Graduate Algorithms

CS673-2016F-01

Algorithm Analysis

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01-0: Syllabus

- Office Hours
- Course Text
- Prerequisites
- Test Dates & Testing Policies
  - Try to combine tests
- Grading Policies

01-1: How to Succeed

- Come to class. Pay attention. Ask questions.
  - A question as vague as “I don’t get it” is perfectly acceptable.
  - If you’re confused, *at least* 4 other people are, too.
- Come by my office
  - I am *very* available to students.
- Start the homework assignments early
- Read the textbook. It’s one of the best textbooks around.
  - Ask questions if you don’t understand the textbook!

01-2: How to Succeed

- Start Early (clarification)
  Some of the homework assignments for this class will be very difficult. I will not expect you to be able to just sit down and complete them.
  - Start working on a problem right when it is given. Work on it until you get stuck.
  - Come by my office, and talk about the problem with me. I will **not** give you the solution, but I will give you a little push in the right direction
  - Work on the problem some more, get stuck again.
  - Come by my office again, get another push.
  - Repeat, as necessary

01-3: What is an Algorithm?

- An algorithm is a step-by-step method for solving a problem.
- Each step must be well defined.
- Algorithm $\neq$ Computer Program.
- A program is an *implementation* of an algorithm.
- Can have different implementations of the same algorithm
  - Different Languages
  - Different Coding Styles

01-4: Comparing Algorithms

- What makes one algorithm better than another?

01-5: Comparing Algorithms

- What makes one algorithm better than another?
  - Space
    - How much memory is used
  - Time
    - How long the algorithm takes to run
- Often see a time/space tradeoff – we can make an algorithm run faster by giving it more space, and vice/versa

01-6: Example: Insertion Sort

	Time / Operation	# of times executed
`i = 2`	C_1	1
`while (i <= N) do`	C_2	N
`  tmp = A[i]`	C_3	(N-1)
`  j = i`	C_4	(N-1)
`  while (j > 1) && (tmp < A[j-1]) do`	C_5	$\sum_{j=1}^{N-1} Itr_j + 1$
`    A[j] = A[j-1]`	C_6	$\sum_{j=1}^{N-1} Itr_j$
`    j--`	C_7	$\sum_{j=1}^{N-1} Itr_j$
`  A[j] = tmp`	C_8	(N-1)
`  i++`	C_9	(N-1)

01-7: Example: Insertion Sort

Adding everything up:

$$C_1 + C_2 N + (C_3 + C_4 + C_8 + C_9)(N-1) + C_5 \sum_{j=1}^{N-1} (Itr_j + 1) + (C_7 + C_8) \sum_{j=1}^{N-1} Itr_j$$

$$(\sum_{j=1}^{N-1} Itr_j)(C_5 + C_7 + C_8) + N(C_2 + C_3 + C_4 + C_5 + C_8 + C_9) + C_1 - (C_3 + C_4 + C_8 + C_9)$$

$$\sum_{j=1}^{N-1} Itr_j * C_{10} + N * C_{11} + C_{12}$$

01-8: Example: Insertion Sort

$$\sum_{j=1}^{N-1} Itr_j * C_{10} + N * C_{11} + C_{12}$$

- Don't know Itr_1, Itr_2, ...
- What can we do?

01-9: Example: Insertion Sort

$$\sum_{j=1}^{N-1} Itr_j * C_{10} + N * C_{11} + C_{12}$$

- Don't know Itr_1, Itr_2, ...
- What can we do?
  - Worst case
  - Best Case
  - Average Case

01-10: Example: Insertion Sort

$$\sum_{j=1}^{N-1} Itr_j * C_{10} + N * C_{11} + C_{12}$$

- Best case

$$N * C_{11} + C_{12}$$

01-11: Example: Insertion Sort

$$\sum_{j=1}^{N-1} Itr_j * C_{10} + N * C_{11} + C_{12}$$

- Worst Case

$$\sum_{j=1}^{N-1} j * C_{10} + N * C_{11} + C_{12} = \frac{(N)(N-1)}{2} * C_{10} + N * C_{11} + C_{12}$$

01-12: Example: Insertion Sort

$$\sum_{j=1}^{N-1} Itr_j * C_{10} + N * C_{11} + C_{12}$$

- Average Case
  - $$\sum_{\texttt{All Possible Occurrences } i} Time(i) * P(i)$$
    - Calculating $P(i)$ can be difficult
    - Often assume that all instances are equally likely
      - Is that always a good assumption?

