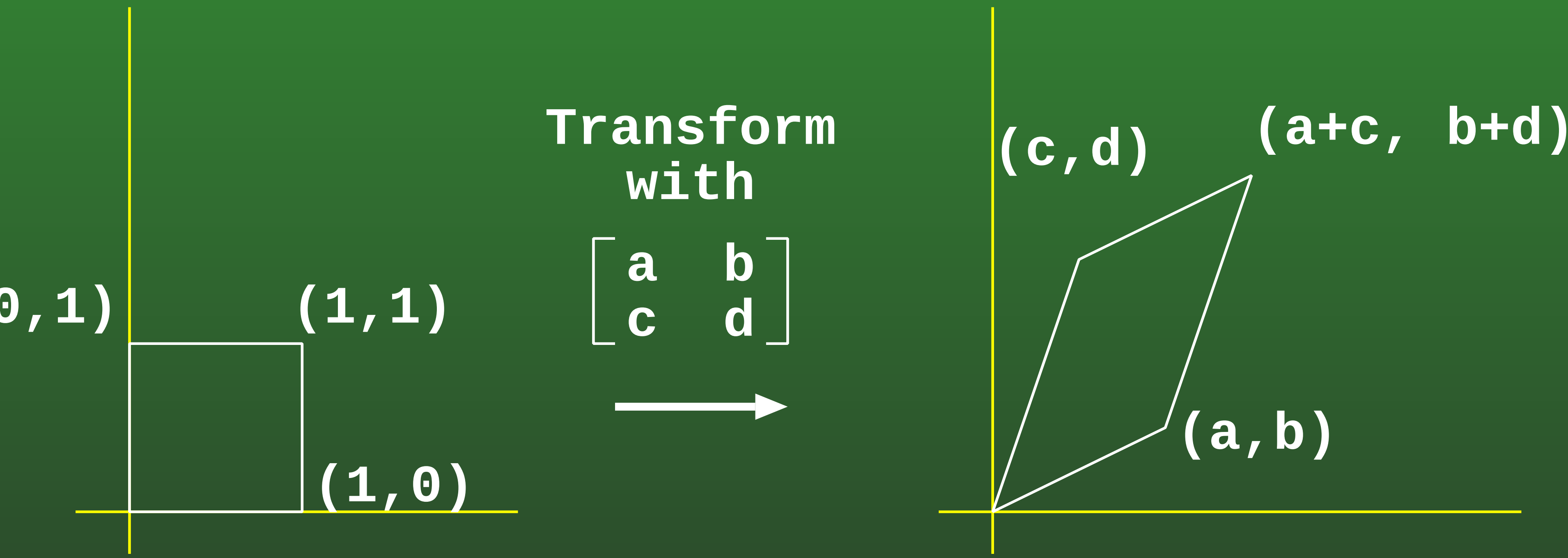
$$\begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \end{bmatrix}$$

06-19: Cube Transformation



06-20: Determinant as Scale

- Determinant gives the volume of a unit square after transformation
  - Determinant gives the scaling factor for the transformation
  - Determinant $< 1 \Rightarrow$ “shrink” object
  - Determinant $= 1 \Rightarrow$ object size (area) unchanged
  - Determinant $> 1 \Rightarrow$ “grow” object

06-21: Determinant as Scale

- Sanity check:
  - Identity transform does not change an object – determinant should be 1

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

  - Matrix that scales by factor s:

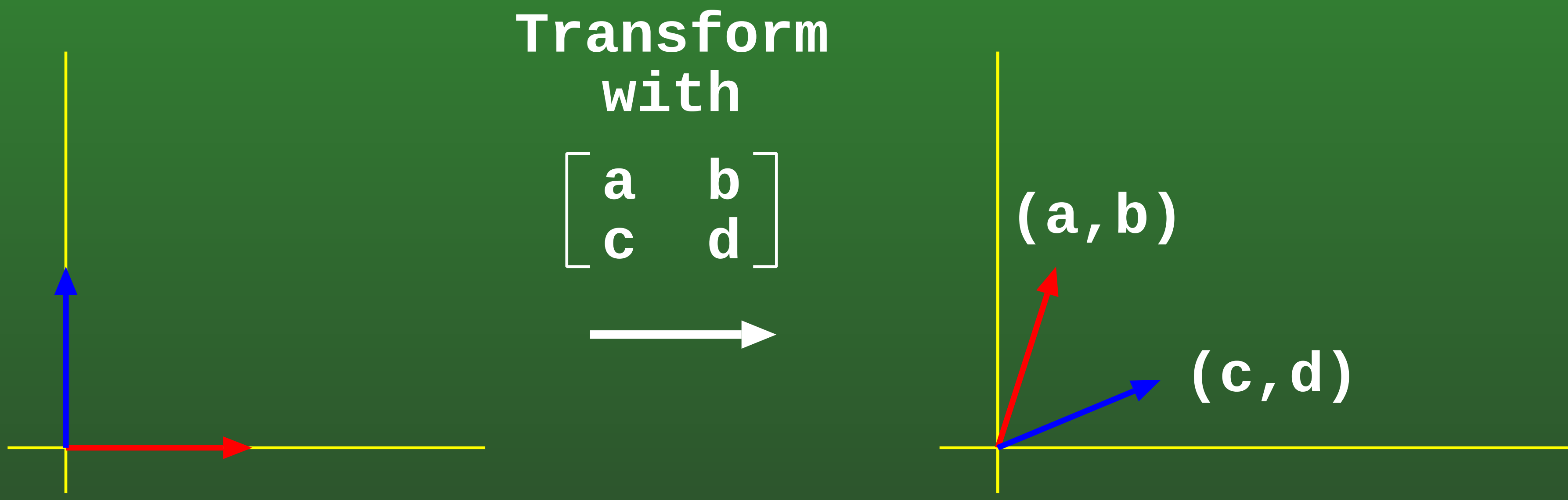
$$\begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix}$$

