AI Programming

CS662-2013S-18

Markov Decision Processes

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18-0: Making Sequential Decisions

- Previously, we've talked about:
  - Making one-shot decisions in a deterministic environment
  - Making sequential decisions in a deterministic environment
    - Search
    - Inference
  - Making one-shot decisions in a stochastic environment
    - Probability and Belief Networks
    - Expected Utility
- What about sequential decisions in a stochastic environment?

18-1: Expected Utility

- Recall that the expected utility of an action is the utility of each possible outcome, weighted by the probability of that outcome occurring.
- More formally, from state s, an agent may take actions $a_1, a_2, ..., a_n$.
- Each action a_i can lead to states $s_{i1}, s_{i2}, ..., s_{im}$, with probability $p_{i1}, p_{i2}, ..., p_{im}$

$$EU(a_i) = \sum p_{ij} s_{ij}$$

- We call the set of probabilities and associated states the state transition model.
- The agent should choose the action a' that maximizes EU.

18-2: Markovian environments

- We can extend this idea to sequential environments.
- Problem: How to determine transition probabilities?
  - The probability of reaching state s given action a might depend on previous actions that were taken.
  - Reasoning about long chains of probabilities can be complex and expensive.
- The Markov assumption says that state transition probabilities depend only on a finite number of parents.
- Simplest: a first-order Markov process. State transition probabilities depend only on the previous state.
  - This is what we'll focus on.

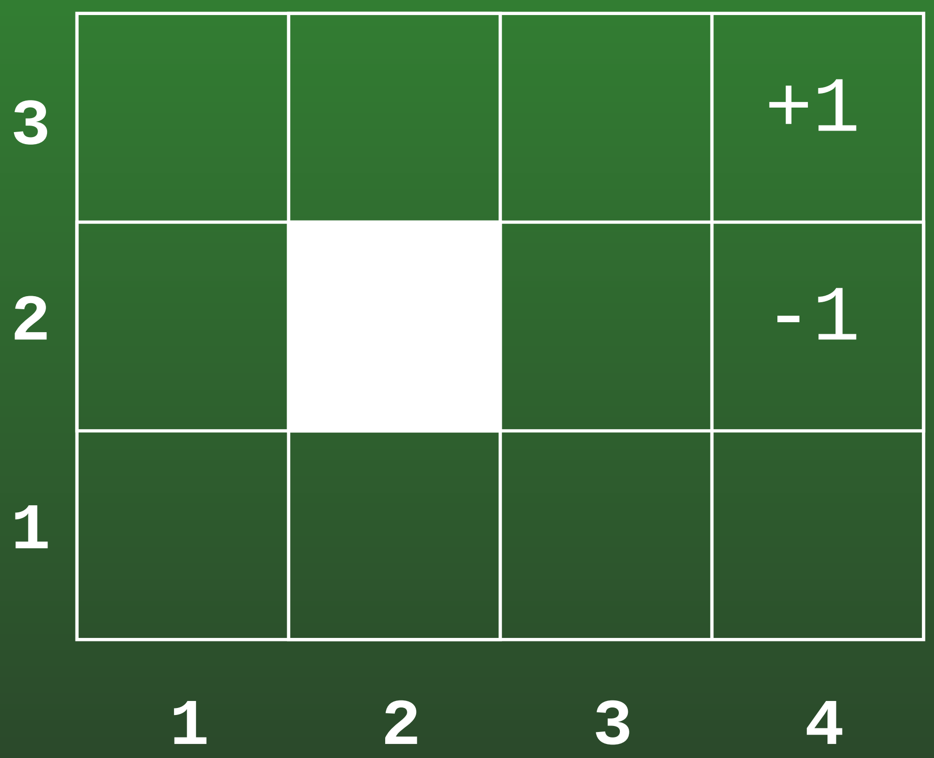
18-3: Stationary Distributions

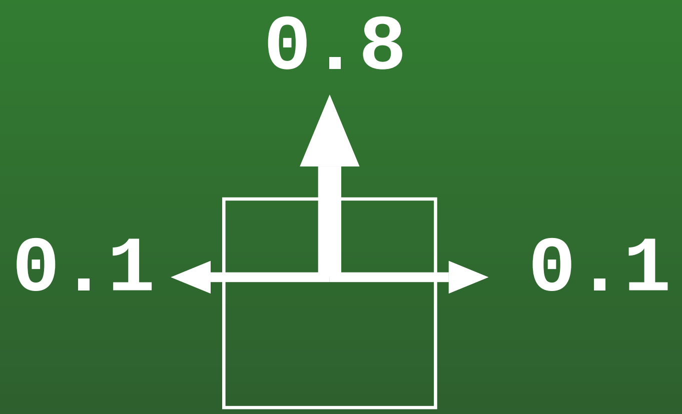
- We'll also assume a *stationary distribution*
- This says that the probability of reaching a state s' given action a from state s with history H does not change.
- Different histories may produce different probabilities
- Given identical histories, the state transitions will be the same.
- We'll also assume that the utility of a state does not change throughout the course of the problem.
  - In other words, our model of the world does not change while we are solving the problem.

18-4: Solving sequential problems

- If we have to solve a sequential problem, the total utility will depend on a sequence of states $s_1, s_2, ..., s_n$.
- Let's assign each state a utility or *reward* $R(s_i)$.
- Agent wants to maximize the sum of rewards.
- We call this formulation a Markov decision process.
  - Formally:
  - An initial state s_0
  - A discrete set of states and actions
  - A Transition model: $T(s, a, s')$ that indicates the probability of reaching state s' from s when taking action a.
  - A reward function: $R(s)$

18-5: Example grid problem





- Agent moves in the “intended” direction with probability 0.8, and at a right angle with probability 0.2
- What should an agent do at each state to maximize reward?

18-6: MDP solutions

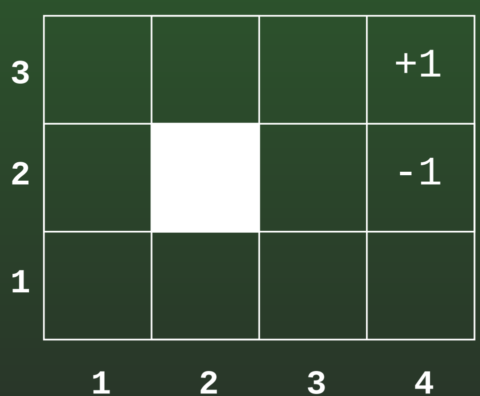
- Since the environment is stochastic, a solution will not be an action sequence.
- Instead, we must specify what an agent should do in any reachable state.
- We call this specification a *policy*
  - "If you're below the goal, move up."
  - "If you're in the left-most column, move right."
- We denote a policy with π, and $\pi(s)$ indicates the policy for state s.

