16-0: **Probability Review**

- Probability allows us to represent a belief about a statement, or a likelihood that a statement is true.
  - $P(rain) = 0.6$ means that we believe it is 60% likely that it is currently raining.
- Axioms:
  - $0 \leq P(a) \leq 1$
  - The probability of $(A \vee B)$ is $P(A) + P(B) - P(A \wedge B)$
  - Tautologies have $P = 1$
  - Contradictions have $P = 0$

16-1: **Conditional Probability**

- Once we begin to make observations about the value of certain variables, our belief in other variables changes.
  - Once we notice that it's cloudy, $P(rain)$ goes up.
- this is called *conditional probability*
- Written as: $P(rain|cloudy)$
- $P(a|b) = \frac{P(a \wedge b)}{P(b)}$
- or $P(a \wedge b) = P(a|b)P(b)$
  - This is called the *product rule*.

16-2: **Conditional Probability**

- Example: $P(Cloudy) = 0.25$
- $P(Rain) = 0.25$
- $P(cloudy \wedge rain) = 0.15$
- $P(cloudy \wedge \neg Rain) = 0.1$
- $P(\neg cloudy \wedge Rain) = 0.1$
- $P(\neg Cloudy \wedge \neg Rain) = 0.65$
  - Initially, $P(Rain) = 0.25$. Once we see that it's cloudy, $P(Rain|Cloudy) = P\frac{(Rain \wedge Cloudy)}{P(Cloudy)} = \frac{0.15}{0.25} = 0.6$

16-3: **Combinations of events**

- The probability of $(A \wedge B)$ is $P(A|B)P(B)$
- What if A and B are independent?
- Then $P(A|B)$ is $P(A)$, and $P(A \wedge B)$ is $P(A)P(B)$.
- Example:
  - What is the probability of "heads" five times in a row?
  - What is the probability of at least one "head"?

16-4: **Bayes' Rule**

- Often, we want to know how a probability changes as a result of an observation.
- Recall the Product Rule:
  - $P(a \wedge b) = P(a|b)P(b)$
  - $P(a \wedge b) = P(b|a)P(a)$
- We can set these equal to each other
  - $P(a|b)P(b) = P(b|a)P(a)$
- And then divide by $P(a)$
  - $P(b|a) = \frac{P(a|b)P(b)}{P(a)}$
- This equality is known as Bayes' theorem (or rule or law).

16-5: **Bayes' Rule**

- We can generalize Bayes' rule, by adding in some more evidence:
  - $P(b|a) = \frac{P(a|b)P(b)}{P(a)}$
  - $P(b|a, e) = \frac{P(a|b,e)P(b|e)}{P(a|e)}$

16-6: **Bayes' Rule**

- We can also avoid the pesky $P(a)$:
  - $P(b|a) = \frac{P(a|b)P(b)}{P(a)}$
  - $P(b|a) = \alpha P(a|b)P(b)$
- Where α is a normalizing constant, so that $P(b|a)$ and $P(\neg b|a)$ sum to 1.
  - Generally, so that $\sum_{b \in B} P(b|a)$ sums to 1
- Example:
  - $P(t|d) = 0.8, P(\neg t|\neg d) = 0.9, P(d) = 0.2$
  - $P(d|t)$?

16-7: **Bayes' Rule**

- $P(t|d) = 9/10, P(\neg t|\neg d) = 8/10, P(d) = 1/10$

$$\begin{aligned}
P(d|t) &= \alpha P(t|d)P(d) \\
&= \alpha 9/10 * 1/10 \\
&= \alpha 9/100 \\
P(\neg d|t) &= \alpha P(t|\neg d)P(\neg d) \\
&= \alpha 2/10 * 9/10 \\
&= \alpha 18/100 \\
\alpha &= \frac{1}{9/100 + 18/100} = \frac{100}{27}
\end{aligned}$$