  This is just multiplying a column (or row!) by the scalar s – multiplies the value of the determinant by s

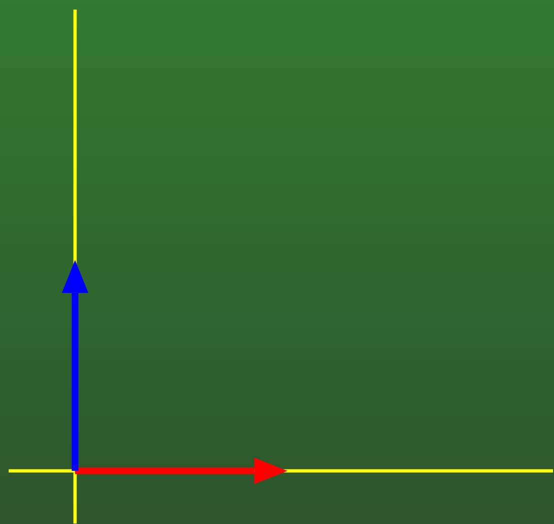
06-22: Negative Determinant

- Determinant gives *signed* area
- Pictures assume that (a, b) is clockwise of (c, d)
- What happens if (a, b) is *counterclockwise* of (c, d)?
- What do you know about the transformation?

06-23: Negative Determinant

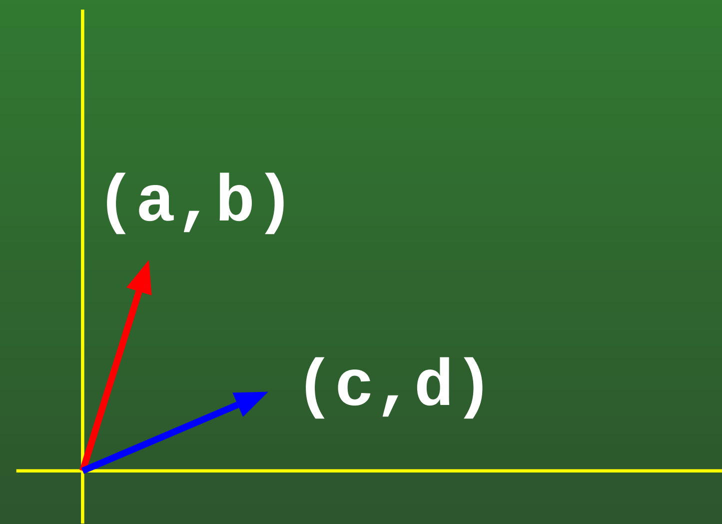


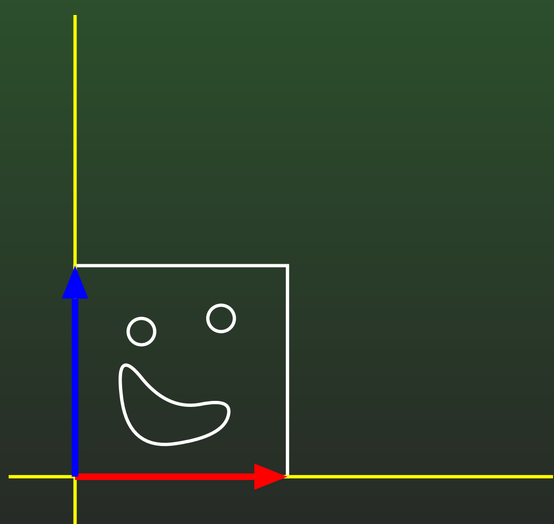
06-24: Negative Determinant



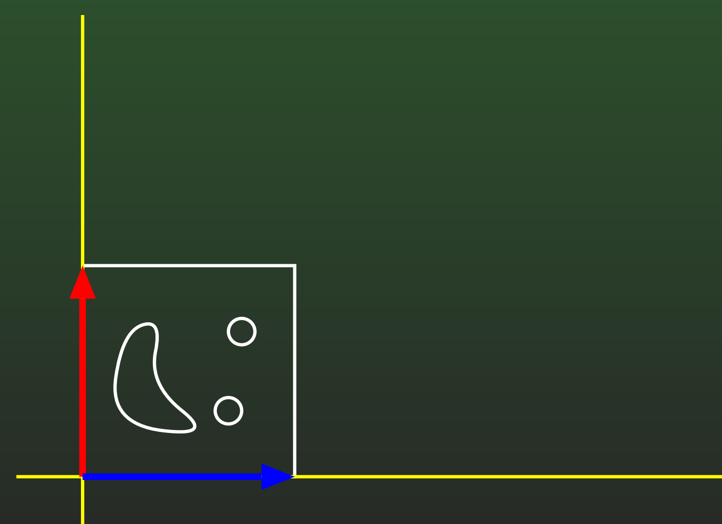
Transform with

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$





$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



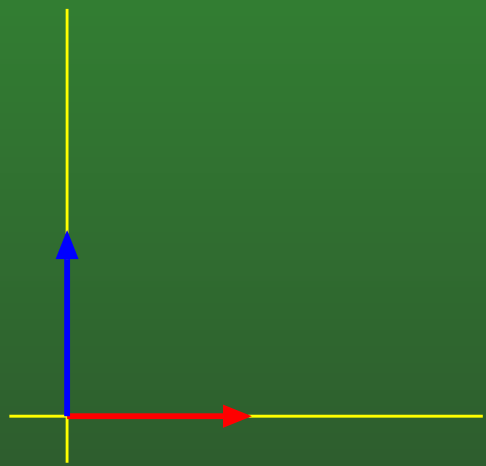
06-25: Negative Determinant

- If the determinant of a transformation matrix is negative
  - Transformation includes a reflection
- Sanity check:
  - Flip the basis vectors in a transformation matrix, cause a reflection
  - Swapping rows (or columns) in a determinant changes the sign

06-26: Negative Determinant

- If the determinant of a transformation matrix is negative
  - Transformation includes a reflection
- Sanity check II:
  - What happens to a transformation matrix when we flip a dimension?
    - That is, multiply a column (assuming row vectors) by -1
  - What does that do to the determinant?

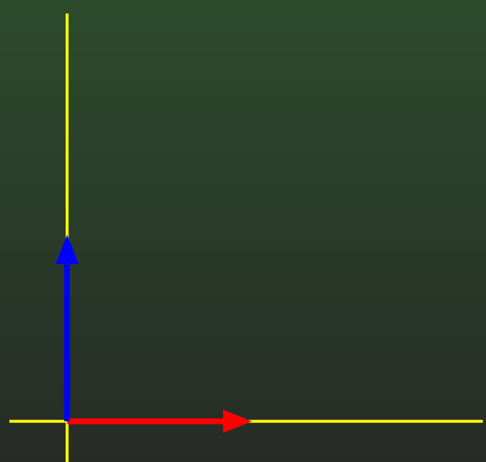
06-27: Negative Determinant



Identity
Transform

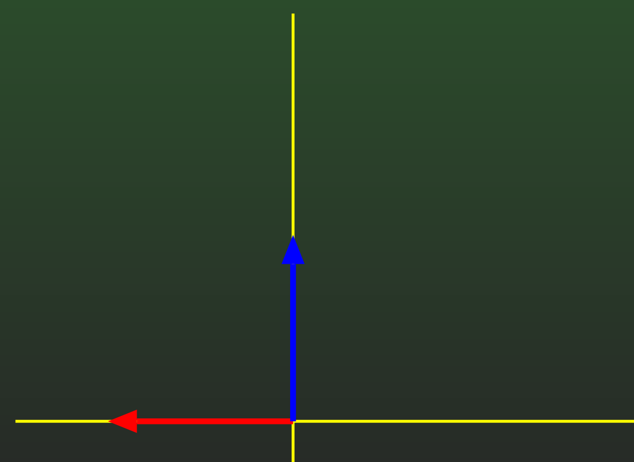
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Modify the transform by flipping the X axis
-- multiply the x column by -1