01-13: Constants

- Constants & lower order terms can be ugly:

$$\sum_{j=1}^{N-1} j*C_{10}+N*C_{11}+C_{12} = \frac{(N-1)(N-2)}{2}*C_{10}+N*C_{11}+C_{12}$$

- Fortunately, these constant & lower order terms don't matter!

01-14: Constants

n	$8n^2 - 2n - 3$	Time	n^2	Time
10	777	0.0007 sec	100	0.0001 sec
100	79797	0.0062 sec	10000	0.01 sec
1000	$7.998 * 10^6$	7.98 sec	10^6	1 sec
10000	$7.9998 * 10^8$	13 min	10^8	1.6 min
100000	$7.99998 * 10^{10}$	22 hours	10^{10}	2 hours
1000000	$8 * 10^{12}$	91 days	10^{12}	11 days

01-15: Constants

n	$6n+3$	Time	$99n$	Time
10	63	0.00006 sec	990	0.00099 sec
100	603	0.0006 sec	9900	0.0099 sec
1000	6003	0.006 sec	99000	0.099 sec
10000	60003	0.06 sec	990000	0.99 sec
100000	600003	0.6 sec	9900000	9.9 sec
1000000	$6 * 10^6$	6 sec	$9.9 * 10^7$	99 sec

01-16: Do Constants Matter?

Comparing a recurive version of Binary Search with iterative version of linear search

- Linear Search requires time $c_1 * n$, for some c_1
- Binary Search requires time $c_2 * lg(n)$, for some c_2

What if there is a *very* high overhead cost for function calls?

What if c_2 is *1000 times larger* than c_1?

01-17: Constants *Do Not* Matter!

Length of list	Time Required for Linear Search	Time Required for Binary Search
10	0.001 seconds	0.3 seconds
100	0.01 seconds	0.66 seconds
1000	0.1 seconds	1.0 seconds
10000	1 second	1.3 seconds
100000	10 seconds	1.7 seconds
1000000	2 minutes	2.0 seconds
10000000	17 minutes	2.3 seconds
10^{10}	11 days	3.3 seconds
10^{15}	30 centuries	5.0 seconds
10^{20}	300 million years	6.6 seconds

01-18: Big-O Notation

$$O(g(n)) = \{f(n) \mid \exists c, n_0, s.t. \; f(n) \leq cg(n) \text{ whenever } n > n_0\}$$

$f(n) \in O(g(n))$ means:

- f is bound from above by g
- f grows no faster than g
- g is an upper bound on f

01-19: Big-Ω Notation

$$\Omega(g(n)) = \{f(n) \mid \exists c, n_0, s.t. \; cf(n) \geq g(n) \text{ whenever } n > n_0\}$$

$f(n) \in \Omega(g(n))$ means:

- f is bound from below by g
- g grows no faster than f
- g is a lower bound on f

01-20: Big-Θ Notation

$$\begin{aligned}\Theta(g(n)) \;=\; \{f(n) \mid \exists c_1, c_2, n_0, s.t.\\ c_1 g(n) \le f(n) \le c_2 g(n)\\ \text{whenever } \; n > n_0\}\end{aligned}$$

Alternately,

$$\Theta(g(n)) = O(g(n)) \bigcap \Omega(g(n))$$

01-21: Big-Θ Notation

Show:

$$3n^2 + 4n \in \Theta(n^2)$$

01-22: Big-Θ Notation

Show:

$$3n^2 + 4n \in \Theta(n^2)$$

$c_1 * n^2 \leq 3n^2 + 4n \leq c_2 * n^2$

True, as long as $c_1 \leq 3 + 4/n$, $c + 2 \geq 3 + 4/n$

(since $n > n_0$, we can assume $4/n \leq 1$)

01-23: Big-Θ Notation

Verify:

$$4n^3 \notin \Theta(n^2)$$

01-24: Big-Θ Notation

Verify:

$$4n^3 \notin \Theta(n^2)$$

$$\begin{aligned} 4n^3 &\leq c_1 n^2 \\ 4n &\leq c_1 \end{aligned}$$

which is not true for any constant c_1

01-25: Big-Θ Notation

- We can drop all constants and lower order terms when finding the Θ running time of a quadratic

$$\sum_{i=0}^{k} a_i n^i \in \Theta(n^k)$$

01-26: Big-Θ in Equations

- $f(n) = \Theta(g(n))$ is shorthand for $f(n) \in \Theta(g(n))$
- $n^2 + 3n + 2 = n^2 + \Theta(n)$ Means $\exists f(n) \in \Theta(n)$ such that $n^2 + 3n + 2 = n^2 + f(n)$