16-8: **Bayes' Rule**

- $P(t|d) = 9/10, P(\neg t|\neg d) = 8/10, P(d) = 1/10$

$$
\begin{aligned}
P(d|t) &= \alpha P(t|d)P(d) \\
&= \alpha 9/10 * 1/10 \\
&= \alpha(9/100) \\
&= (100/27)(9/100) \\
&= 1/3 \\
P(\neg d|t) &= \alpha P(t|\neg d)P(\neg d) \\
&= \alpha 2/10 * 9/10 \\
&= \alpha(18/100) \\
&= (100/27)(18/100) \\
&= 2/3
\end{aligned}
$$

16-9: **More Probability**

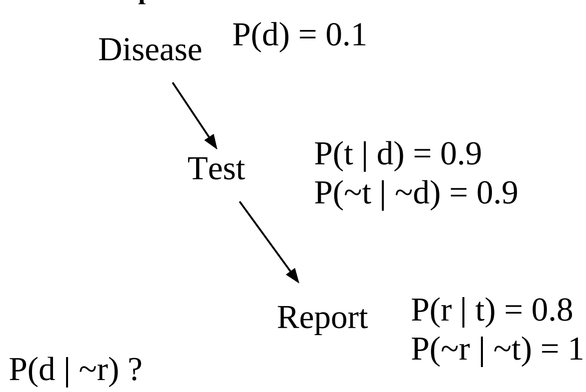
$$P(a) = \sum_{b \in B} P(a|b)P(b)$$

- Intuitively: It's always either day or night (and never both). Probability of of something happening is the probability that it happens at night, plus the probability that it happens during the day

$$P(a|e) = \sum_{b \in B} P(a|b,e)P(b|e)$$

- For any probability function, we can always condition *everything* for some additional evidence

16-10: **Example**



16-11: **Example**

- $P(d|\neg r) = \alpha P(\neg r|d)P(d)$
  - We know $P(d)$, we just need $P(\neg r|d)$ and $P(\neg r)$.
  - $P(\neg r|d)$ We only know $P(R)$ in terms of T ...
- $P(\neg r|d) = P(\neg r|d,t)P(t|d) + P(\neg r|d,\neg t)P(\neg t|d)$
  - Conditional Independence to the rescue!

- $P(\neg r|d, t) = P(\neg r|t)$, which we know.

16-12: **Example**

$$\begin{aligned} P(d|\neg r) &= \alpha(P(\neg r|t)P(t|d) + P(\neg r|\neg t)P(\neg t|d)) * P(d) \\ &= \alpha((0.2) * (0.9) + (1) * 0.1) * 0.1 \\ &= \alpha * 0.019 \end{aligned}$$

$$\begin{aligned} P(\neg d|\neg r) &= \alpha(P(\neg r|t)P(t|\neg d) + P(\neg r|\neg t)P(\neg t|\neg d)) * P(\neg d) \\ &= \alpha((0.2) * (0.1) + (1) * 0.9) * 0.9 \\ &= \alpha(0.828) \end{aligned}$$

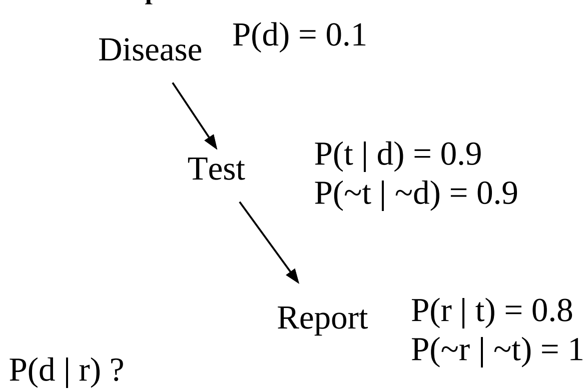
$$\alpha = \frac{1}{0.828 + 0.019} = \frac{1}{0.847}$$

16-13: **Example**

$$\begin{aligned} P(d|\neg r) &= \alpha(P(\neg r|t)P(t|d) + P(\neg r|\neg t)P(\neg t|d)) * P(d) \\ &= \alpha((0.2) * (0.9) + (1) * 0.1) * 0.1 \\ &= \alpha * 0.019 \\ &= 0.022 \end{aligned}$$

$$\begin{aligned} P(\neg d|\neg r) &= \alpha(P(\neg r|t)P(t|\neg d) + P(\neg r|\neg t)P(\neg t|\neg d)) * P(\neg d) \\ &= \alpha((0.2) * (0.1) + (1) * 0.9) * 0.9 \\ &= \alpha(0.828) \\ &= 0.978 \end{aligned}$$