Identity
Transform

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



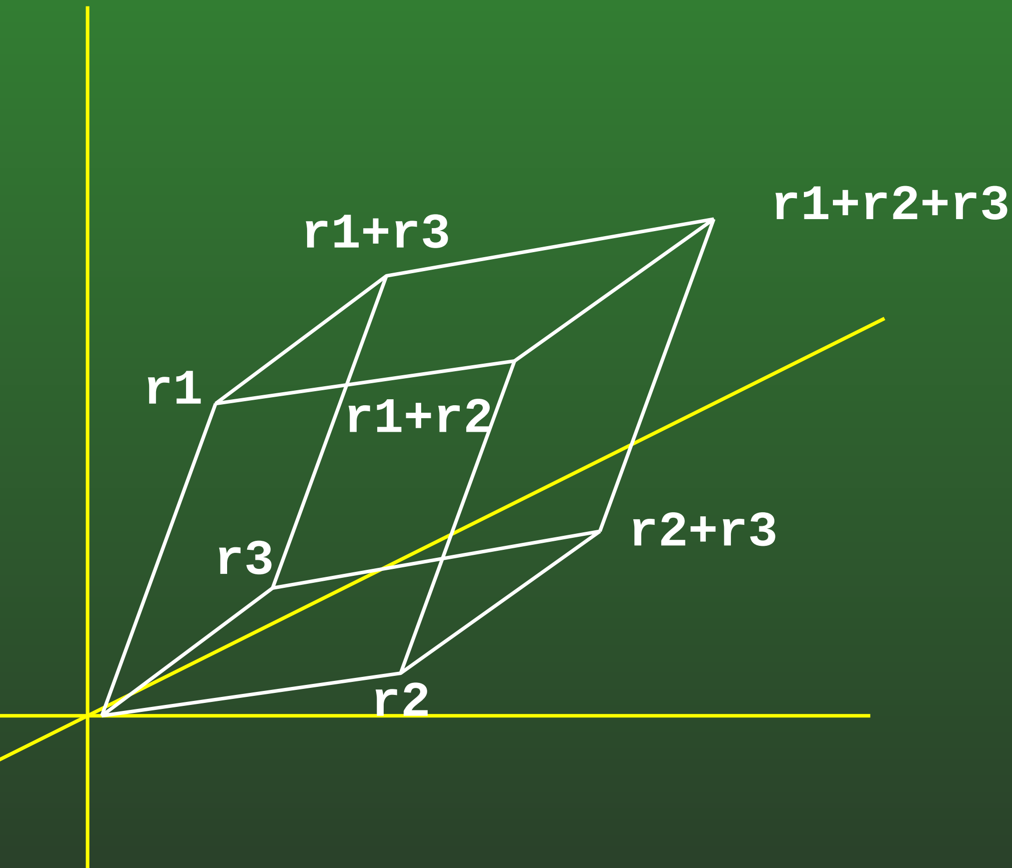
06-28: Negative Determinant

- Multiplying one column of the matrix flips that axis
- Multiplies the determinant by -1
    - Multiplying all elements of a row or column by k changes the value of the determinant by a factor of k

06-29: Determinant Geom 3D

- 3x3 determinants have the same properties as 2x2 determinants
- Determinant of a 3x3 matrix is the scale factor for how the volume of the transformed object changes
- -1 == Reflection, just like 2D case

06-30: Determinant Geom 3D



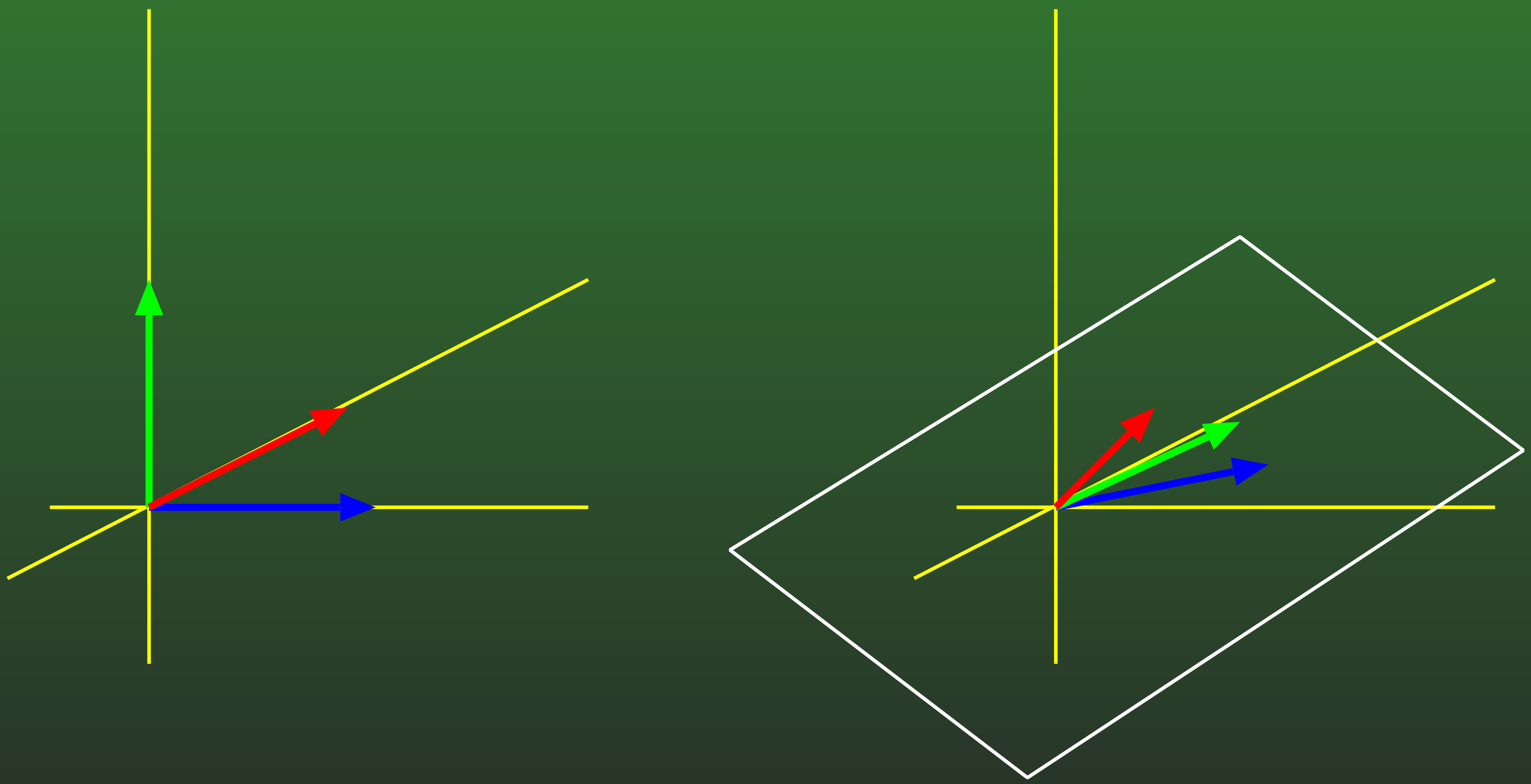
$$\begin{vmatrix} r1_x & r1_y & r1_z \\ r2_x & r2_y & r2_z \\ r3_x & r3_y & r3_z \end{vmatrix}$$

