04-0: **Right-Handed vs. Left-Handed**

- Hold out your left hand (really, do it!):
  - Thumb to the right
  - Index finder up
  - Middle finger straight ahead
- This forms a basis for a 3D coordinate system

04-1: **Right-Handed vs. Left-Handed**

- Hold out your left hand (really, do it!):
  - Thumb to the right (+ x)
  - Index finder up (+ y)
  - Middle finger straight ahead (+ z)
- This forms a basis for a 3D coordinate system – Left-Handed Coordinate system

04-2: **Right-Handed vs. Left-Handed**

- Now, Hold out your *right* hand (yes, really do it!):
  - Thumb to the left (+ x)
  - Index finder up (+ y)
  - Middle finger straight ahead (+ z)
- This forms the other basis for 3D coordinate system – Right-Handed Coordinate system

04-3: **Right-Handed vs. Left-Handed**

- Any basis can be rotated to be either left-handed or right-handed
- Swap between systems by flipping any one axis
- Flipping *two* axes leaves handedness unchanged (why?)
  - What about flipping all 3?

04-4: **Right-Handed vs. Left-Handed**

- Computer Graphics typically uses Left-Handed coordinate system
  - Book does, too
- "Pure" linear algebra often uses Right-Handed coordiate system
  - Ogre also uses a Right-Handed coordinate system
  - Easy transformation, just invert the sign of one axis

04-5: **Multiple Cooridinate Systems**

- OK, so we've decided on a right-handed coordinate system (given Ogre), with y pointing "Up"

- Pick an arbitrary location for the origin
  - Often in the middle of the world
  - Can place it off in some corner
- Not quite done – can use multiple coordinate systems!

04-6: **Multiple Cooridinate Systems**

- World Space
- Object Space
- Camera Space (Special case of Object Space)
- Intertial Space

04-7: **World Space**

- Assume that the origin of the world is the middle of the field between SI and K Hall
  - 2130 Fulton, the official University address is there
  - +x is East (along Fulton), +y is straight up, +z is North
- What direction is "forward" from me in world space?
- What is the point 5 feet in front of me in world space?
  - What if I rotate 15 deg. to the left?

04-8: **Object Space**

- Define a new coordinate system
- Origin is at my center
- +x to my right
- +y is up through my head
- +z is straight ahead

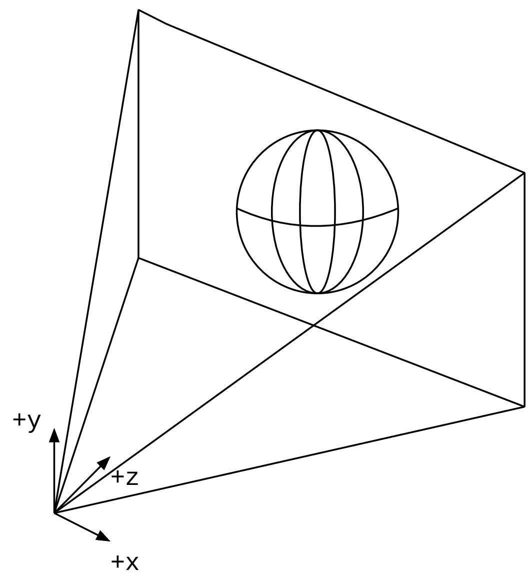
04-9: **Object Space**

- In my object space, finding a point right ahead of me is trivial
- Given a coordinate in my object space, determining where I have to look (to aim, for instance) is trivial
- Of course, we will need a way to translate between world space and object space
  - Say tuned!
- Define an "Object Space" for each object in our world

04-10: **Camera Space**

- Camera Space is a special case of object space
  - Object is the camera

- We'll use left-handed coordinates (+z into the screen), swapping to right-hand is easy (invert Z)
- Why is camera space useful?



04-11: **Camera Space**

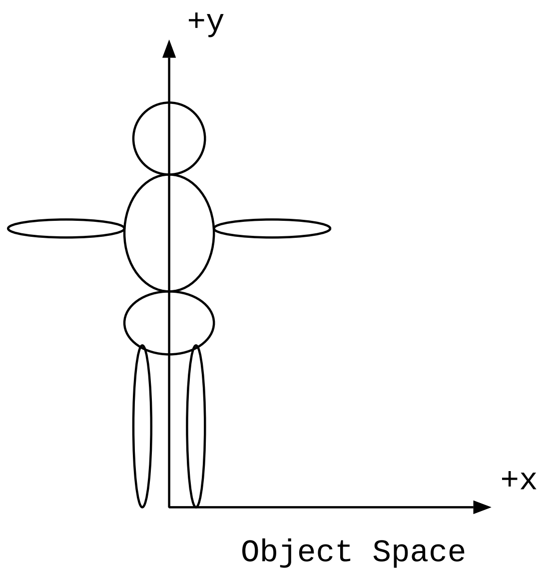
04-12: **Camera Space**

- Is an object within the camera's frustum?
- Is object A in front of object B, or vice-versa?
- Is an object close enough to the camera to render?
- ... etc

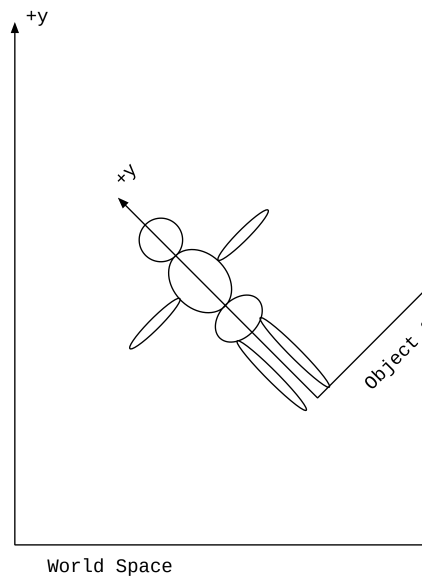
04-13: **Intertial Space**

- Halfway betwen object space and world space
- Axes parallel to world space
- Origin same as object space

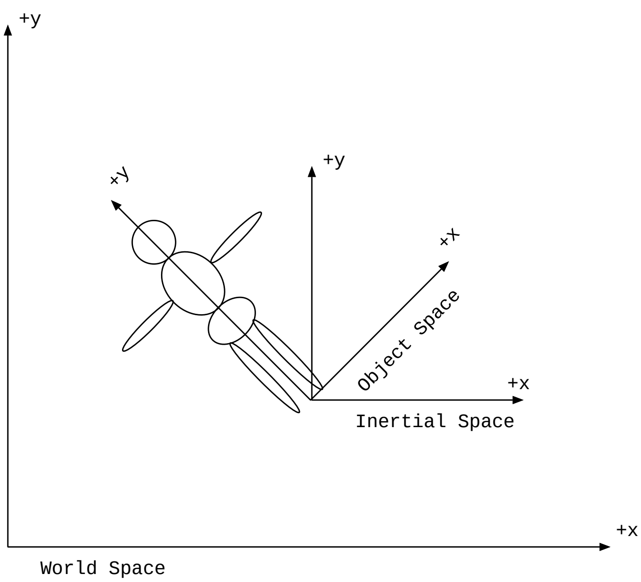
04-14: **Inertial Space**



04-15: **Inertial Space**



04-16: **Inertial Space**

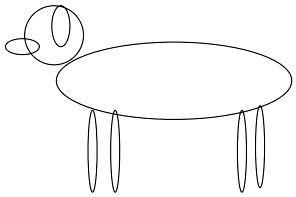


04-17: **Nested Coordinate Spaces**

- Each object needs to be oriented in world space
- That is, the axes for the local space of the object need to be oriented in world space.
- We could use a different object's local space instead of global space
  - Easiest to see with an example