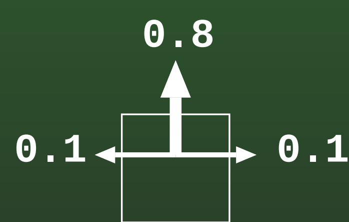
18-7: MDP solutions

- Things to note:
  - We've wrapped the goal formulation into the problem
    - Different goals will require different policies.
  - We are assuming a great deal of (correct) knowledge about the world.
    - State transition models, rewards
    - We'll touch on how to learn these without a model.

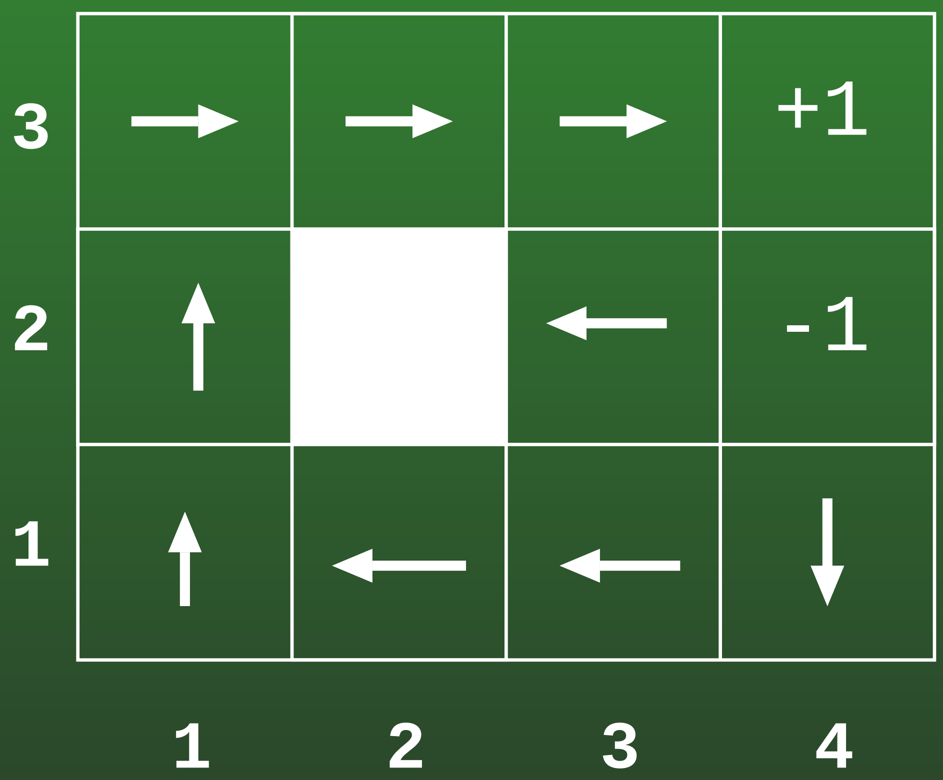
18-8: Comparing policies

- We can compare policies according to the expected utility of the histories they produce.
- The policy with the highest expected utility is the *optimal policy*.
- Once an optimal policy is found, the agent can just look up the best action for any state.





Example grid problem

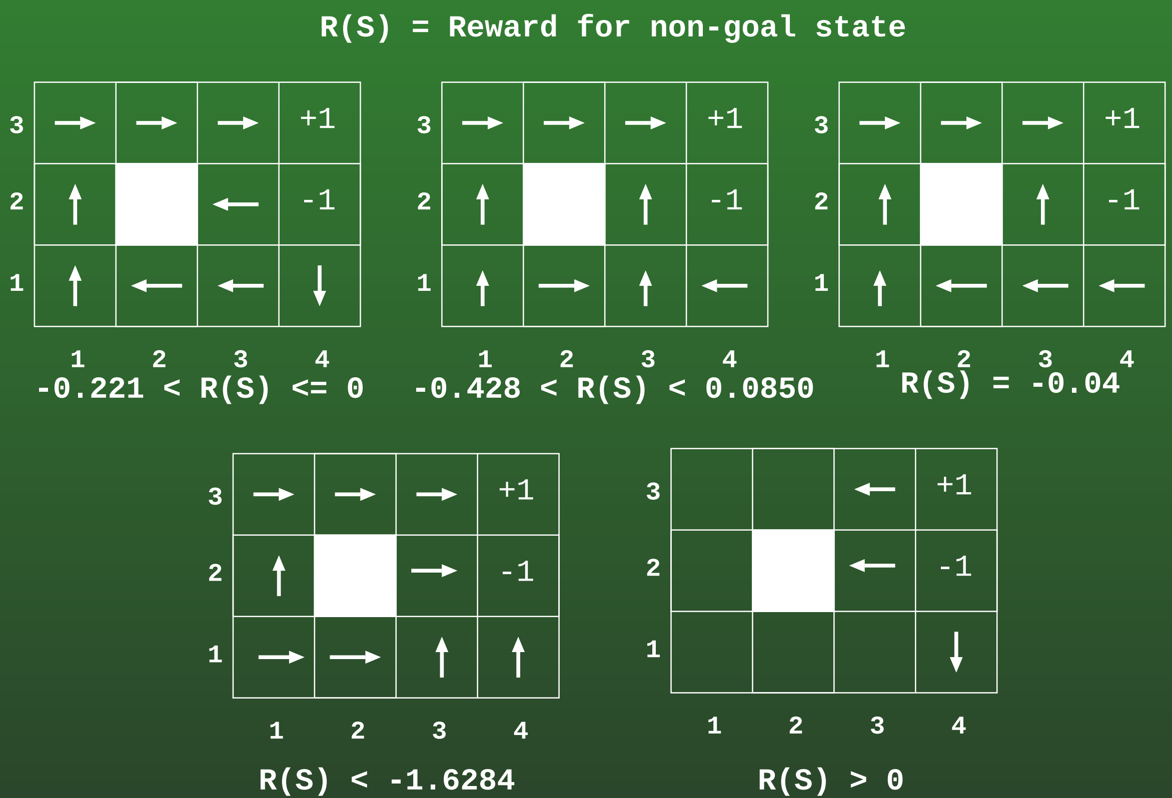


- Assumes no cost for non-goal states
- No benifit for faster solutions

18-10: Non-Goal Costs

- Spending unlimited time trying to find th best solution is not always the best idea.
- We can give a cost (negative utility) to each non-goal state
- penalized for taking too long to find the goal state

18-11: Non-Goal Costs



- Very high cost: Agent tries to exit immediately
- Middle ground: Agent tries to avoid bad exit
- Positive reward: Agent doesn't try to exit.