This can lead to some weirdness:

- $n^2 + 2n + 3 = 5n + \Theta(n^2)$
- $n^2 + 2n + 3 = 6n + O(n^4)$

01-27: Loose & Tight Bounds

- $O()$ may or may not be tight:
  - $n^2 + 2n + 3 \in O(n^2)$
  - $n^2 + 2n + 3 \in O(n^4)$
- We have a notation for tight bound (Θ), and we also have a notation for a bound that is not tight:

$$o(g(n)) = \{f(n) \mid \forall c > 0, \exists n_0 > 0 \; s.t. \; 0 \leq f(n) < cg(n) \text{ whenever } n \geq n_0\}$$

01-28: Loose & Tight Bounds

$$o(g(n)) = \{f(n) \mid \forall c > 0, \exists n_0 > 0 \; s.t. \; 0 \leq f(n) < cg(n) \text{ whenever } n \geq n_0\}$$

- $4n \in o(n^2)$
- $3n^2 \notin o(n^2)$
- $n^2 \notin o(n^2)$

01-29: little-o

$$o(g(n)) = \{f(n) \mid \forall c > 0, \exists n_0 > 0 \; s.t. \; 0 \leq f(n) < cg(n) \text{ whenever } n \geq n_0\}$$

$$f(n) \in o(g(n)) \Rightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

01-30: little-ω

$$\omega(g(n)) = \{f(n) \mid \forall c > 0, \exists n_0 > 0 \; s.t. \; 0 \leq cg(n) < f(n) \text{ whenever } n \geq n_0\}$$

$$f(n) \in \omega(g(n)) \Rightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$$

01-31: $O, o, \Omega, \omega, \Theta$

True or false:

- $f(n) \in \Theta(g(n)) \rightarrow f(n) \in O(g(n))$
- $f(n) \in O(g(n)) \rightarrow g(n) \in \Omega(f(n))$
- $f(n) \in O(g(n)) \rightarrow f(n) \in o(g(n))$
- $f(n) \in o(g(n)) \rightarrow f(n) \in O(g(n))$
- $f(n) \in o(g(n)) \rightarrow f(n) \in \Theta(g(n))$
- For any two functions $f(n), g(n)$, either $f(n) \in O(g(n))$ or $f(n) \in \Omega(g(n))$

01-32: $O, o, \Omega, \omega, \Theta$

True or false:

- $f(n) \in \Theta(g(n)) \rightarrow f(n) \in O(g(n))$ True
- $f(n) \in O(g(n)) \rightarrow g(n) \in \Omega(f(n))$ True
- $f(n) \in O(g(n)) \rightarrow f(n) \in o(g(n))$ False, $n \in O(n)$, $n \notin o(n)$
- $f(n) \in o(g(n)) \rightarrow f(n) \in O(g(n))$ True.
- $f(n) \in o(g(n)) \rightarrow f(n) \in \Theta(g(n))$ False. In fact, $f(n) \in o(g(n)) \rightarrow f(n) \notin \Theta(g(n))$
- For any two functions $f(n), g(n)$, either $f(n) \in O(g(n))$ or $f(n) \in \Omega(g(n))$. False. Consider $f(n) = n$, $g(n) = n^{1+\sin n}$

01-33: Summations

- We will assume you've seen inductive proof that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (check Appendix A in the text otherwise!)
- Can use induction to prove bounds as well. Show:

$$\sum_{i=1}^{n} 3^i = O(3^n)$$

01-34: Summations

$$\sum_{i=1}^{n} 3^i = O(3^n)$$

- Base Case:
  - $\sum_{i=1}^{1} 3^1 = 3 \leq c * 3^1$ as long as $c \geq 1$

01-35: Summations

$$\sum_{i=1}^{n} 3^i = O(3^n)$$

- Recursive Case:

$$\begin{aligned}
\sum_{i=1}^{n+1} 3^i &= \sum_{i=1}^{n} 3^i + 3^{n+1} \\
&\leq c3^n + 3^{n+1} \\
&= (1/3 + 1/c)c3^{n+1} \\
&\leq c3^{n+1}
\end{aligned}$$

As long as $(1/3 + 1/c) \leq 1$, or $c \geq 3/2$

01-36: Summations

Beware! What's wrong with this proof?