06-31: Zero Determinant

- What does it mean (geometrically) when a 3x3 matrix has a zero determinant?
- What kind of a transformation is it?

06-32: Zero Determinant

Transformation Matrix with Zero Determinant



Basis vectors transformed to same plane

06-33: Matrix Inverse

- Given a square matrix M, the inverse M^{-1} is the matrix such that
  - $\mathbf{M}\mathbf{M}^{-1} = \mathbf{I}$
  - $\mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$
- Since matrix multiplication is associative, for any vector $\mathbf{v}$:
  - $(\mathbf{v}\mathbf{M})\mathbf{M}^{-1} = \mathbf{v}$
- Matirx Inverse undoes the transformation

06-34: Matrix Inverse

- Do all square matrices have an inverse?

06-35: Matrix Inverse

- Do all square matrices have an inverse?
  - Consider:

$$\mathbf{M} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

  - What happens when we multiply $\mathbf{M}$ by a vector $\mathbf{v}$? Why can't we undo this operation?

06-36: Singular

- A matrix that has an inverse is *Non-Singular* or *Invertable*
- A matrix without an inverse is *Singlular* or *Non-Invertable*
- A matrix is singular if and only if it has a determinant of 0

06-37: Calculating Inverse

- To calculate the inverse, we will need cofactor matrix
- We've already seen cofactors (just not with that name), when calculating determinant

06-38: Cofactors

+	-	+
-	+	-
+	-	+

a	b	c
d	e	f
g	h	i

a	b	c
d	e	f
g	h	i

a	b	c
d	e	f
g	h	i

$$+\begin{vmatrix} e & f \\ h & i \end{vmatrix} \qquad - \begin{vmatrix} d & f \\ g & i \end{vmatrix} \qquad + \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

06-39: Cofactors

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$-\begin{vmatrix} b & c \\ h & i \end{vmatrix} \qquad +\begin{vmatrix} a & c \\ g & i \end{vmatrix} \qquad -\begin{vmatrix} a & b \\ g & h \end{vmatrix}$$

06-40: Cofactor Matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Cofactor Matrix

$$\begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

06-41: Calculating Inverse

- Adjoint of a matrix $\mathbf{M}$, $adj(\mathbf{M})$ is the transpose of the cofactor matrix
- The inverse of a matrix is the adjoint divided by the determinant
  - $\mathbf{M}^{-1} = \frac{adj(\mathbf{M})}{|\mathbf{M}|}$
- (can see how a matrix a determinant of 0 would have a hard time having an inverse)
- Other ways of computing inverses (i.e. Gaussian elimination)
  - Fewer arithmetic operations
  - Algoritmns require branches (expensive!)
  - Vector hardware makes adjoint method fast

06-42: Inverse Example

First, Calculate determinant

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 1 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = -3$$

06-43: Inverse Example

Next, calculate cofactor matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} \end{bmatrix}$$

06-44: Inverse Example

Next, calculate cofactor matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Cofactor matrix:

$$\begin{bmatrix} 3 & -3 & 0 \\ -3 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

06-45: Inverse Example

Transpose the cofactor array to get the adjoint (in this example the adjoint is equal to its transpose, but that doesn't always happen)

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Adjoint:

$$\begin{bmatrix} 3 & -3 & 0 \\ -3 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

06-46: Inverse Example

Finally, divide adjoint by the determinant to get the inverse:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Inverse:

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

06-47: Inverse Example

Sanity check

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

06-48: Orthogonal Matrices

- A matrix M is orthogonal if:
  - $\mathrm{MM}^T = \mathrm{I}$
  - $\mathrm{M}^T = \mathrm{M}^{-1}$
- Orthogonal matrices are handy, because they are easy to invert
- Is there a geometric interpretation of orthogonality?

06-49: Orthogonal Matrices

$$\mathbf{M}\mathbf{M}^T = \mathbf{I}$$

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} m_{11} & m_{21} & m_{31} \\ m_{12} & m_{22} & m_{32} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Do all the multiplications ...