04-18: **Nested Coordinate Spaces**

- Assume that we have a dog, which has a head and ears
  - The head can wag back and forth (in relation to the body)
  - The ears can flap up and down (in relation to the head)



- We don’t want to decribe the position of the ears in world space, or even in dog space
    - Head space is much more convienent

04-19: **Nested Coordinate Spaces**

- Dog’s ears are described in head space
    - Up and down in relation to the head
- Dog’s head is described in dog space
    - Back and forth in relation to the dog
- Dog’s position is described in world space

04-20: **Nested Coordinate Spaces**

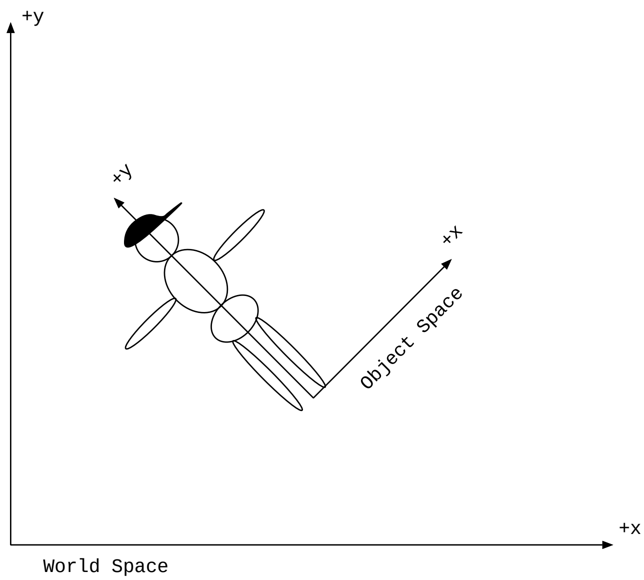
- To render the dog (with ears!)
    - Translate the ear location from head space to dog space
    - Translate the ear location (and the head location) from do space to world space
    - Translate all the dog from world space to camera space
    - Project the objects from 3-space to a plane

04-21: **Nested Coordinate Spaces**

- The Head space is a child of the Dog space
    - The Dog space is the parent of the Head space
- The Ear space is a child of the head space
    - The Head space is the parent of the ear space
- We could also dynamically parent and unparent objects

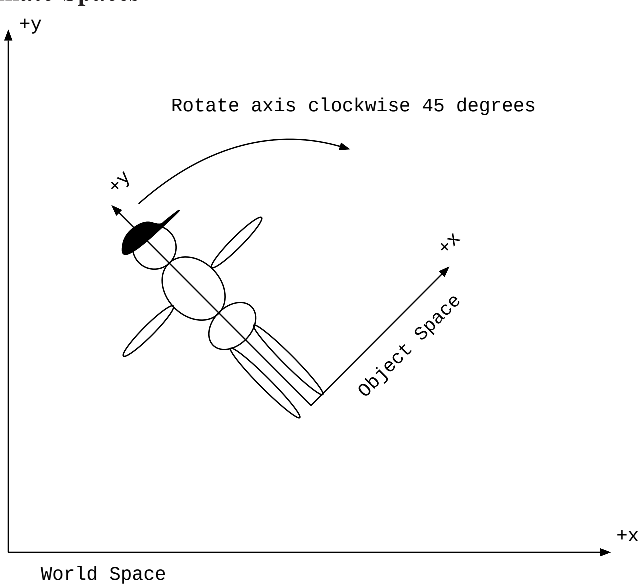
04-22: **Changing Coordinate Spaces**

- Our character is wearing a red hat
- The hat is at position (0,100) in object space
- What is the position of the hat in world space?
- To make life easier, we will think about rotating the axes, instead of moving the objects

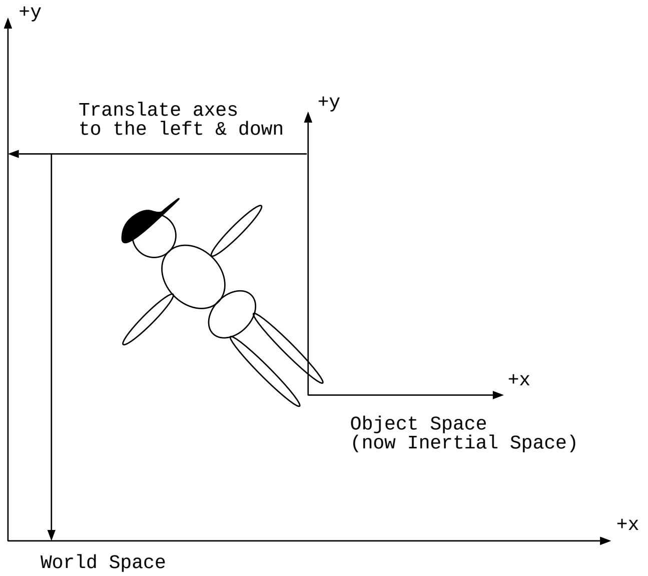


04-23: **Changing Coordinate Spaces**

04-24: **Changing Coordinate Spaces**

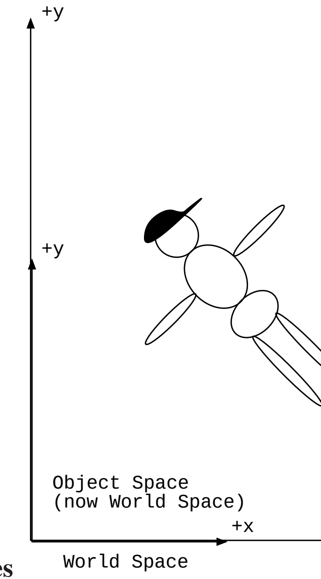


04-25: **Changing Coordinate**



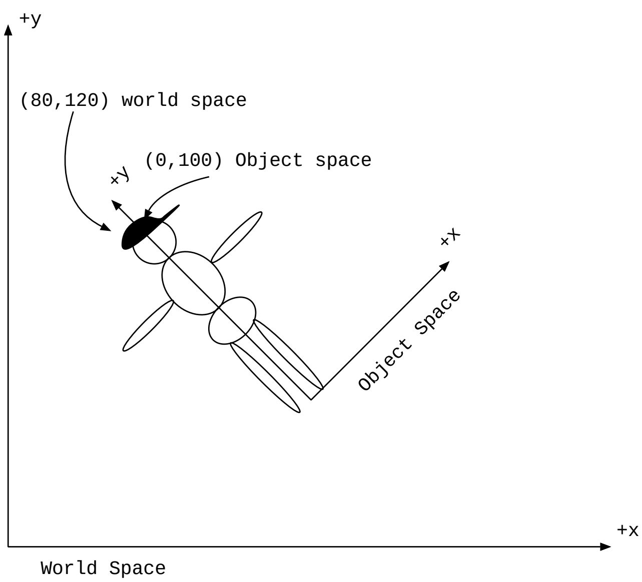
Spaces

04-26: **Changing Coordinate Spaces**



04-27: **Changing Coordinate Spaces**

- Rotate axes to the right 45 degrees
  - Hat rotates the the left 45 degrees, from (0,100) to (-70, 70)
- Translate axes to the left 150, and down 50
  - Hat rotates to the right 150 and up 50, to (80, 120)
- We'll see how to do those rotations using matrices later ...