16-14: **Example II**



16-15: **Example**

- $P(d|r) = \alpha P(r|d)P(d)$
  - $P(r|d) = P(r|d, t)P(t|d) + P(r|d, \neg t)P(\neg t|d)$

- Conditional Independence to the rescue!
- $P(r|d, t) = P(r|t)$, which we know.

16-16: **Example**

$$\begin{aligned} P(d|r) &= \alpha(P(r|t)P(t|d) + P(r|\neg t)P(\neg t|d)) * P(d) \\ &= \alpha((0.8) * (0.9) + (0) * 0.1) * 0.1 \\ &= \alpha * 0.072 \end{aligned}$$

$$\begin{aligned} P(\neg d|r) &= \alpha(P(r|t)P(t|\neg d) + P(r|\neg t)P(\neg t|\neg d)) * P(\neg d) \\ &= \alpha((0.8) * (0.1) + (0) * 0.9) * 0.9 \\ &= \alpha(0.072) \end{aligned}$$

$P(d|r) = P(\neg d|r) = 0.5$

16-17: **Learning and Classification**

- An important sort of learning problem is the *classification* problem.
- This involves placing examples into one of two or more classes.
  - Should/shouldn't play tennis
  - Spam/not spam.
  - Wait/don't wait at a restaurant
- Classification is a *supervised* learning task.
  - Requires access to a set of labeled training examples
- From this we induce a hypothesis that describes how to determine what class an example should be in.

16-18: **Probabilistic Learning**

- Decision trees are one way to do this.
  - They tell us the most likely classification for given data.
  - Work best with tabular data, where each attribute has a known number of possible values.
- What if we want to know how likely a hypothesis is?
- We can apply our knowledge of probability to learn a hypothesis.

16-19: **Bayes' Theorem**

- Recall the definition of Bayes' Theorem
- $P(b|a) = \frac{P(a|b)P(b)}{P(a)}$
- Let's rewrite this a bit.
- Let D be the data we've seen so far.
- Let h be a possible hypothesis

- $P(h|D) = \frac{P(D|h)P(h)}{P(D)}$

16-20: **MAP Hypothesis**

- Often, we're not so interested in the particular probabilities for each hypothesis.
- Instead, we want to know: Which hypothesis is most likely, given the data?
  - Which classification is the most probable?
  - Is $PlayTennis$ or $\neg PlayTennis$ more likely?
- We call this the *maximum a posteriori hypothesis* (MAP hypothesis).
- In this case, we can ignore the denominator ($P(D)$) in Bayes' Theorem, since it will be the same for all h.
- $h_{MAP} = argmax_{h \in H} P(D|h)P(h)$

16-21: **MAP Hypothesis**

- Advantages:
  - Simpler calculation
  - No need to have a prior for $P(D)$

16-22: **ML Hypothesis**

- In some cases, we can simplify things even further.
- What are the priors $P(h)$ for each hypothesis?
- Without any other information, we'll often assume that they're equally possible.
  - Each has probability $\frac{1}{H}$
- In this case, we can just consider the conditional probability $P(D|h)$.
- We call the hypothesis that maximizes this conditional probability the *maximum likelihood* hypothesis.
- $h_{ML} = argmax_{h \in H} P(D|h)$

16-23: **Example**

- Imagine that we have a large bag of candy. We want to know the ratio of cherry to lime in the bag.
- We start with 5 hypotheses:
  1. h_1: 100% cherry
  2. h_2 75% cherry, 25% lime.
  3. h_3 50% cherry, 50% lime
  4. h_4 25% cherry, 75% lime
  5. h_5 100% lime
- Our agent repeatedly draws pieces of candy.
- We want it to correctly pick the type of the next piece of candy.