06-50: Orthogonal Matrices

$$
\begin{aligned}
m_{11}m_{11} + m_{12}m_{12} + m_{13}m_{13} &= 1 \\
m_{11}m_{21} + m_{12}m_{22} + m_{13}m_{23} &= 0 \\
m_{11}m_{31} + m_{12}m_{32} + m_{13}m_{33} &= 0 \\
m_{21}m_{11} + m_{22}m_{12} + m_{23}m_{13} &= 0 \\
m_{21}m_{21} + m_{22}m_{22} + m_{23}m_{23} &= 1 \\
m_{21}m_{31} + m_{22}m_{32} + m_{23}m_{33} &= 0 \\
m_{31}m_{11} + m_{32}m_{12} + m_{33}m_{13} &= 0 \\
m_{31}m_{21} + m_{32}m_{22} + m_{33}m_{23} &= 0 \\
m_{31}m_{31} + m_{32}m_{32} + m_{33}m_{33} &= 1
\end{aligned}
$$

- Hmmm... that doesn’t seem to help much

06-51: Orthogonal Matrices

- Recall that rows of matrix are basis after rotation
  - $\mathbf{v}_x = [m_{11}, m_{12}, m_{13}]$
  - $\mathbf{v}_y = [m_{21}, m_{22}, m_{23}]$
  - $\mathbf{v}_z = [m_{31}, m_{32}, m_{33}]$
- Let's rewrite the previous equations in terms of $\mathbf{v}_x$, $\mathbf{v}_y$, and $\mathbf{v}_z$ ...

06-52: Orthogonal Matrices

$$
\begin{aligned}
m_{11}m_{11} + m_{12}m_{12} + m_{13}m_{13} &= 1 \quad \mathbf{v}_x \cdot \mathbf{v}_x = 1 \\
m_{11}m_{21} + m_{12}m_{22} + m_{13}m_{23} &= 0 \quad \mathbf{v}_x \cdot \mathbf{v}_y = 0 \\
m_{11}m_{31} + m_{12}m_{32} + m_{13}m_{33} &= 0 \quad \mathbf{v}_x \cdot \mathbf{v}_z = 0 \\
m_{21}m_{11} + m_{22}m_{12} + m_{23}m_{13} &= 0 \quad \mathbf{v}_y \cdot \mathbf{v}_x = 0 \\
m_{21}m_{21} + m_{22}m_{22} + m_{23}m_{23} &= 1 \quad \mathbf{v}_y \cdot \mathbf{v}_y = 1 \\
m_{21}m_{31} + m_{22}m_{32} + m_{23}m_{33} &= 0 \quad \mathbf{v}_y \cdot \mathbf{v}_z = 0 \\
m_{31}m_{11} + m_{32}m_{12} + m_{33}m_{13} &= 0 \quad \mathbf{v}_z \cdot \mathbf{v}_x = 0 \\
m_{31}m_{21} + m_{32}m_{22} + m_{33}m_{23} &= 0 \quad \mathbf{v}_z \cdot \mathbf{v}_y = 0 \\
m_{31}m_{31} + m_{32}m_{32} + m_{33}m_{33} &= 1 \quad \mathbf{v}_z \cdot \mathbf{v}_z = 1
\end{aligned}
$$

06-53: Orthogonal Matrices

- What does it mean if $\mathbf{u} \cdot \mathbf{v} = 0$?
  - (assuming both $\mathbf{u}$ and $\mathbf{v}$ are non-zero)
- What does it mean if $\mathbf{v} \cdot \mathbf{v} = 1$?

06-54: Orthogonal Matrices

- What does it mean if $\mathbf{u} \cdot \mathbf{v} = 0$?
  - (assuming both $\mathbf{u}$ and $\mathbf{v}$ are non-zero)
  - $\mathbf{u}$ and $\mathbf{v}$ are perpendicular to each other (orthogonal)
- What does it mean if $\mathbf{v} \cdot \mathbf{v} = 1$?
  - $||\mathbf{v}|| = 1$
- So, transformed basis vectors must be mutually perpendicular unit vectors

06-55: Orthogonal Matrices

- If a transformation matrix is orthogonal,
  - Transformed basis vectors are mutually perpendicular unit vectors
- What kind of transformations are done by orthogonal matrices?