04-28: **Changing Coordinate Spaces**

04-29: **Back to Basics**

- A Vector is a displacement
- Vector has both *direction* and *length*

- Can also think of a vector as a position (just a displacement from the origin)
- Can be written as a row or column vector
  - Differnence can be important for multiplication

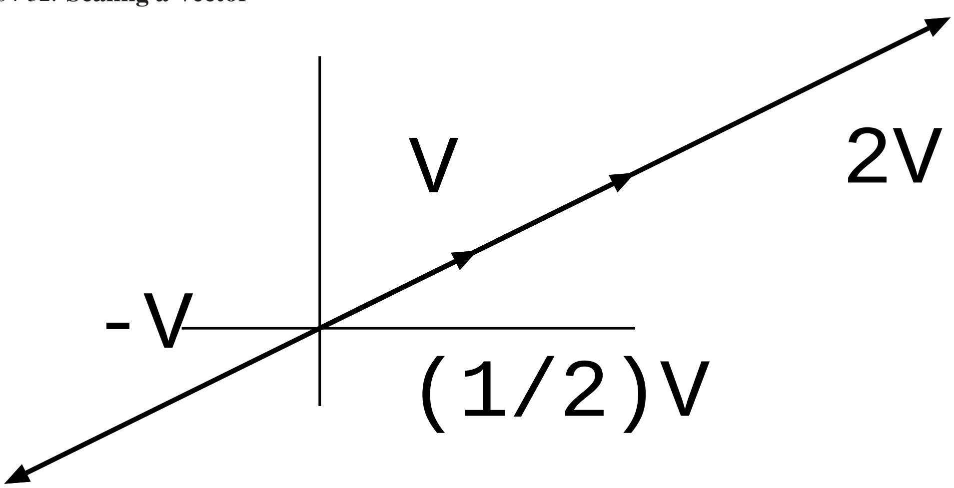
04-30: **Vector Operations**

- Multiplying by a scalar
  - To multiply a vector $\mathbf{v}$ by a scalar s, multiply each component of the vector by s
  - Effect is scaling the vector – multiplying by 2 maintains the direction of the vector, but makes the length twice as long
  - Works the same for 2D and 3D vectors (and highter dimemsion vectors, too, for that matter)

04-31: **Vector Operations**

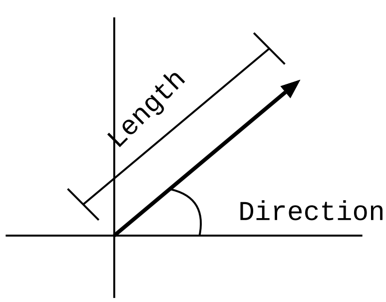
- Multiplying by a scalar
  - Multiplying a vector by -1 flips the direction of the vector
  - Works for 2D and 3D
  - Multiplying a vector by -2 both flips the direction, and scales the vector

04-32: **Scaling a Vector**



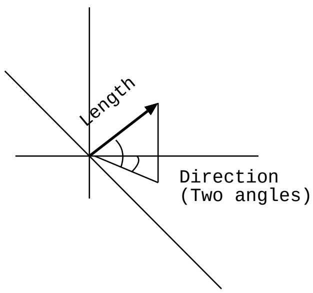
04-33: **Length**

- Vector has both direction and length



04-34: **Length**

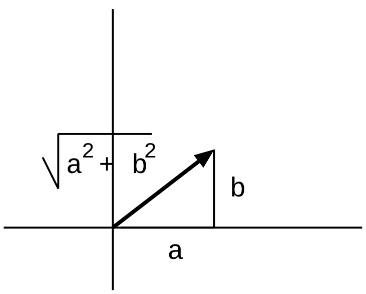
- Vector has both direction and length

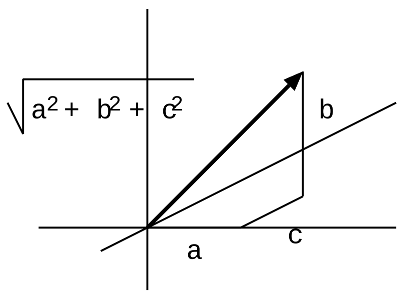


04-35: **Length**

- Vector $\mathbf{v} = [v_1, v_2, \ldots v_n]$
- Length of **v**:

$$||\mathbf{v}|| = \sqrt{\sum_{i=1}^{n} v_i^2}$$



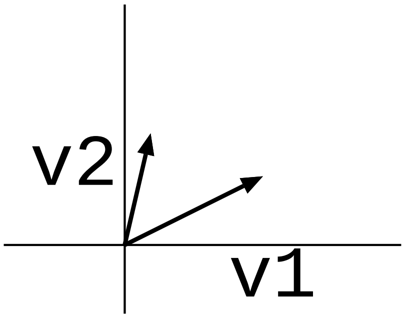


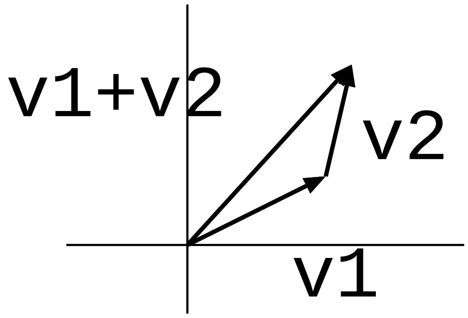
04-36: **Normalizing a Vector**

- Normalize a vector by setting its length to 1, but maintining its direction.
- Multiply by 1/length
- $\mathbf{v}_{norm} = \frac{\mathbf{v}}{||\mathbf{v}||}$
- Of course, v can't be the zero vector
  - Zero vector is the only vector without a direction

04-37: **Vector Addition**

- Add two vectors by adding their components
- $[u_1, u_2, u_3] + [v_1, v_2, v_3] = [u_1 + v_1, u_2 + v_2, u_3 + v + 3]$

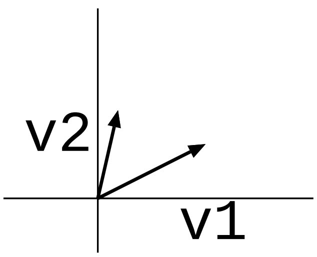


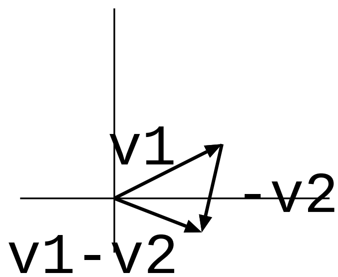


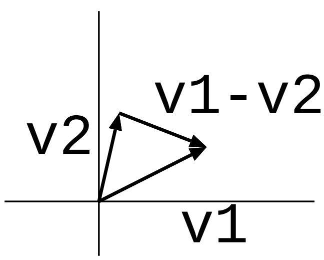
04-38: **Vector Subtraction**

- Vector subtraction is the same as multiplying by -1 and adding
- $\mathbf{v_1} - \mathbf{v_2}$ is the displacement from the point at $\mathbf{v_2}$ to the point at $\mathbf{v_1}$
    - *not* the displacement from $\mathbf{v_1}$ to $\mathbf{v_2}$

04-39: **Vector Subtraction**







04-40: **Point Distance**

- We can use subtraction and length to find the distance between two points
- Represent points as vectors – displacement from the origin
- Distance from $\mathbf{v}$ to $\mathbf{u}$ is $||\mathbf{v} - \mathbf{u}|| = ||\mathbf{u} - \mathbf{v}||$
    - Where $||\mathbf{v}||$ is the length of the vector $\mathbf{v}$.