16-24: **Example**

- Let's assume our priors for the different hypotheses are:
- $(0.1, 0.2, 0.4, 0.2, 0.1)$
- Also, we assume that the observations are i.i.d.
  - Independent and Identically Distributed – each choice is independent of the others, and order doesn't matter.
- In that case, we can multiply probabilities.
- $P(D|h_i) = \Pi_j P(d_j|h_i)$
- Suppose we draw 10 limes in a row. $P(D|h_3)$ is $(\frac{1}{2})^{10}$, since the probability of drawing a lime under h_3 is $\frac{1}{2}$.

16-25: **Example**

- How do the hypotheses change as data is observed?
- Initially, we start with the priors: $(0.1, 0.2, 0.4, 0.2, 0.1)$
- Then we draw a lime.
  - $P(h_1|lime) = \alpha P(lime|h_1)P(h_1) = 0.$
  - $P(h_2|lime) = \alpha P(lime|h_2)P(h_2) = \alpha\frac{1}{4} * 0.2 = \alpha 0.05.$
  - $P(h_3|lime) = \alpha P(lime|h_3)P(h_3) = \alpha\frac{1}{2} * 0.4 = \alpha 0.2$
  - $P(h_4|lime) = \alpha P(lime|h_4)P(h_4) = \alpha\frac{3}{4} * 0.2 = \alpha 0.15.$
  - $P(h_5|lime) = \alpha P(lime|h_5)P(h_5) = \alpha 1 * 0.1 = \alpha 0.1.$
  - $\alpha = 2.$

16-26: **Example**

- Then we draw a second lime.
  - $P(h_1|lime, lime) = \alpha P(lime, lime|h_1)P(h_1) = 0.$
  - $P(h_2|lime, lime) = \alpha P(lime, lime|h_2)P(h_2) = \alpha\frac{1}{4}\frac{1}{4} * 0.2 = \alpha 0.0125.$
  - $P(h_3|lime, lime) = \alpha P(lime, lime|h_3)P(h_3) = \alpha\frac{1}{2}\frac{1}{2} * 0.4 = \alpha 0.1$
  - $P(h_4|lime, lime) = \alpha P(lime, lime|h_4)P(h_4) = \alpha\frac{3}{4}\frac{3}{4} * 0.2 = \alpha 0.1125.$
  - $P(h_5|lime) = \alpha P(lime|h_5)P(h_5) = \alpha 1 * 0.1 = \alpha 0.1.$
  - $\alpha = 3.07.$
- Strictly speaking, we don't really care what α is.
- We can just select the MAP hypothesis, since we just want to know the most likely hypothesis.

16-27: **Bayesian Learning**

- Eventually, the true hypothesis will dominate all others.
  - Caveat: assuming the data is noise-free, or noise is uniformly distributed.
- Notice that we can use Bayesian learning (in this case) either as a batch algorithm or as an incremental algorithm.
- We can always easily update our hypotheses to incorporate new evidence.

  - This depends on the assumption that our observations are independent.

16-28: **Learning bias**

- What sort of bias does Bayesian Learning use?
- Typically, simpler hypotheses will have larger priors.
- More complex hypotheses will fit data more exactly (but there's many more of them).
  - Under these assumptions, h_{MAP} will be the simplest hypothesis that fits the data.
  - This is Occam's razor, again.

16-29: **Bayesian Concept Learning**

- Bayesian Learning involves estimating the likelihood of each hypothesis.
- In a more complex world where observations are not independent, this could be difficult.
- Our first cut at doing this might be a brute force approach:
  1. For each h in H, calculate $P(h|D) = \frac{P(D|h)P(h)}{P(D)}$
  2. From this, output the hypothesis h_{MAP} with the highest posterior probability.
- This is what we did in the example.
  - Challenge - Bayes' Theorem can be computationally expensive to use when observations are not i.i.d.
  - $P(h|o_1, o_2) = \frac{P(o_1|h, o_2)P(h|o_2)}{P(o_1|o_2)}$

16-30: **Bayesian Optimal Classifiers**

- There's one other problem with the formulation as we have it.
- Usually, we're not so interested in the hypothesis that fits the data.
- Instead, we want to classify some unseen data, given the data we've seen so far.
- One approach would be to just return the MAP hypothesis.
- We can do better, though.