18-12: More on reward functions

- In solving an MDP, an agent must consider the value of future actions.
- There are different types of problems to consider:
- Horizon - does the world go on forever?
  - Finite horizon: after N actions, the world stops and no more reward can be earned.
  - Infinite horizon; World goes on indefinitely, or we don't know when it stops.
    - Infinite horizon is simpler to deal with, as policies don't change over time.

18-13: More on reward functions

- We also need to think about how to value future reward.
- \$100 is worth more to me today than in a year.
- We model this by *discounting* future rewards.
  - γ is a *discount factor*
- $U(s_0, s_1, s_2, s_3, ...) =$ $R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \gamma^3 R(s_3) + ..., \gamma \in [0, 1]$
- If γ is large, we value future states
- if γ is low, we focus on near-term reward
- In monetary terms, a discount factor of γ is equivalent to an interest rate of $(1/\gamma) - 1$

18-14: More on reward functions

- Discounting lets us deal sensibly with infinite horizon problems.
  - Otherwise, all EUs would approach infinity.
- Expected utilities will be finite if rewards are finite and bounded and $\gamma < 1$.
- We can now describe the optimal policy π^* as:
-

$$\pi^* = argmax_\pi EU(\sum_{t=0}^{\infty} \gamma^t R(s_t)|\pi)$$

18-15: Value iteration

- How to find an optimal policy?
- We'll begin by calculating the expected utility of each state and then selecting actions that maximize expected utility.
- In a sequential problem, the utility of a state is the expected utility of all the state sequences that follow from it.
- This depends on the policy π being executed.
- Essentially, $U(s)$ is the expected utility of executing an optimal policy from state s.

Utilities of States

0.812	0.868	0.918	+1
0.762		0.660	-1
0.705	0.655	0.611	0.388
1	2	3	4

γ = 1

R(S) = -0.04 (non-goals)

- Notice that utilities are highest for states close to the +1 exit.

18-17: Utilities of States

- The utility of a state is the immediate reward for that state plus the expected discounted utility of the next state, assuming that the agent chooses the optimal action.
- $$U(s) = R(s) + \gamma max_a \sum_{s'} T(s, a, s')U(s')$$
- This is called the Bellman equation
- Example:

$$\begin{aligned} U(1,1) = -0.04 + \gamma max(&0.8U(1,2) + 0.1U(2,1) + 0.1U(1,1) \\ &0.9U(1,1) + 0.1U(1,2), \\ &0.9U(1,1) + 0.1U(2,1), \\ &0.8U(2,1) + 0.1U(1,2) + 0.1U(1,1)) \end{aligned}$$

18-18: Dynamic Programming

- Solving the Bellman equation is a dynamic programming problem
- In an acyclic transition graph, you can solve these recursively by working backward from the final state to the initial states.
- Can't do this directly for transition graphs with loops.

18-19: Value Iteration

- Since state utilities are defined in terms of other state utilities, how to find a closed-form solution?
- We can use an iterative approach:
  - Give each state random initial utilities.
  - Calculate the new left-hand side for a state based on its neighbors' values.
  - Propagate this to update the right-hand-side for other states,
  - Update rule:
    $U_{i+1}(s) = R(s) + \gamma max_a \sum_{s'} T(s, a, s')U_i(s')$
- This is guaranteed to converge to the solutions to the Bellman equations.

18-20: Value Iteration algorithm

```
Assing random utilities to each state
do
   for s in states
      U(s) = R(s) + gamma * max T(s,a,s') U(s')
until
     all utilities change by less then delta
```

- where $\delta = error * (1 - \gamma)/\gamma$

Example

3	0.1	-0.1	0.05	+1
2	-0.02		0.15	-1
1	0.0	0.1	-0.1	0.15
	1	**2**	**3**	**4**

γ = 0.8
R(S) = -0.04
error = 0.01
δ = 0.0025

- Initially, use random values

Example

	1	2	3	4
3	0.03	-0.02	0.62	+1
2	0.02		0.05	-1
1	0.02	0.02	0.08	0.06

γ = 0.8
R(S) = -0.04
error = 0.01
δ = 0.0025

- After 1 iteration

Example

3	-0.02	0.35	0.65	+1
2	-0.02		0.28	-1
1	-0.02	0.01	0.02	0.01
	1	2	3	4

γ = 0.8
R(S) = -0.04
error = 0.01
δ = 0.0025

- After 2 iterations

18-24: Example

	1	2	3	4
3	0.19	0.43	0.69	+1
2	-0.06		0.32	-1
1	-0.04	-0.03	0.14	-0.03

γ = 0.8

R(S) = -0.04

error = 0.01

δ = 0.0025

- After 3 iterations

Example

	1	2	3	4
3	0.25	0.47	0.68	+1
2	0.07		0.34	-1
1	-0.07	0.04	0.16	-0.03

γ = 0.8
R(S) = -0.04
error = 0.01
δ = 0.0025

- After 4 iterations

Example

	1	2	3	4
3	0.27	0.47	0.68	+1
2	0.13		0.34	-1
1	0.0	0.07	0.18	-0.02

γ = 0.8
R(S) = -0.04
error = 0.01
δ = 0.0025

- After 5 iterations

Example

3	0.29	0.47	0.68	+1
2	0.15		0.34	-1
1	0.04	0.08	0.18	-0.01
	1	2	3	4

γ = 0.8
R(S) = -0.04
error = 0.01
δ = 0.0025

- After 6 iterations – almost converged

18-28: Discussion

- Strengths of Value iteration
  - Guaranteed to converge to correct solution
  - Simple iterative algorithm
- Weaknesses:
  - Convergence can be slow
  - We really don’t need all this information
  - Just need *what to do* at each state.