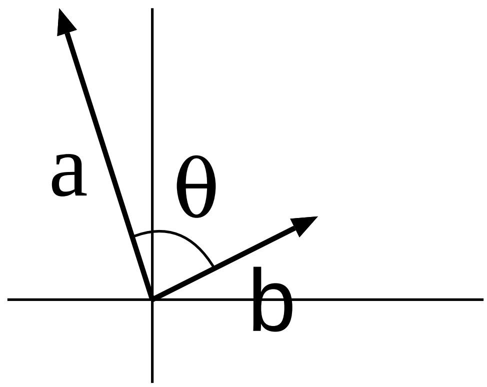
04-41: **Dot Product**

- $a = [a_1, a_2, \ldots, a_n]$
- $b = [b_1, b_2, \ldots, b_n]$
- $a \cdot b = \sum_{i=1}^{n} a_i b_i$

- $v_1 = [x_1, y_1, z_1], v_2 = [x_2, y_2, z_2]$
- $v_1 \cdot v_2 = x_1 x_2 + y_1 y_2 + z_1 z_2$

04-42: **Dot Product**

$$a \cdot b = ||a|| * ||b|| * \cos\theta$$



04-43: **Dot Product**

$$\theta = \arccos\left(\frac{a \cdot b}{||a||||b||}\right)$$

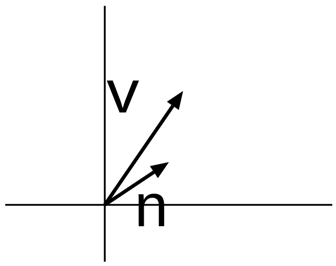
If a and b are unit vectors:

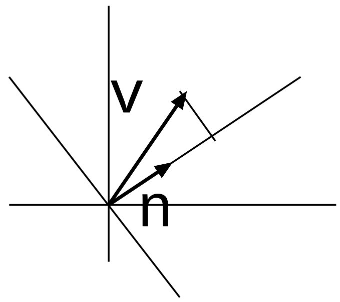
$$\theta = \arccos\left(a \cdot b\right)$$

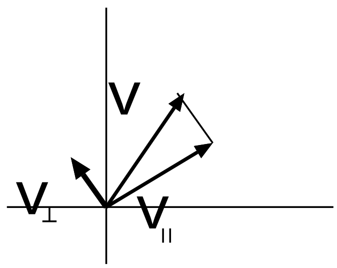
04-44: **Dot Product**

- If we don't need the exact angle, we can just use the sign
  - If $\theta < 90$, $\cos\theta > 0$
  - If $\theta = 90$, $\cos\theta = 0$
  - If $90 < \theta < 180$, $\cos\theta < 0$
- Since $a \cdot b = ||a||||b|| \cos\theta$:
  - If $a \cdot b > 0$, $\theta < 90(\frac{\pi}{2})$
  - If $a \cdot b = 0$, $\theta = 90(\frac{\pi}{2})$
  - If $a \cdot b < 0$, $90 < \theta < 180$
    - $\frac{\pi}{2} < \theta < \pi$

04-45: **Projecting Vectors**







04-46: **Projecting Vectors**

- Given a vector v and n, we want to decompose v into two vectors, $v_{\|}$ (parallel to n) and $v_{\perp}$ (perpendicular to n)
- $v_{\|} = n\frac{||v_{\|}||}{||n||}$
  - So all we need is $||v_{\|}||$

$$\begin{aligned}
\cos\theta &= \frac{||v_{\|}||}{||v||} \\
||v_{\|}|| &= \cos\theta ||v||
\end{aligned}$$

04-47: **Projecting Vectors**

- $v_{\|} = n\frac{||v_{\|}||}{||n||}$
- $||v_{\|}|| = \cos\theta ||v||$

$$\begin{aligned}
v_{\|} &= n\frac{||v_{\|}||}{||n||} \\
&= n\frac{\cos\theta ||v||}{||n||} \\
&= n\frac{\cos\theta ||v|| ||n||}{||n||^2} \\
&= n\frac{v \cdot n}{||n||^2}
\end{aligned}$$

04-48: **Projecting Vectors**

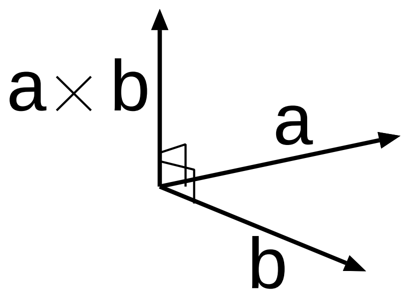
- Once we have $v_{\|}$, finding $v_{\perp}$ is easy, since $v = v_{\|} + v_{\perp}$

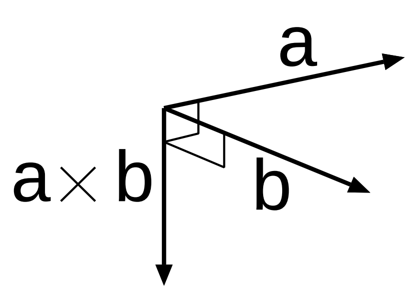
$$\begin{aligned} v_{\parallel} + v_{\perp} &= v \\ v_{\perp} &= v - v_{\parallel} \\ v_{\perp} &= v - n\frac{v \cdot n}{||n||^2} \end{aligned}$$

04-49: **Cross Product**

- $v_1 = [x_1, y_1, z_1]$, $v_2 = [x_2, y_2, z_2]$
- $v_1 \times v_2 = [y_1 z_2 - z_1 y_2, z_1 x_2 - x_1 z_2, x_1 y_2 - y_1 x_2]$
- Cross product of two vectors is a new vector perpendicular to the other two vectors

04-50: **Cross Product**



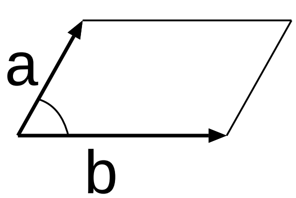


04-51: **Cross Product**

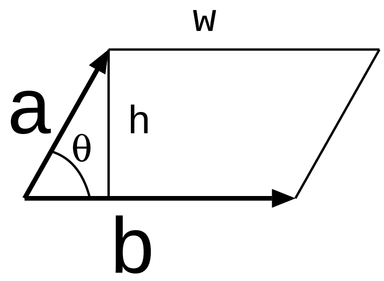
- Which way does the cross product $a \times b$ point?
  - It depends upon your coordinate system – right-handed vs. left-handed
- For right-handed coordinate systems, take your right hand, move your fingers from a to b – thumb points along $a \times b$
- For left-handed coordinate systems, take your right hand, move your fingers from a to b – thumb points along $a \times b$

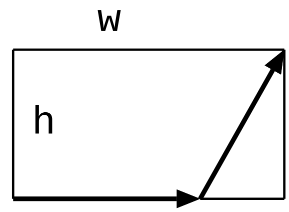
04-52: **Cross Product**

- Magnitude of cross product:
  - $||a \times b|| = ||a||||b|| \sin\theta$
- Same as the area of the parallelogram defined by a and b



04-53: **Cross Product**





- Area of parallelogram = $w * h$
- $w = ||b||$
- $\sin\theta = h/||a||$, $h = ||a|| \sin\theta$
- $w * h = ||a|| ||b|| \sin\theta = ||a \times b||$

04-54: **Matrices**

- A 4x3 matrix M:

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \end{bmatrix}$$

04-55: **Matrices**

- A Square matrix has the same width and height

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

- A diagonal matrix is a square matrix with non-diagonal elements equal to zero

$$\mathbf{M} = \begin{bmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{bmatrix}$$

04-56: **Matrices**

- The *Identity Matrix* is a diagonal matrix with all diagonal elements = 1

$$\mathbf{I_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

04-57: **Matrices**

- Matrices and vectors

- Vectors are a special case of matrices
- Row vectors (as we've seen so far) $[x, y, z]$
- Column vectors : $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