$$\sum_{i=1}^{n} i \in O(n)$$

- Base case: $\sum_{i=1}^{1} i = O(1)$
- Inductive case:

$$\begin{aligned}\sum_{i=1}^{n+1} i &= \sum_{i=1}^{n} i + (i+1) \\ &= O(n) + (n+1) \\ &= O(n)\end{aligned}$$

01-37: Bounding Summations

$$\sum_{i=1}^{n} a_i \leq n * a_{max}$$

for instance: $\sum_{i=1}^{n} i \leq n^2 \in O(n^2)$

and

$$\sum_{i=1}^{n} a_i \geq n * a_{min}$$

for instance: $\sum_{i=1}^{n} i \geq 1 * n \in \Omega(n)$

(note that the bounds are not always tight!)

01-38: Splitting Summations

We can sometimes get tighter bounds by splitting the summation:

$$\begin{aligned}
\sum_{i=1}^{n} i &= \sum_{i=1}^{\lfloor n/2 \rfloor} i + \sum_{i=\lfloor n/2 \rfloor + 1}^{n} i \\
&\geq n/2 * 1 + n/2 * n/2 \\
&\geq (n/2)^2 \\
&\in \Omega(n^2)
\end{aligned}$$

01-39: Splitting Summations

We can split summations in more tricky ways, as well. Consider the harmonic series:

$$H_n = \sum_{i=1}^{n} \frac{1}{n}$$

How could we split this to get a good upper bound?

01-40: Splitting Summations

We can split summations in more tricky ways, as well. Consider the harmonic series:

$$H_n = \sum_{i=1}^{n} \frac{1}{n}$$

How could we split this to get a good upper bound?
HINT: The solution we are looking for is $H_n \in \lg(n)$

01-41: Splitting Summations

$$
\begin{aligned}
H_n &= \sum_{i=1}^{n} \frac{1}{i} \\
&\le \sum_{i=0}^{\lfloor \lg n \rfloor} \sum_{j=0}^{2^i-1} \frac{1}{2^i + j} \\
&\le \sum_{i=0}^{\lfloor \lg n \rfloor} \sum_{j=0}^{2^i-1} \frac{1}{2^i} \\
&\le \sum_{i=0}^{\lfloor \lg n \rfloor} 1 \\
&\le \lg n + 1
\end{aligned}
$$

01-42: Summations & Code

```
for (i = 1; i <= n; i++)
    for (j = 1; j <= i; j++)
        sum++;
```

01-43: Summations & Code

```
for (i = 1; i <= n; i++)
    for (j = 1; j <= i; j++)
        sum++;
```

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \in \Theta(n^2)$$

01-44: Summations & Code

```
for (i = 1; i < n; i = i * 2)
    for (j = 1; j < i; j++)
        sum++;
```

vs

```
for (i = 1; i < n; i = i + 1)
    for (j = 1; j < i; j = j * 2)
        sum++;
```

01-45: Summations & Code

```
for (i = 1; i < n; i = i * 2)
    for (j = 1; j < i; j++)
        sum++;
```

$$\sum_{i=1}^{\lg n} 2^i \in \Theta(n)$$

```
for (i = 1; i < n; i = i + 1)
    for (j = 1; j < i; j = j * 2)
        sum++;
```

$$\sum_{i=1}^{n} \lg n \in \Theta(n \lg n)$$

01-46: More Summations

- More information on manipulation of summations is in Appendix A
  - (pages 1147 – 1157 in the text)
- Read over Appendix A for review
- (Should also read Chapter 2 – I'm assuming that is all review for all of you. Let me know if it is not)

01-47: Recursive Algorithms

- Summations are used to calculate running times for iterative programs
- Recursive Algorithms use Recurrence Relations
  - $T(n)$ is a function that returns the time it takes to solve a problem of size n, for a particular recursive algorithm
  - Definition of $T(n)$ is similar to (but not the same as!) the recursive function itself
  - Usually have a base case and a recursive case

01-48: Recurrence Relations

```
MergeSort(A,low,high) {
    if (low < high) {
        mid = floor ( (low + high / 2) )
        MergeSort(A,low,mid)
        MergeSort(A,mid+1,high)
        Merge(A,low,mid,high)
    }
}
```

01-49: Recurrence Relations

```
MergeSort(A,low,high) {
    if (low < high) {
        mid = floor ( (low + high / 2) )
        MergeSort(A,low,mid)
        MergeSort(A,mid+1,high)
        Merge(A,low,mid,high)
    }
}
```

$$
\begin{aligned}
T(0) &= \Theta(1) \\
T(1) &= \Theta(1) \\
T(n) &= T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \Theta(n)
\end{aligned}
$$