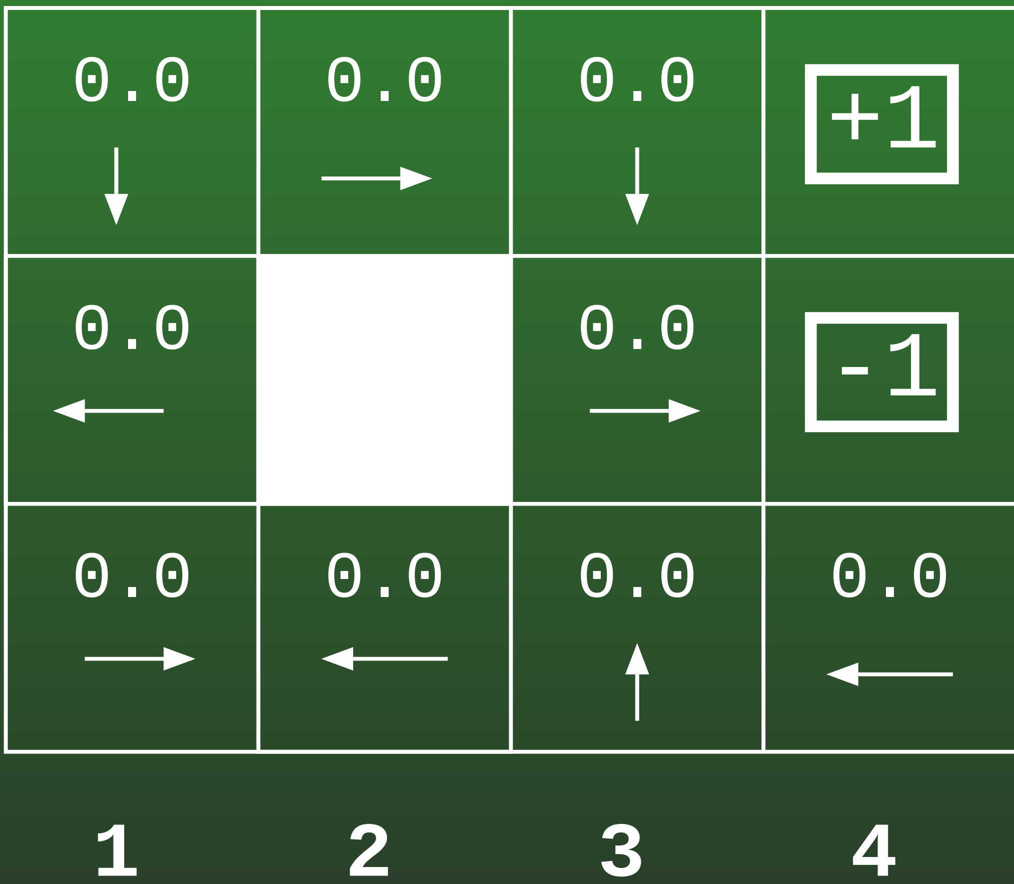
18-29: Policy iteration

- Policy iteration helps address these weaknesses.
- Searches directly for optimal policies, rather than state utilities.
- Same idea: iteratively update policies for each state.
- Two steps:
  - Given a policy, compute the utilities for each state.
  - Compute a new policy based on these new utilities.

Policy iteration algorithm

```
Initialize all state utilities to zero
Pi = random policy vector indexed by state
do
  U = evaluate the utility of each state for Pi
  for s in states
    a = find action that maximizes expected
          utility for that state
    Pi(s) = a
while some action changed
```