04-58: **Matrices**

- Transpose
  - Written $\mathbf{M^T}$
  - Exchange rows and colums

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \\ j & k & l \end{bmatrix}^T = \begin{bmatrix} a & d & g & j \\ b & e & h & k \\ c & f & i & l \end{bmatrix}$$

04-59: **Transpose**

- The transpose of a row vector is a column vector
- For any matrix M, $(M^T)^T = M$
- For a diagonal matrix D, $D^T = ?$

04-60: **Transpose**

- The transpose of a row vector is a column vector
- For any matrix M, $(M^T)^T = M$
- For a diagonal matrix D, $D^T = D$
  - True for any matrix that is symmetric along the diagonal

04-61: **Matrix Multiplication**

- Multiplying a Matrix by a scalar
  - Multiply each element in the Matrix by the scalar
  - Just like multiplying a vector by a scalar

$$k\mathbf{M} = k \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \end{bmatrix} = \begin{bmatrix} km_{11} & km_{12} & km_{13} \\ km_{21} & km_{22} & km_{23} \\ km_{31} & km_{32} & km_{33} \\ km_{41} & km_{42} & km_{43} \end{bmatrix}$$

04-62: **Matrix Multiplication**

- Multiplying two matrices $\mathbf{A}$ and $\mathbf{B}$
- A dimensions $n \times m$, B dimensions $m \times p$

- $\mathbf{C} = \mathbf{AB}$
    - $\mathbf{C}$ dimensions $n \times p$

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

04-63: **Matrix Multiplication**

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

04-64: **Matrix Multiplication**

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} \boxed{c_{11}} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$c_{11} = a_{11} b_{11} + a_{12} b_{21} + a_{13} b_{31}$$

04-65: **Matrix Multiplication**

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ \boxed{c_{21}} & c_{22} \end{bmatrix}$$

$$c_{21} = a_{21} b_{11} + a_{22} b_{21} + a_{23} b_{31}$$

04-66: **Matrix Multiplication**

$$\begin{bmatrix} \boxed{a_{11} \quad a_{12} \quad a_{13}} \\ a_{21} \quad a_{22} \quad a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & \boxed{\begin{matrix} b_{12} \\ b_{22} \\ b_{32} \end{matrix}} \\ b_{21} & \\ b_{31} & \end{bmatrix} = \begin{bmatrix} c_{11} & \boxed{c_{12}} \\ c_{21} & c_{22} \end{bmatrix}$$

$$c_{12} = a_{11} b_{12} + a_{12} b_{22} + a_{13} b_{32}$$

04-67: **Matrix Multiplication**

$$\begin{bmatrix} a_{11} \quad a_{12} \quad a_{13} \\ \boxed{a_{21} \quad a_{22} \quad a_{23}} \end{bmatrix} \begin{bmatrix} b_{11} & \boxed{\begin{matrix} b_{12} \\ b_{22} \\ b_{32} \end{matrix}} \\ b_{21} & \\ b_{31} & \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & \boxed{c_{22}} \end{bmatrix}$$

$$c_{22} = a_{21} b_{12} + a_{22} b_{22} + a_{23} b_{32}$$

04-68: **Matrix Multiplication**

- Vectors are special cases of matrices
- Multiplying a vector and a matrix is just like multiplying two matrices

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} =$$

$$\begin{bmatrix} xm_{11} + ym_{21} + zm_{31} & xm_{12} + ym_{22} + zm_{32} & xm_{13} + ym_{23} + zm_{33} \end{bmatrix}$$

04-69: **Matrix Multiplication**

- Vectors are special cases of matrices
- Multiplying a vector and a matrix is just like multiplying two matrices

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} xm_{11} + ym_{12} + zm_{13} \\ xm_{21} + ym_{22} + zm_{23} \\ xm_{31} + ym_{32} + zm_{33} \end{bmatrix}$$

04-70: **Matrix Multiplication**

- Note that the following multiplications are not legal:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix}$$

04-71: **Matrix Multiplication**

- Matrix Multiplicaton is not commutative: $AB \neq BA$ (at least not for all A and B – is it true for at least one A and B?)
- Matrix Multiplication is associative: $(AB)C = A(BC)$
- Transposing product is the same as the product of the transpose, in reverse order: $(AB)^T = B^T A^T$

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

04-72: **Matrix Multiplication**

- Matrix Multiplicaton is not commutative: $AB \neq BA$ (at least not for all A and B – is it true for at least one A and B?)
- Matrix Multiplication is associative: $(AB)C = A(BC)$
- Transposing product is the same as the product of the transpose, in reverse order: $(AB)^T = B^T A^T$

$$\left(\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)^T = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} m_{11} & m_{21} & m_{31} \\ m_{12} & m_{22} & m_{32} \\ m_{13} & m_{23} & m_{33} \end{bmatrix}$$

04-73: **Matrix Multiplication**

- Identity Matrix I:
  - $AI = A$ (for appropriate I)
  - $IA = A$ (for appropriate I)

$$\begin{bmatrix} m_{11} & m_{21} & m_{31} \\ m_{12} & m_{22} & m_{32} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{21} & m_{31} \\ m_{12} & m_{22} & m_{32} \\ m_{13} & m_{23} & m_{33} \end{bmatrix}$$

04-74: **Matrix Multiplication**

- Identity Matrix I:
  - $AI = A$ (for appropriate I)
  - $IA = A$ (for appropriate I)

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

04-75: **Matrix Multiplication**

- Identity Matrix I:

- $AI = A$ (for appropriate I)
- $IA = A$ (for appropriate I)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

04-76: **Matrix Multiplication**

- Identity Matrix I:
  - $AI = A$ (for appropriate I)
  - $IA = A$ (for appropriate I)

$$\begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$$

04-77: **Row vs. Column Vectors**

- A vector can be reresented as a row vector or a column vector
- This makes a difference when using matrices
  - Row: $\mathbf{vA}$, Column $\mathbf{Av}$
- It gets even more fun when using matrices to do several transformations of a vector:
  - Row $\mathbf{vABC}$, Column $\mathbf{CBAv}$ (note that to get the same transformation, you need to take the transpose of A, B, and C when swapping between row and column vectors

04-78: **Row vs. Column Vectors**

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax + by + cz \\ dx + ey + fz \\ gx + hy + iz \end{bmatrix}$$

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

$$= \begin{bmatrix} ax + by + cz & dx + ey + fz & gx + hy + iz \end{bmatrix}$$

04-79: **Row vs. Column Vectors**

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} =$$

$$\begin{bmatrix} xa + yc & xb + yd \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} =$$

$$\begin{bmatrix} (xa + yc)e + (xb + yd)g & (xa + yc)f + (xb + yd)h \end{bmatrix}$$

$$\begin{bmatrix} e & g \\ f & h \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} =$$

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} ax + cy \\ bx + dy \end{bmatrix} =$$

$$\begin{bmatrix} e(ax + cy) + g(ax + cy) \\ f(ax + cy) + h(ax + yd) \end{bmatrix}$$

04-80: **Row vs. Column Vectors**

- DirectX and the text use row vectors
- OpenGL and Ogre use column vectors

- Ogre has a back end for both OpenGL and Direct3D
- Ogre transposes matrices before sending them to D3D libraries

- Lecture will use both
  - This is on purpose
  - I want you to really understand what's going on, not just memorize formulas