01-50: Recurrence Relations

- How do we solve recurrence relations?
  - Substitution Method
    - Guess a solution
    - Prove the guess is correct, using induction

$$
\begin{aligned}
T(1) &= 1 \\
T(n) &= 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n
\end{aligned}
$$

01-51: Substitution Method

- Inductive Case

$$
\begin{aligned}
T(n) &= 2T\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + n \\
&\leq 2\left(c\frac{n}{2}\lg\frac{n}{2}\right) + n \\
&= cn\lg n - cn\lg 2 + n \\
&= cn\lg n - cn + n \\
&\leq cn\lg n
\end{aligned}
$$

01-52: Substitution Method

- Base Case

$$
\begin{aligned}
T(1) &= 1 \\
T(n) &\leq cn \lg n \\
T(1) &\leq c * 1 * \lg 1 \\
T(1) &\leq c * 1 * 0 = 0
\end{aligned}
$$

Whoops! If the base case doesn't work the inductive proof is broken! What can we do?

01-53: Substitution Method

- Fixing the base case

Note that we only care about $n > n_0$, and for $n > 3$, recurrence does not depend upon $T(1)$ except through $T(2)$ and $T(3)$

$$\begin{aligned} T(2) &= 4 \leq 2 * c * \lg 2 \\ T(3) &= 5 \leq 3 * c * \lg 3 \end{aligned}$$

(for $c > 2$)

01-54: Substitution Method

- Sometimes, the math doesn't work out in the substitution method:

$$\begin{aligned} T(1) &= 1 \\ T(n) &= T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lceil \frac{n}{2} \right\rceil\right) + 1 \end{aligned}$$

(Work on board)

01-55: Substitution Method

Try $T(n) \leq cn$:

$$\begin{aligned} T(n) &= T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lceil \frac{n}{2} \right\rceil\right) + 1 \\ &\leq c\left\lfloor \frac{n}{2} \right\rfloor + c\left\lceil \frac{n}{2} \right\rceil + 1 \\ &\leq cn + 1 \end{aligned}$$

We did not get back $T(n) \leq cn$ – that extra +1 term means the proof is not valid. We need to get back *exactly* what we started with (see invalid proof of $\sum_{i=1}^{n} i \in O(n)$ for why this is true)

01-56: Substitution Method

Try $T(n) \leq cn - b$:

$$\begin{aligned} T(n) &= T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lceil \frac{n}{2} \right\rceil\right) + 1 \\ &\leq c\left\lfloor \frac{n}{2} \right\rfloor - b + c\left\lceil \frac{n}{2} \right\rceil - b + 1 \\ &\leq cn - 2b + 1 \\ &\leq cn - b \end{aligned}$$

As long as $b \geq 1$

01-57: Substitution Method

- Substitution method can *verify* the solution to a recurrence relation, but how can we get our original guess?
  - Compare to similar problems
    - $T(n) = 2T(\lfloor \frac{n}{2} \rfloor - 1) + 3n + 2$ similar to $T(n) = 2T\left(\lfloor \frac{n}{2} \rfloor\right) + n$
  - Start with loose bounds, tighten them to get a tight bound
  - Recursion Trees