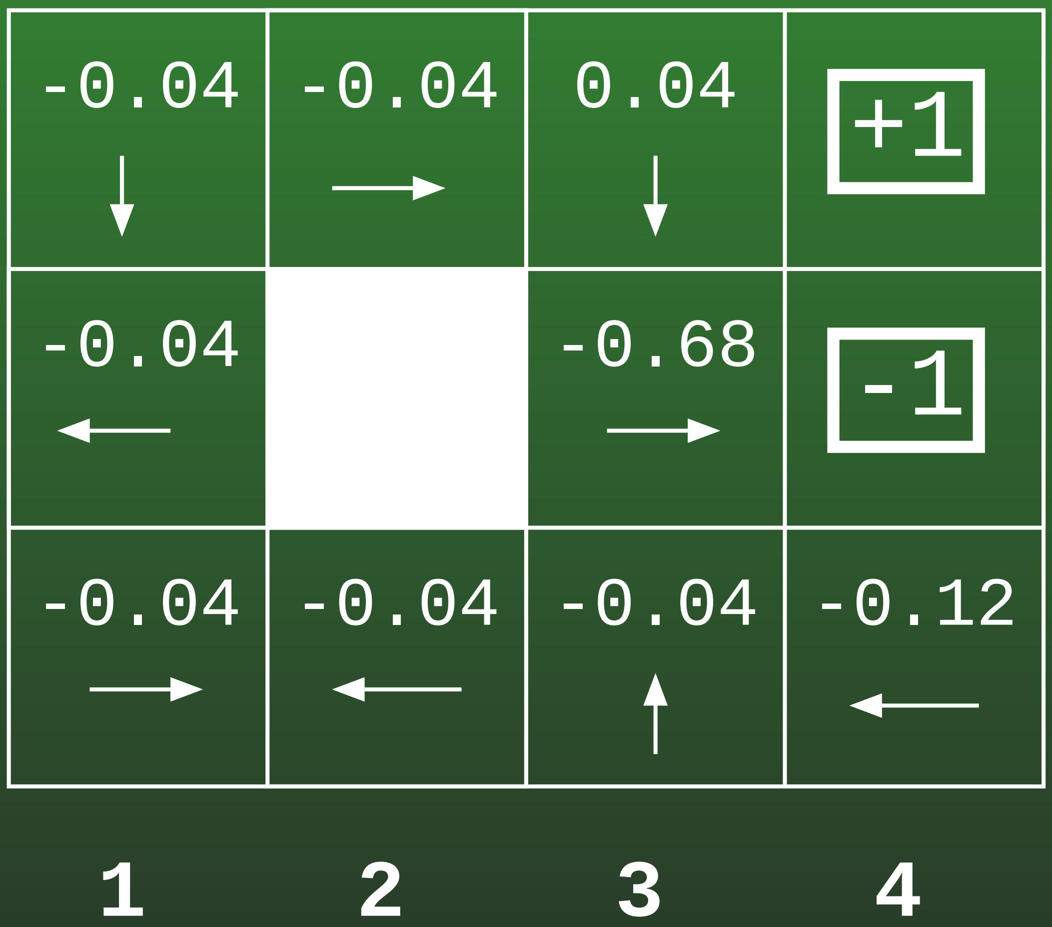
18-31: Policy Iter. Example



All non-goal utilities 0

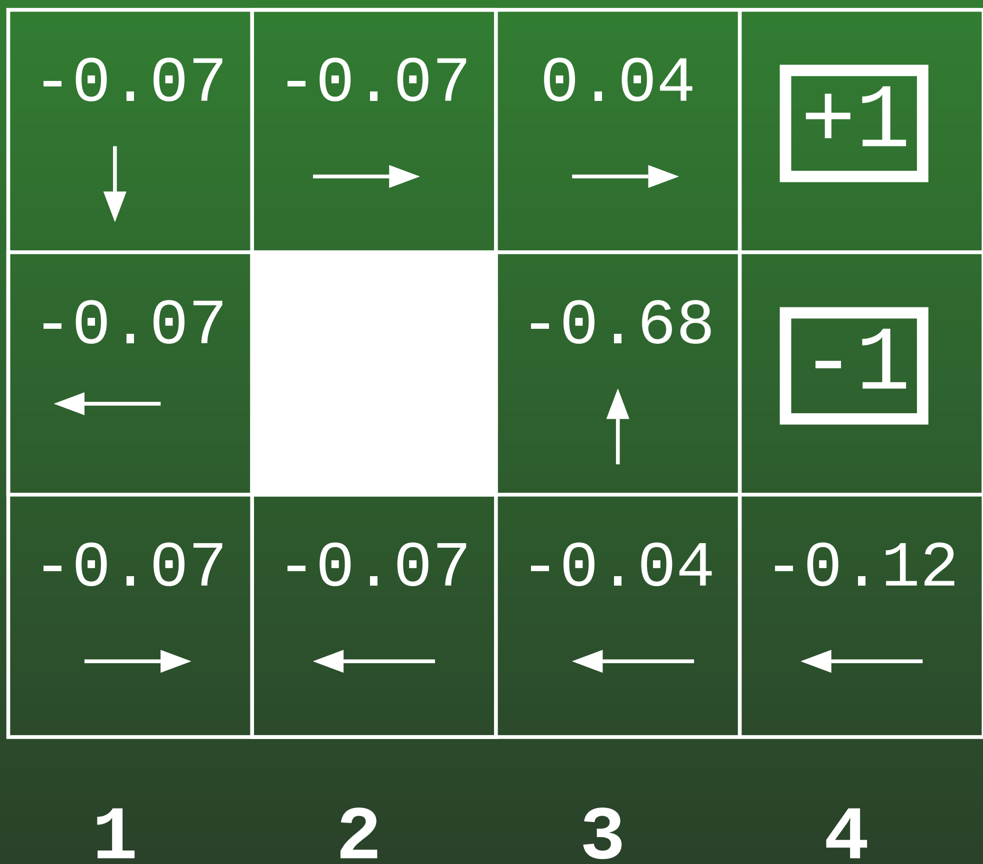
Random policies

18-32: Policy Iter. Example



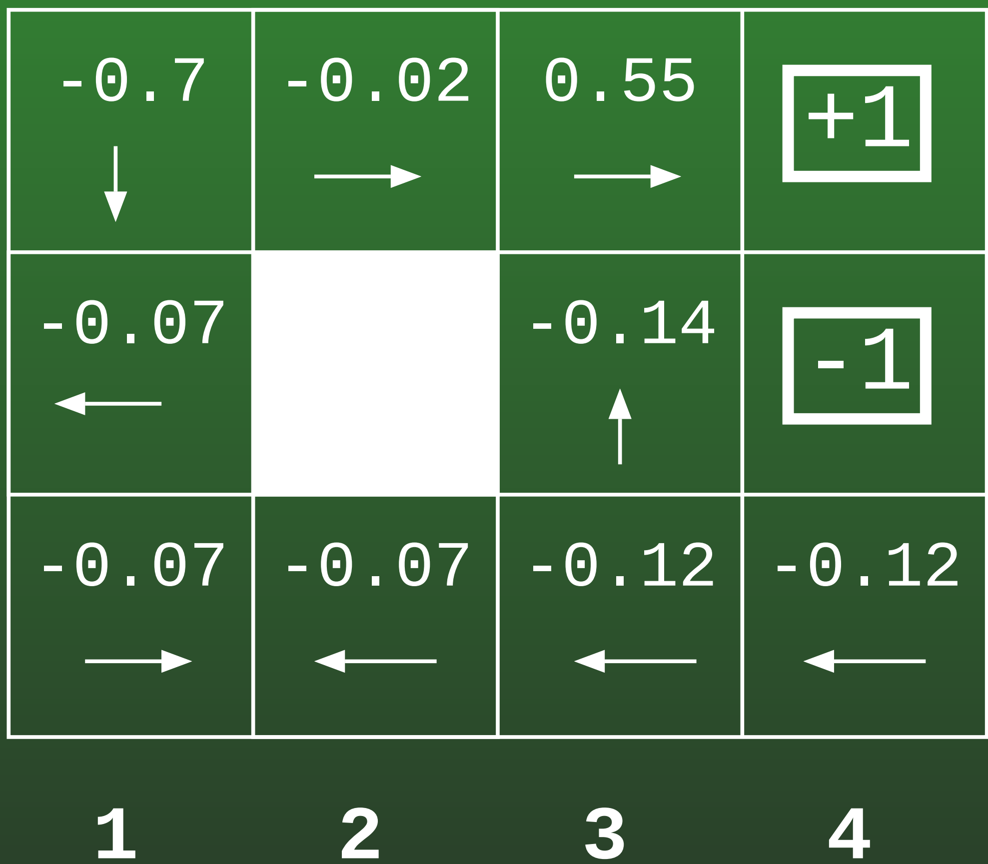
Assign new utilities based on old utilies and policy

Policy Iter. Example



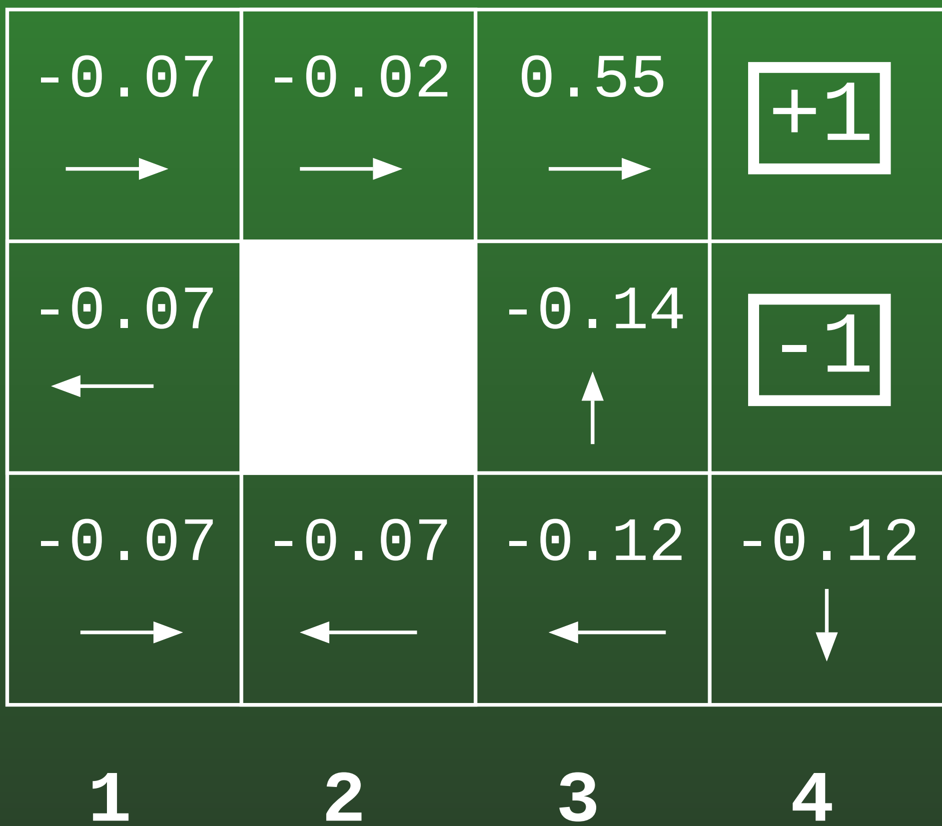
Create a new policy based on new Utilities