04-81: **More Matrices**

- Consider the vector $[x, y, z]$

$$V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$$

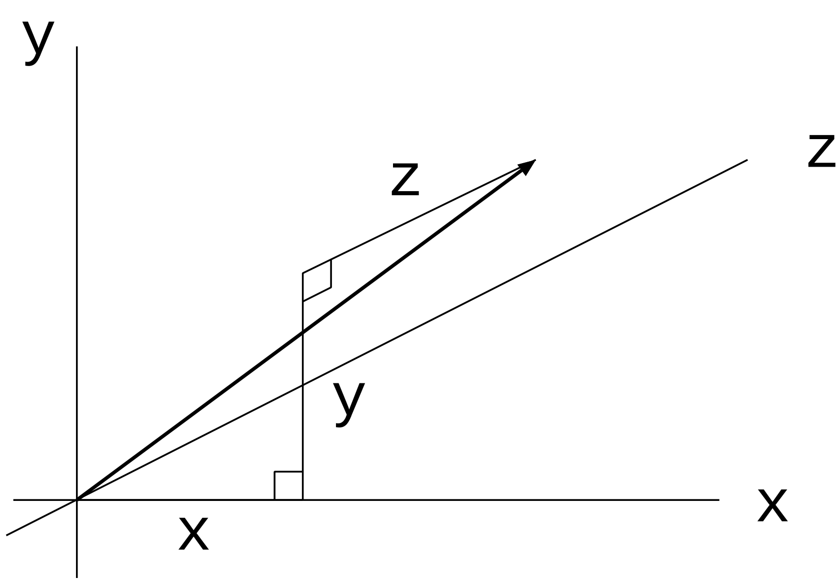
- Rewrite as:

$$V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

04-82: **More Matrices**

- let $\mathbf{p}, \mathbf{q}, \mathbf{r}$ be unit vectors for $+x, +y$ and $+z$
- $\mathbf{v} = x\mathbf{p} + y\mathbf{q} + z\mathbf{r}$
- We have defined v as a linear combination of $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$.
  - $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ are *basis vectors*

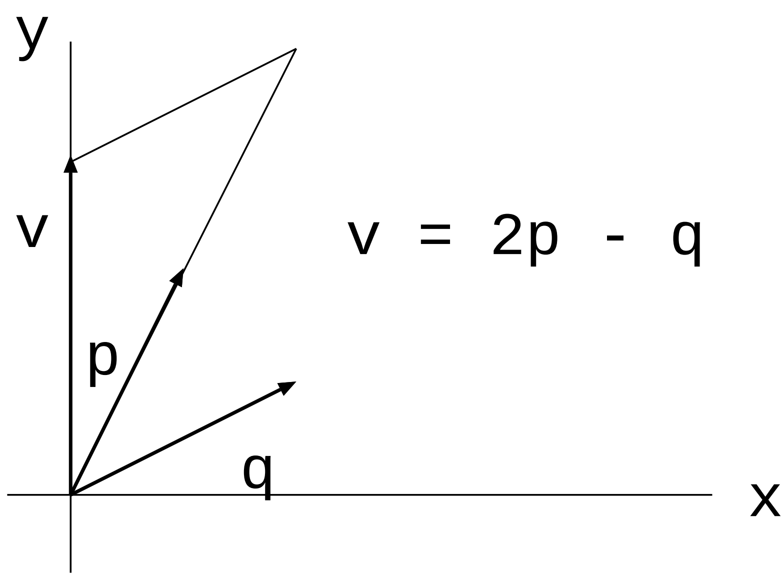
04-83: **Basis Vectors**



04-84: **Basis Vectors**

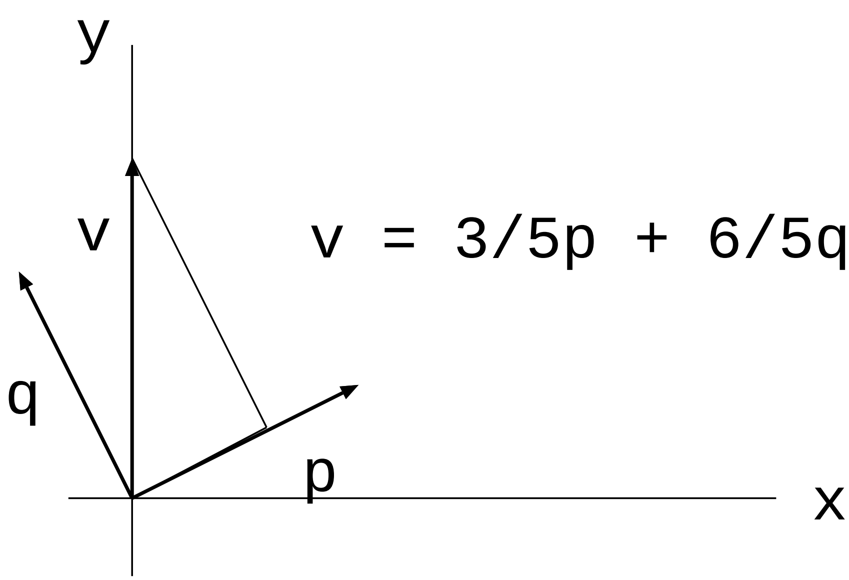
- $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ are unit vectors along the X, Y and Z axes – we're used to seeing vectors decomposed this way

- Technically, *any* 3 linearly-independent vectors could be used as basis vectors
- Typically, mutually perpendicular vertices are used as basis vectors
- Basis vectors not aligned with axes: Object space rotated from world space



04-85: **Non-Perpendicular Basis**

04-86:



Perpendicular Basis & Basis

04-87: **Marices**

- Look back at our basis vectors $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$.
- Create a 3x3 matrix M using $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ as rows:

$$\mathbf{M} = \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \\ \mathbf{r} \end{bmatrix} = \begin{bmatrix} p_x & p_y & p_z \\ q_x & q_y & q_z \\ r_x & r_y & r_z \end{bmatrix}$$

- Multiply a vector by this matrix:

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} p_x & p_y & p_z \\ q_x & q_y & q_z \\ r_x & r_y & r_z \end{bmatrix} =$$

04-88: **Marices & Basis**

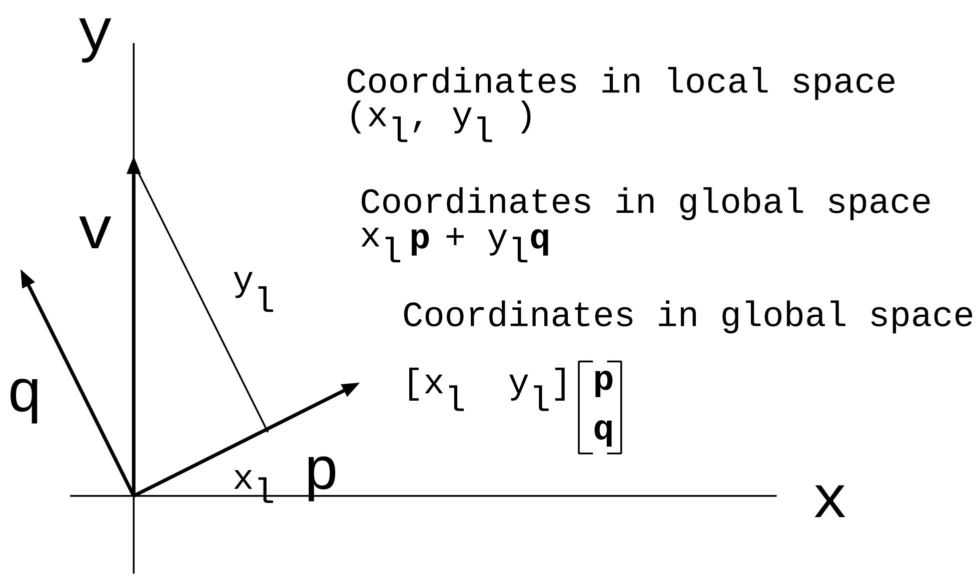
$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} p_x & p_y & p_z \\ q_x & q_y & q_z \\ r_x & r_y & r_z \end{bmatrix} =$$

$$\begin{bmatrix} xp_x + yq_x + zr_x & xp_y + yq_y + zr_y & xp_z + yq_z + zr_z \end{bmatrix} =$$

$$x\mathbf{p} + y\mathbf{q} + z\mathbf{r}$$

04-89: **Marices & Basis**

- This is really cool. Why?
  - Take a local space, defined by 3 basis vectors
    - Rotation only (no translation)
  - Create a matrix with these vectors as rows (or cols)
  - Matrix transforms from local space into global space



04-90: **Matrices & Basis**

04-91: **Matrices as Transforms**

- A 3x3 matrix is a transform
  - Transforms a vector
  - Since a 3D model is just a series of points, can also transform a model
    - Transforming each point in the model
- What does the transformation look like?
- Can you look at the matrix, and see what the transformation will be?