01-58: Recursion Trees

$$T(n) = 2T(n/2) + cn$$

01-59: Recursion Trees

$$T(n) = T(n-1) + cn$$

01-60: Recursion Trees

$$T(n) = T(n/2) + c$$

01-61: Recursion Trees

$$T(n) = 3T(n/4) + cn^2$$

01-62: Recursion Trees

$$T(n) = \quad cn^2 \quad +$$

$$T(n/4) \quad + \quad T(n/4) \quad + \quad T(n/4)$$

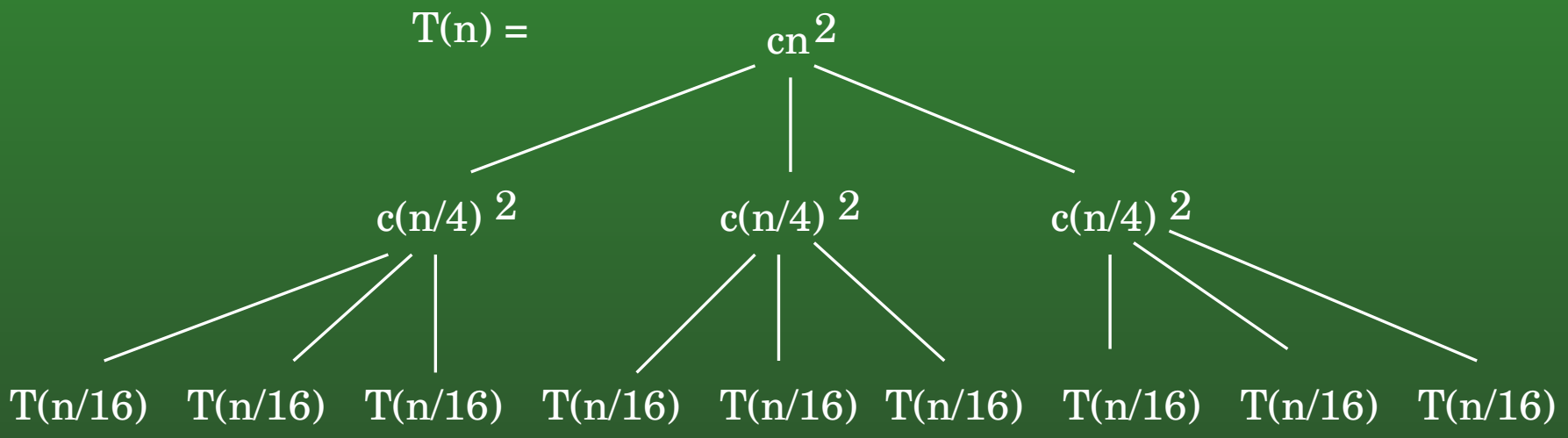
01-63: Recursion Trees

$$T(n) = \quad cn^2 \quad +$$

$$T(n/4) \quad + \quad T(n/4) \quad + \quad T(n/4)$$

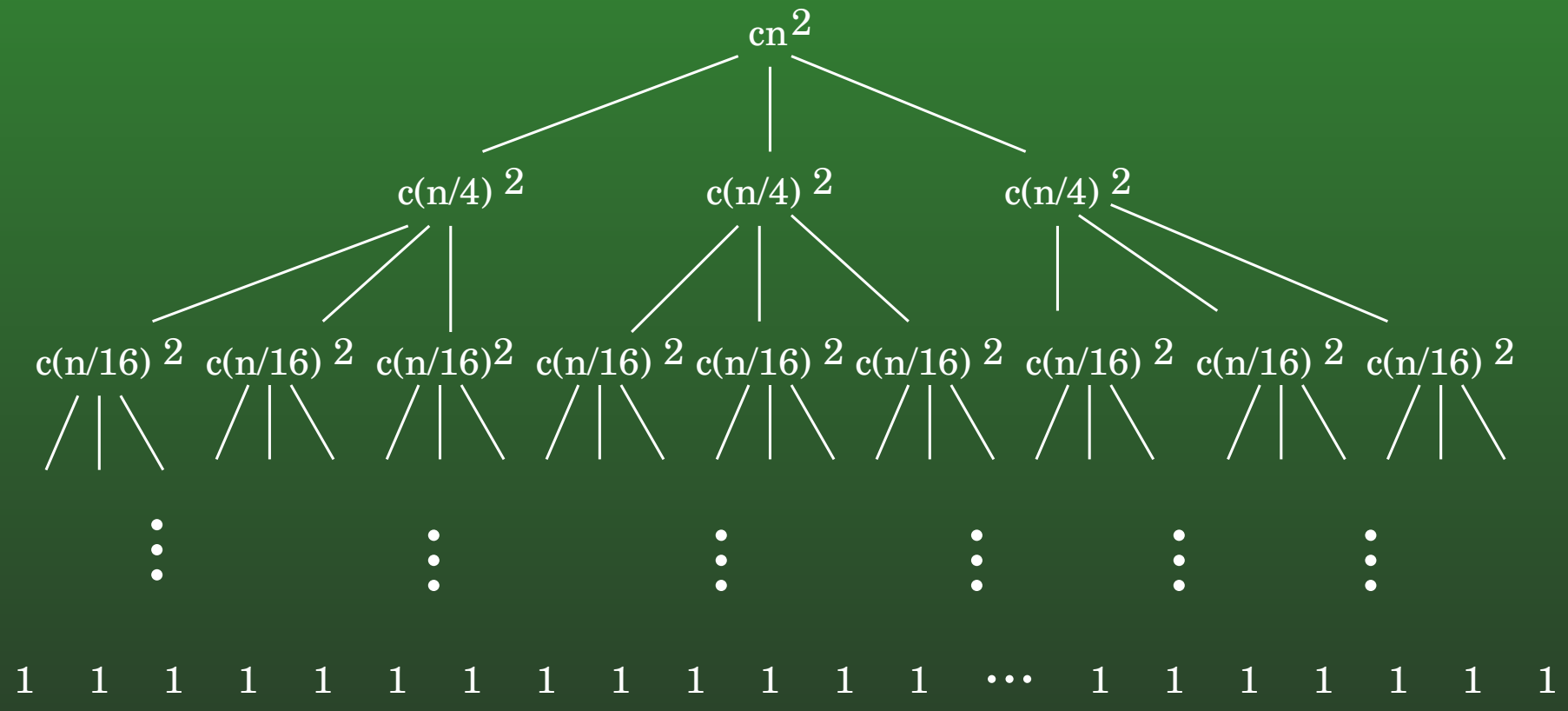
$$T(n/4) = 3T(n/16) + c(n/4)^2$$

01-64: Recursion Trees

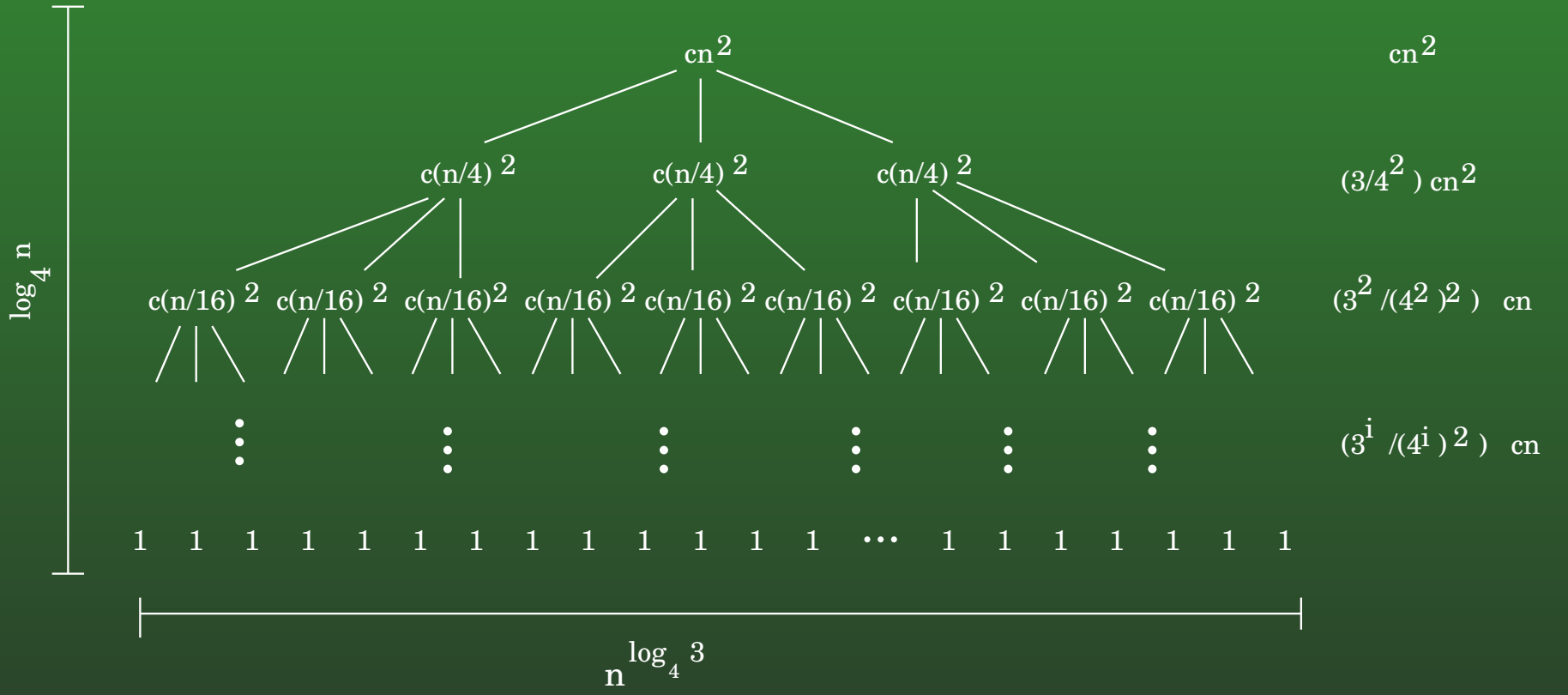