Policy Iter. Example



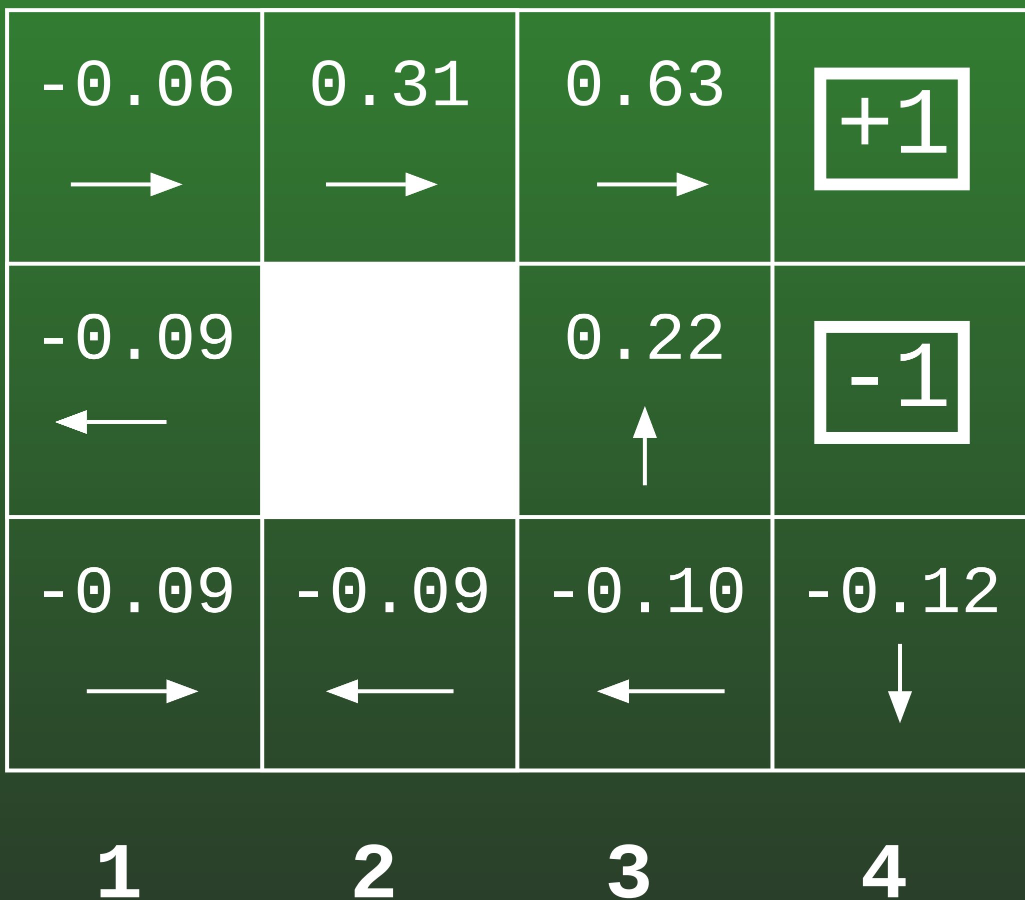
Create new Utilities based on policy and previous Utilities

Policy Iter. Example



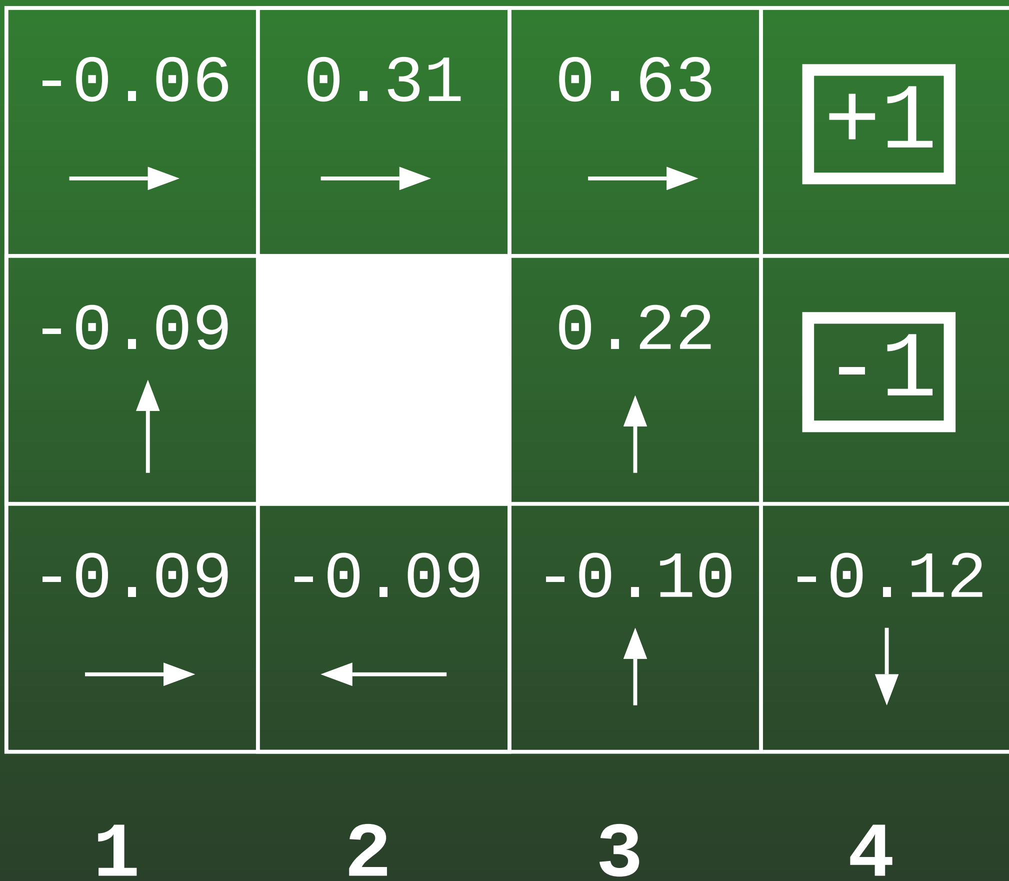
Create policies based on previous Utilities