04-92: **Matrices as Transforms**

- Let's look at what happens when we multiply the basis vectors $[1, 0, 0]$, $[0, 1, 0]$ and $[0, 0, 1]$ by an arbitrary matrix:

04-93: **Matrices as Transforms**

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} = \begin{bmatrix} m_{21} & m_{22} & m_{23} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} = \begin{bmatrix} m_{31} & m_{32} & m_{33} \end{bmatrix}$$

04-94: **Matrices as Transforms**

- Each row of the matrix is a basis vector after transformation
  - (Or each column of the matrix, if we're using column vectors)
- Let's look at an example in 2D:

$$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

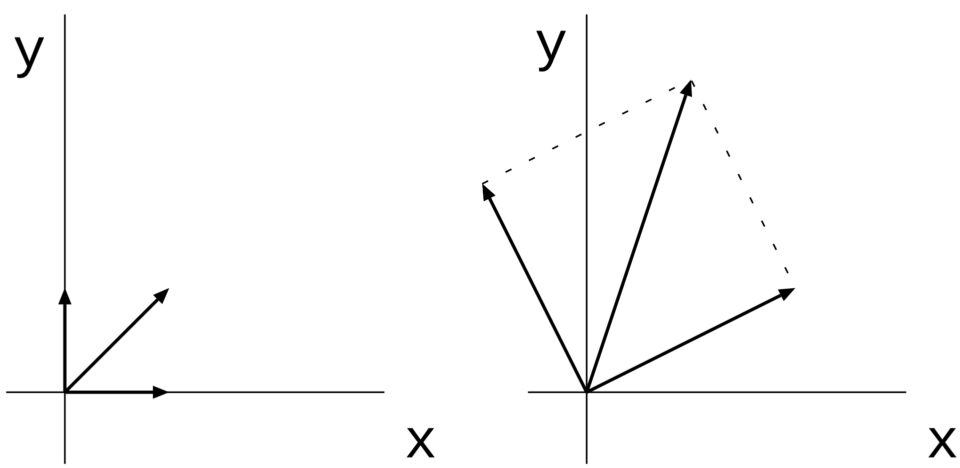
- What happens when we transform a vector (or a 2D polgon) using this matrix?
- Assume row vectors for the moment ...

04-95: **Matrices as Transforms**

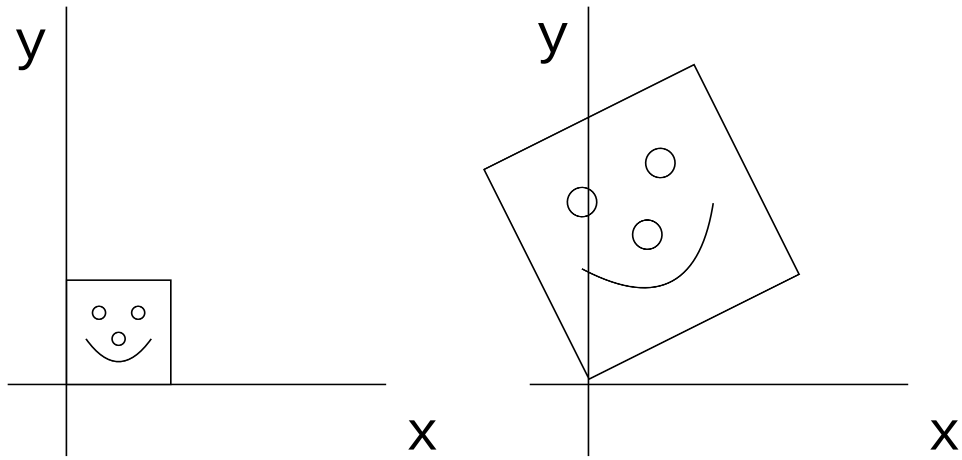
$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \end{bmatrix}$$

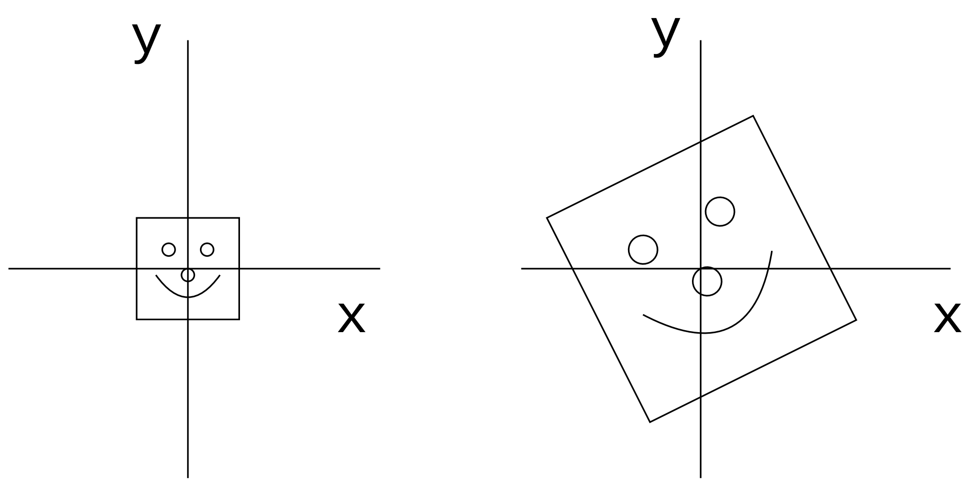
$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \end{bmatrix}$$