06-56: Orthogonal Matrices

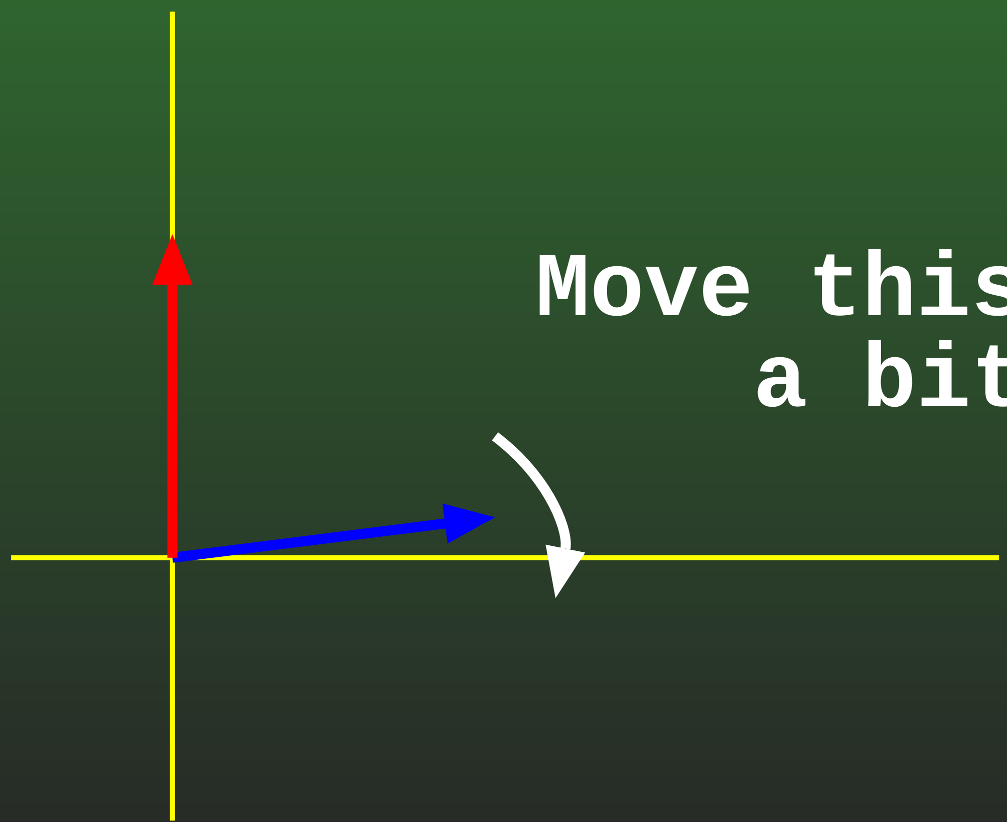
- If a transformation matrix is orthogonal,
  - Transformed basis vectors are mutually perpendicular unit vectors
- What kind of transformations are done by orthogonal matrices?
  - Rotations & Reflections

06-57: Orthogonalizing a Matrix

- It is possible to have a matrix that *should* be orthogonal that is *not quite* orthogonal
  - Bad Data
  - Accumulated floating point error (matrix creep)
- If a matrix is just slightly non-orthogonal, we can modify it to be orthogonal