Policy Iter. Example



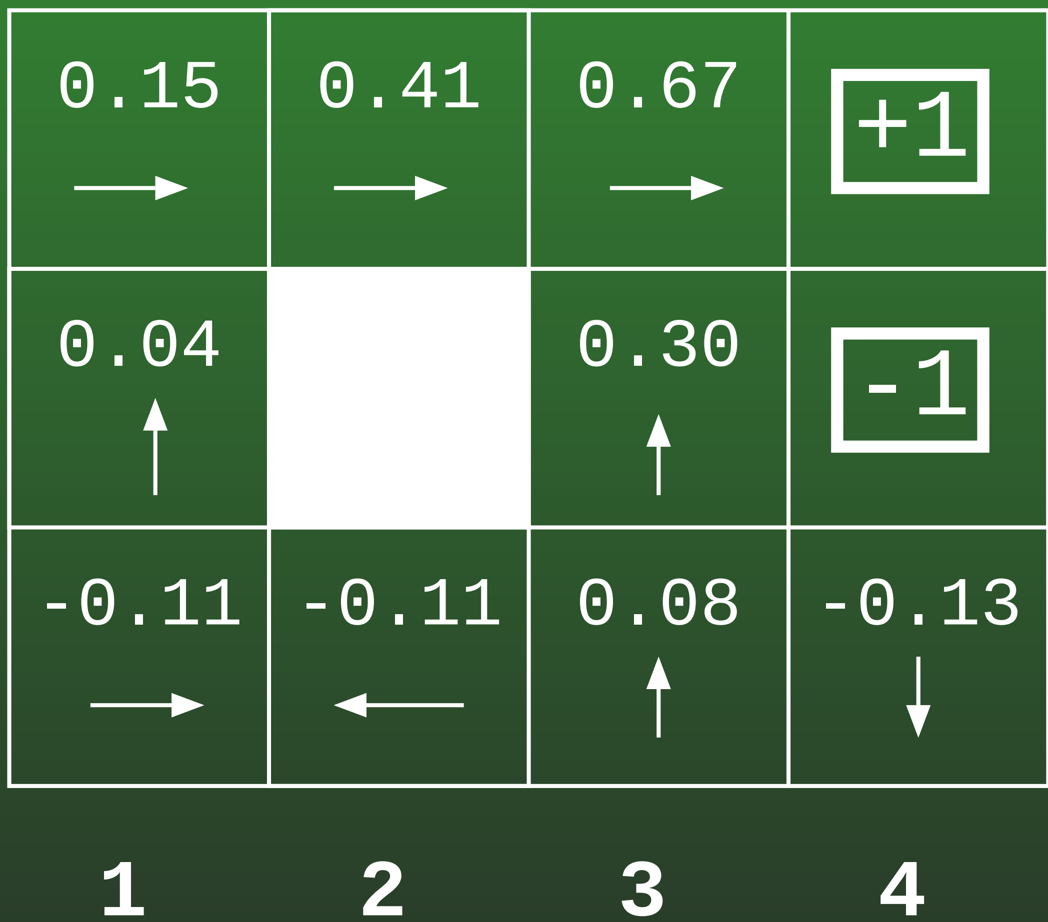
Create new Utilities using old Utilities and Policy

18-37: Policy Iter. Example



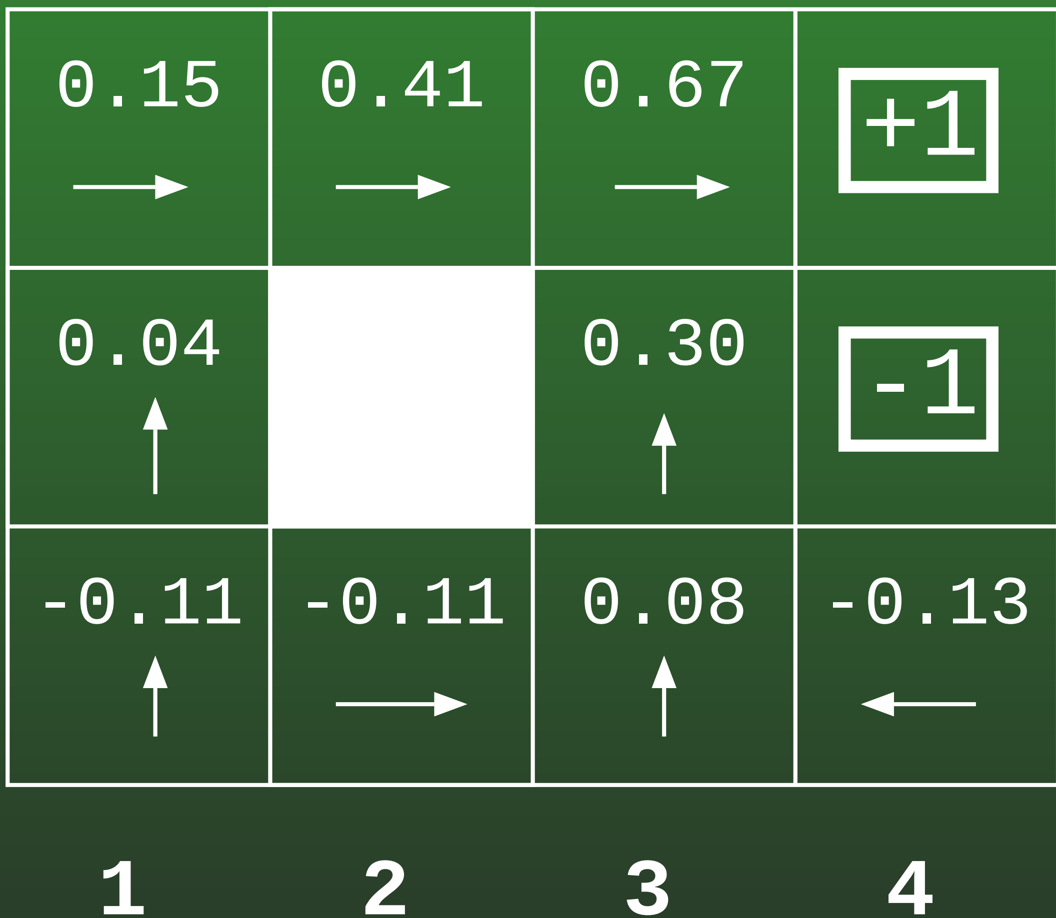
Use new utility estimates to construct new policies