16-31: **Bayesian Optimal Classifiers**

- Suppose we have three hypotheses and posteriors: $h_1 = 0.4, h_2 = 0.3, h_3 = 0.3$.
- We get a new piece of data - h_1 says it's positive, h_2 and h_3 negative.
- h_1 is the MAP hypothesis, yet there's a 0.6 chance that the data is negative.
- By combining weighted hypotheses, we improve our performance.

16-32: **Bayesian Optimal Classifiers**

- By combining the predictions of each hypothesis, we get a Bayesian optimal classifier.
- More formally, let's say our unseen data belongs to one of v classes.
- The probability $P(v_j|D)$ that our new instance belongs to class v_j is:
- $\sum_{h_i \in H} P(v_j|h_i)P(h_i|D)$

- Intuitively, each hypothesis gives its prediction, weighted by the likelihood that that hypothesis is the correct one.
- This classification method is provably optimal - on average, no other algorithm can perform better.

16-33: **Problems**

- However, the Bayes optimal classifier is mostly interesting as a theoretical benchmark.
- In practice, computing the posterior probabilities is exponentially hard.
- This problem arises when instances or data are conditionally dependent upon each other.
- Can we get around this?

16-34: **Naive Bayes classifier**

- The Naive Bayes classifer makes a strong assumption that makes the algorithm practical:
  - Each attribute of an example is independent of the others.
  - $P(a \wedge b) = P(a)P(b)$ for all a and b.
- This makes it straightforward to compute posteriors.

16-35: **Bayesian Learning Problem**

- Given: a set of labeled, multivalued examples.
- Find a function $F(x)$ that correctly classifies an unseen example with attributes $(a_1, a_2, ..., a_n)$.
- Call the most probable category v_{map}.
- $v_{map} = argmax_{v_i \in V} P(v_i | a_1, a_2, ..., a_n)$
- We rewrite this with Bayes' Theorem as: $v_{map} = argmax_{v_i \in V} P(a_1, a_2, ..., a_n | v_i) P(v_i)$
- Estimating $P(v_i)$ is straightforward with a large training set; count the fraction of the set that are of class v_i.
- However, estimating $P(a_1, a_2, ..., a_n | v_i)$ is difficult unless our training set is *very* large. We need to see every possible attribute combination many times.

16-36: **Naive Bayes assumption**

- Naive Bayes assumes that all attributes are conditionally independent of each other.
- In this case, $P(a_1, a_2, ..., a_n | v_i) = \Pi_i P(a_i | v_i)$.
- This can be estimated from the training data.
- The classifier then picks the class with the highest probability according to this equation.
- Interestingly, Naive Bayes performs well even in cases where the conditional independence assumption fails.

16-37: **Example**

- Recall your tennis-playing problem from the decision tree homework.
- We want to use the training data and a Naive Bayes classifier to classify the following instance:
- Outlook = Sunny, Temperature = Cool, Humidity = high, Wind = Strong.

16-38: **Example**

Day	Outlook	Temperature	Humidity	Wind	PlayTennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

16-39: **Example**

- Our priors are:
  - $P(PlayTennis = yes) = 9/14 = 0.64$
  - $P(PlayTennis = no) = 5/14 = 0.36$
- We can estimate:
  - $P(wind = strong|PlayTennis = yes) = 3/9 = 0.33$
  - $P(wind = strong|PlayTennis = no) = 3/5 = 0.6$
  - $P(humidity = high|PlayTennis = yes) = 3/9 = 0.33$
  - $P(humidity = high|PlayTennis = no) = 4/5 = 0.8$
  - $P(outlook = sunny|PlayTennis = yes) = 2/9 = 0.22$
  - $P(outlook = sunny|PlayTennis = no) = 3/5 = 0.6$
  - $P(temp = cool|PlayTennis = yes) = 3/9 = 0.33$
  - $P(temp = cool|PlayTennis = no) = 1/5 = 0.2$

16-40: **Example**

- $v_{yes} = P(yes)P(sunny|yes)P(cool|yes)P(high|yes)P(strong|yes) = 0.005$
- $v_{no} = P(no)P(sunny|no)P(cool|no)P(high|no)P(strong|no) = 0.0206$
- So we see that not playing tennis is the maximum likelihood hypothesis.
- Further, by normalizing, we see that the classifier predicts a $\frac{0.0206}{0.005+0.0206} = 0.80$ probability of not playing tennis.