 # Orthogonalizing a Matrix

Not quite orthogonal

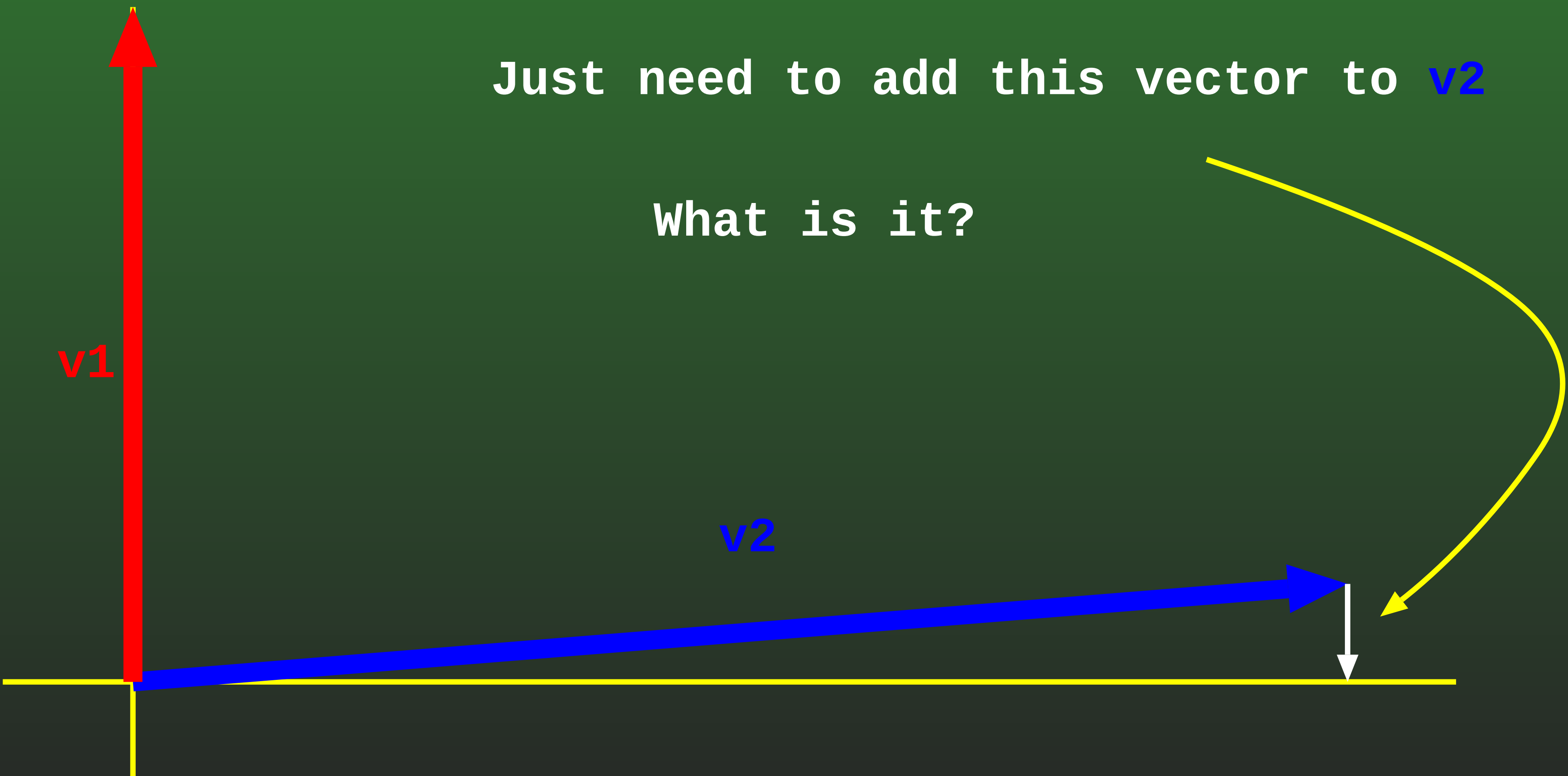


Move this vector down a bit

06-59: Orthogonalizing a Matrix

Just need to add this vector to v2

What is it?



06-60: Orthogonalizing a Matrix

- Given two vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ that are *nearly* orthogonal, we subtract from $\mathbf{v_2}$ the component of $\mathbf{v_2}$ that is parallel to $\mathbf{v_1}$

$$\begin{aligned}\mathbf{v_2} &= \mathbf{v_2} - \frac{\mathbf{v_1} \cdot \mathbf{v_2}}{||\mathbf{v_1}||^2}\mathbf{v_1} \\ &= \mathbf{v_2} - \frac{\mathbf{v_1} \cdot \mathbf{v_2}}{\mathbf{v_1} \cdot \mathbf{v_1}}\mathbf{v_1}\end{aligned}$$

06-61: Orthogonalizing a Matrix

- We can easily extend this to 3 dimensions:
  - Leave the first vector alone
  - Tweak the second vector to be perpendicular to the first vector
  - Tweak the third vector to be perpendicular to first two vectors

06-62: Orthogonalizing a Matrix

- Given a *nearly* ortrhogonal matrix with rows r_1, r_2, and r_3:

$$
\begin{aligned}
\mathbf{r}'_1 &= \mathbf{r}_1 \\
\mathbf{r}'_2 &= \mathbf{r}_2 - \frac{\mathbf{r}_2 \cdot \mathbf{r}'_1}{\mathbf{r}'_1 \cdot \mathbf{r}'_1}\mathbf{r}'_1 \\
\mathbf{r}'_3 &= \mathbf{r}_3 - \frac{\mathbf{r}_3 \cdot \mathbf{r}'_1}{\mathbf{r}'_1 \cdot \mathbf{r}'_1}\mathbf{r}'_1 - \frac{\mathbf{r}_3 \cdot \mathbf{r}'_2}{\mathbf{r}'_2 \cdot \mathbf{r}'_2}\mathbf{r}'_2
\end{aligned}
$$

- Need to normalize $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$