04-96: **Matrices as Transforms**



04-97: **Matrices as Transforms**



04-98: **Matrices as Transforms**

04-99: **Matrices as Transforms**

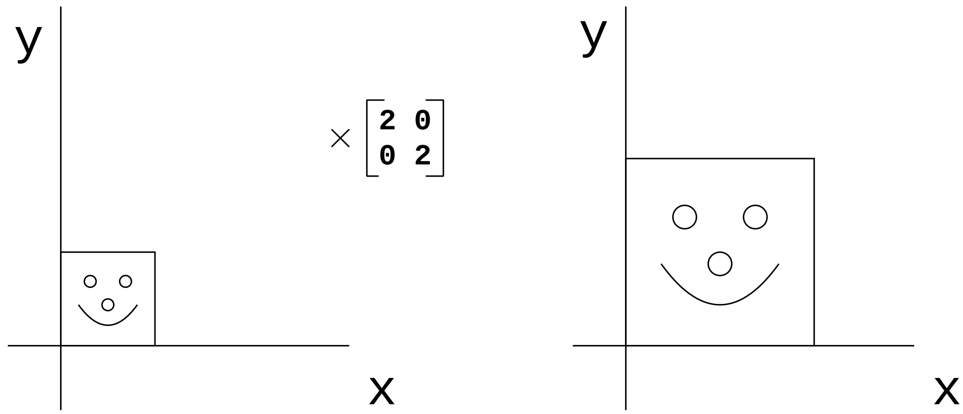
- The matrix:

$$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

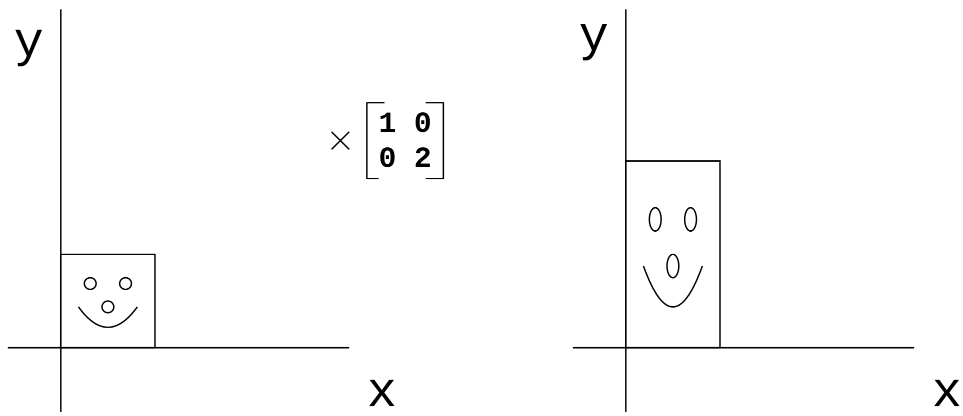
  both scaled and rotated a 2D image

- It is possible, of course for a matrix to just scale, or just rotate an image as well

04-100: **Matrices as Transforms**



04-101: **Matrices as Transforms**



04-102: **Matrices as Transforms**

- Can a matrix do something other than scale and rotate?

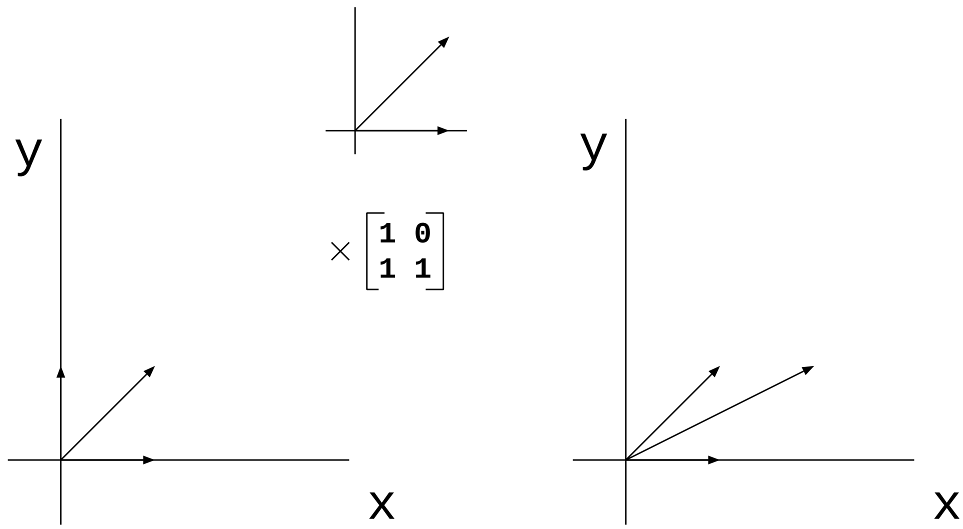
04-103: **Matrices as Transforms**

- Can a matrix do something other than scale and rotate?
    - Yes!
- What would a matrix that did something other than scale or rotate look like? (stay 2D, for the moment)

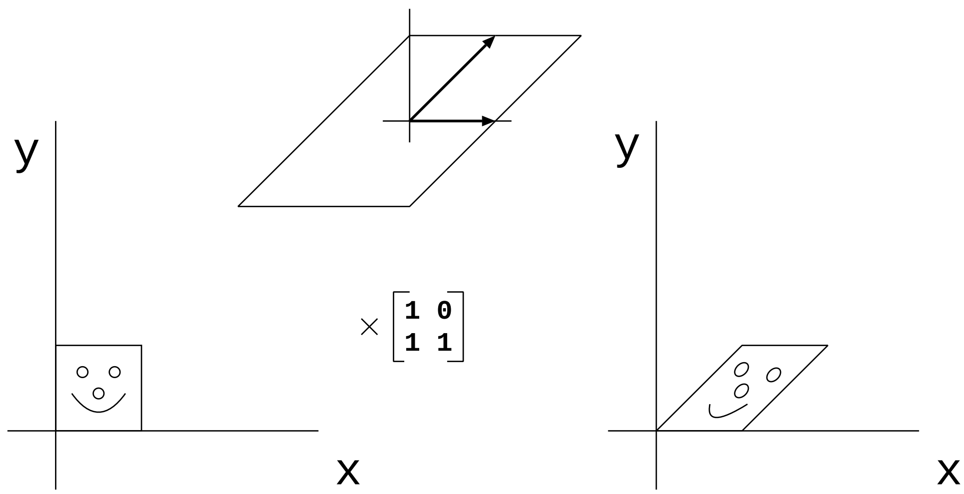
04-104: **Matrices as Transforms**

- Can a matrix do something other than scale and rotate?
    - Yes!
- What would a matrix that did something other than scale or rotate look like? (stay 2D, for the moment)
    - “Basis vectors” in matrix non-orthogonal

04-105: **Matrices as Transforms**



04-106: **Matrices as Transforms**



04-107: **Matrices as Transforms**

- This translates (reasonably) easily into 3D
- Instead of stretching, rotating, or skewing part of a plane, stretch, rotate, or skew a cube

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

No transformation (or identity transformation)

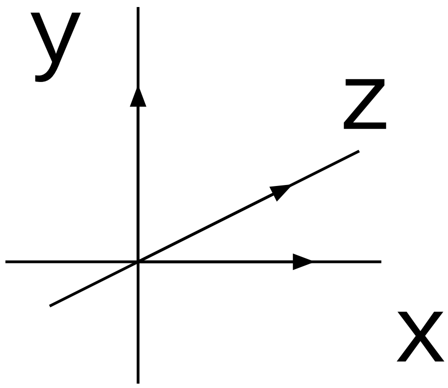
04-108: **Matrices as Transforms**

- This translates (reasonably) easily into 3D
- Instead of stretching, rotating, or skewing part of a plane, stretch, rotate, or skew a cube

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

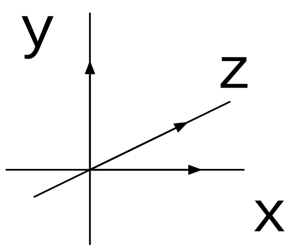
What is this? 04-109: **Matrices as Transforms**

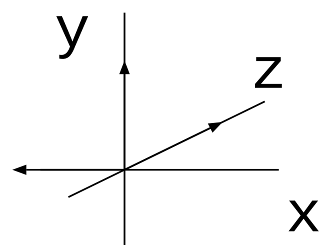
$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$



04-110: **Matrices as Transforms**

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$





04-111: **Matrices as Transforms**

- This translates (reasonably) easily into 3D
- Instead of stretching, rotating, or skewing part of a plane, stretch, rotate, or skew a cube

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Rotation about the Y axis, $\frac{\pi}{2}$ (90 degrees)