16-41: **Estimating Probabilities**

- As we can see from this example, estimating probabilities through frequency is risky when our data set is small.
- We only have 5 negative examples, so we may not have an accurate estimate.
- A better approach is to use the following formula, called an *m*-estimate:
- $\frac{n_c+mp}{n+m}$
- Where n_c is the number of individuals with the characteristic of interest (say Wind = strong), n is the total number of positive/negative examples, p is our prior estimate, and m is a constant called the *equivalent sample size*.

16-42: **Estimating Probabilities**

- m determines how heavily to weight p.
- p is assumed to be uniform.
- So, in the Tennis example, $P(wind = strong|playTennis = no) = \frac{3+0.2m}{5+m}$
- We'll determine an m based on sample size.
  - If m is zero, we just use observed data.
  - If $m >> n$, we use the prior.
  - Otherwise m lets us weight these parameters' relative influence.

16-43: **Naive Bayes: Classify Spam**

- One are where Naive Bayes has been very successful is in text classification.
  - Despite the violation of independence assumptions.
- Classifying spam is just a special case of text classification.
- Problem - given some emails labled ham or spam, determine the category of new and unseen documents.
- Our features will be the tokens that appear in a document.
- Based on this, we'll predict a category.

16-44: **Classifying spam**

- Naive Bayes is only one possible way to classify spam.
  - Rule-based systems (SpamAssassin)
  - Examining headers (broken From or Content-Type)
  - Blacklist/Whitelist
  - Challenge/response

16-45: **Naive Bayes: Classify Spam**

- Naive Bayes has several properties that make it nice as a spam classifier:
  - We don't need to encode specific rules
  - We can adapt as the types of spam change
  - Somewhat robust to spammers adding in extra text

16-46: **Naive Bayes: Classify Spam**

- For a given email, we'll want to compute the MAP hypothesis - that is, is:
  - $P(spam|t1, t2, .., tn)$ greater than
  - $P(ham|t1, t2, .., tn)$
- We can use Bayes' rule to rewrite these as:
  - $\alpha P(t1, t2, ..., tn|spam)P(spam)$
  - $\alpha P(t1, t2, ..., tn|ham)P(ham)$

16-47: **Naive Bayes: Classify Spam**

- We can then use the Naive Bayes assumption to rewrite these as:
  - $\alpha P(t1|spam)P(t2|spam)...P(tn|spam)P(spam)$
  - $\alpha P(t1|ham)P(t2|ham)...P(tn|ham)P(ham)$
- And this we know how to compute.

16-48: **Naive Bayes: Classify Spam**

- We can get the conditional probabilities by counting tokens in the training set.
- We can get the priors from the training set, or through estimation.

16-49: **Naive Bayes: Classify Spam**

- This is a case of a problem where we can tolerate occasional false negatives (spam classified as ham) but we cannot tolerate false positives (ham classified as spam).
- Plain old vanilla Naive Bayes will do fairly well, but there's a lot of tuning and tweaking that can be done to optimize performance.

16-50: **Naive Bayes: Classify Spam**

- There are a lot of wrinkles to consider:
  - What should be treated as a token? All words? All strings? Only some words?
  - Should headers be given different treatment? Greater or less emphasis? What about subject?
  - What about HTML?

16-51: **Naive Bayes: Classify Spam**

- There are a lot of wrinkles to consider:
  - When classifying an email, should you consider all tokens, or just the most significant?
  - When computing conditional probabilities, should you could the fraction of documents a token appear in, or the fraction of words represented by a particular token?
  - Can you use any of the NLTK tools to better identify structure?