18-38: Policy Iter. Example



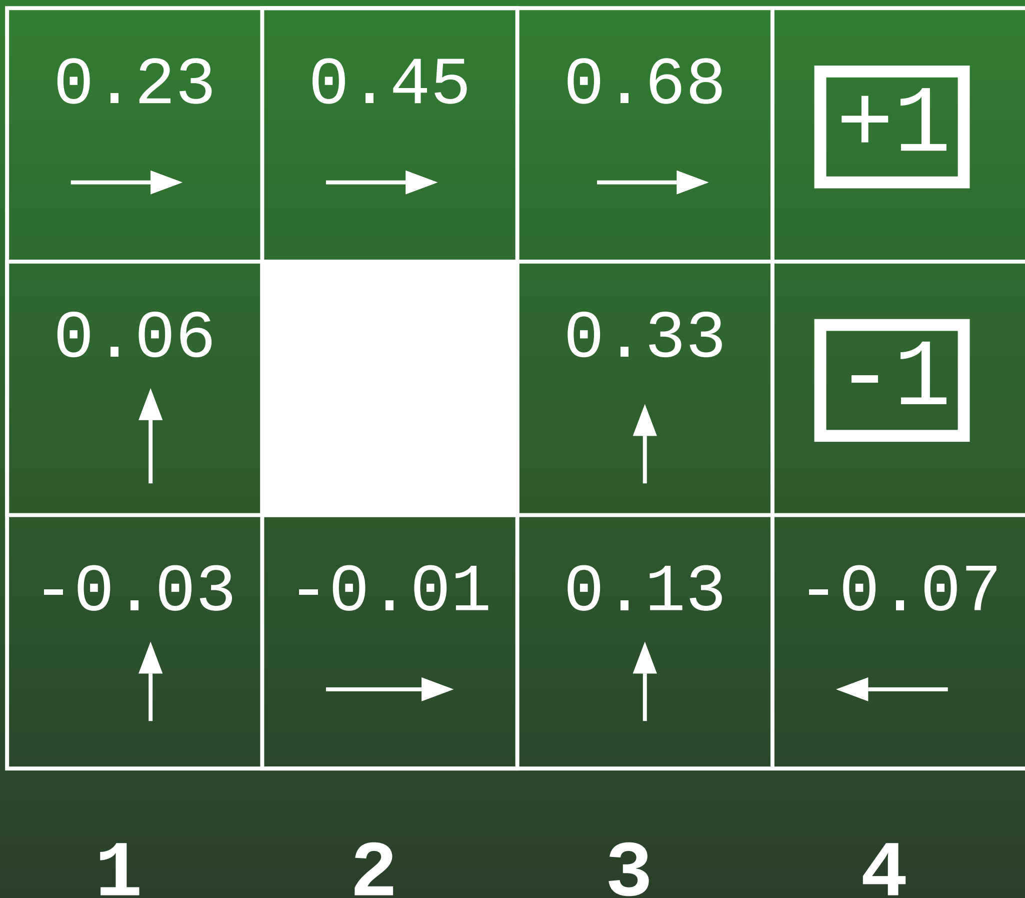
Create new utility esitmates using old Utilities & current policies

Policy Iter. Example



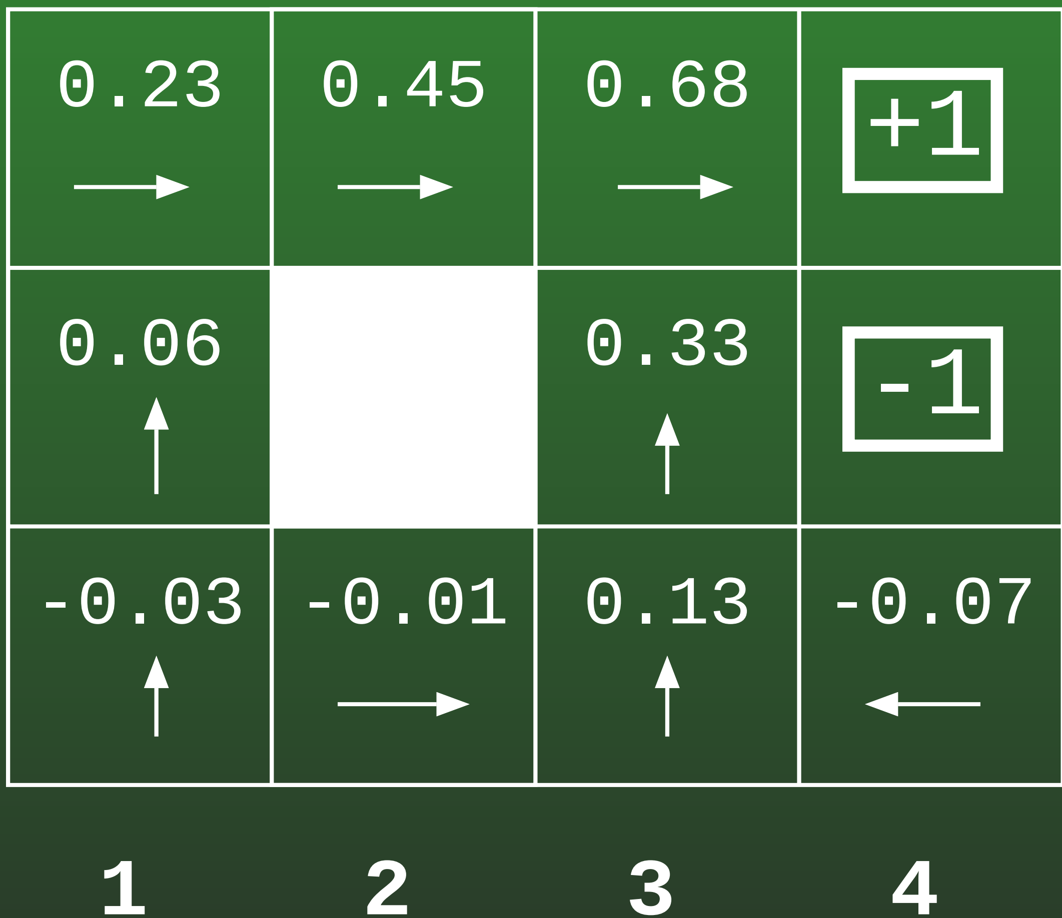
Use new Utilities to update policy

18-40: Policy Iter. Example



Update utilities based on old Utilites and Policies

Policy Iter. Example



Update policices
No change.

Discussion

- Advantages:
  - Faster convergence.
  - Solves the actual problem we're interested in. We don't really care about utility estimates except as a way to construct a policy.

18-43: Learning a Policy

- MDPs assume that we know a model of the world
  - Specifically, the transition function T
- We can also learn a policy through interaction with the environment.
- This is known as *reinforcement learning*.
- We'll talk about this in a couple of weeks.

18-44: Summary

- Markov decision policies provide an agent with a description of *how to act optimally* for any state in a problem.
  - Must know state space, have a fixed goal.
- Value iteration and policy iteration can be applied to solve this.