- If we normalize $\mathbf{r}'_1$ before calculating $\mathbf{r}'_2$, then $\mathbf{r}'_1 \cdot \mathbf{r}'_1 = 1$, and we can remove a division

06-63: Orthogonalizing a Matrix

- The problem with orthogonalizing a matrix this way is that there is a bias
  - First row never changes
  - Third row changes the most
- What if we don't want a bias?

06-64: Orthogonalizing a Matrix

- The problem with orthogonalizing a matrix this way is that there is a bias
  - First row never changes
  - Third row changes the most
- What if we don't want a bias?
  - Change each vector a little bit in the correct direction
  - Repeat until you get close enough
  - Then run the "standard" method

06-65: Orthogonalizing a Matrix

$$
\begin{aligned}
\mathbf{r}_1' &= \mathbf{r}_1 - k\frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{\mathbf{r}_2 \cdot \mathbf{r}_2}\mathbf{r}_2 - k\frac{\mathbf{r}_1 \cdot \mathbf{r}_3}{\mathbf{r}_3 \cdot \mathbf{r}_3}\mathbf{r}_3 \\
\mathbf{r}_2' &= \mathbf{r}_2 - k\frac{\mathbf{r}_2 \cdot \mathbf{r}_1}{\mathbf{r}_1 \cdot \mathbf{r}_1}\mathbf{r}_1 - k\frac{\mathbf{r}_2 \cdot \mathbf{r}_3}{\mathbf{r}_3 \cdot \mathbf{r}_3}\mathbf{r}_3 \\
\mathbf{r}_3' &= \mathbf{r}_3 - k\frac{\mathbf{r}_3 \cdot \mathbf{r}_1}{\mathbf{r}_1 \cdot \mathbf{r}_1}\mathbf{r}_1 - k\frac{\mathbf{r}_3 \cdot \mathbf{r}_2}{\mathbf{r}_2 \cdot \mathbf{r}_2}\mathbf{r}_2
\end{aligned}
$$

- Do several iterations (smallish k)
- Not guaranteed to get exact – run standard method when done