$$T(n/16) = T(n/32) + c(n/16)^2$$

01-65: Recursion Trees



01-66: Recursion Trees



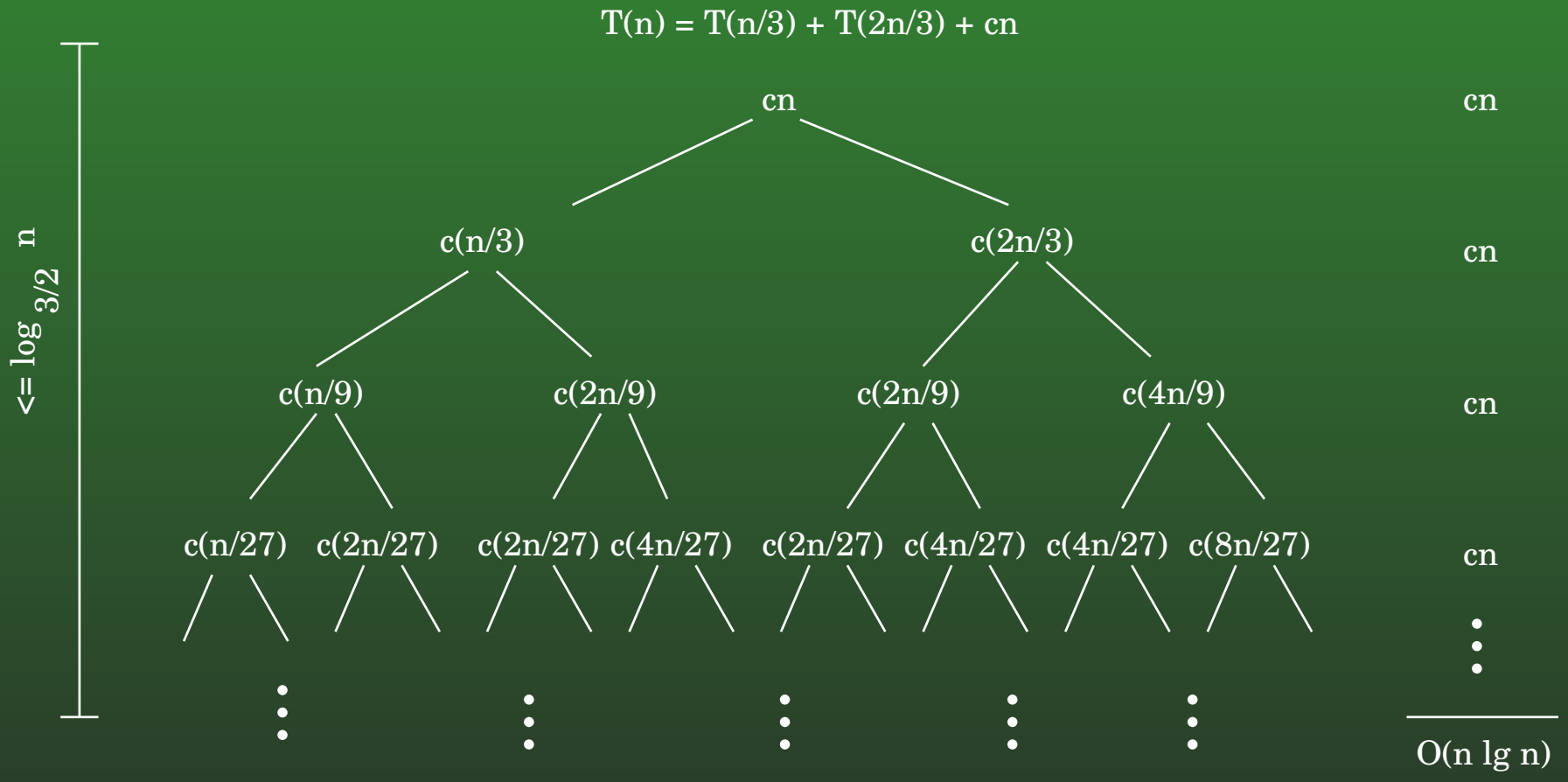
01-67: Recursion Trees

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{\log_4 n} \left(\frac{3^i}{4^{2i}}\right) cn^2 + \sum_{i=0}^{n^{\log_4 3}} 1 \\
&< \sum_{i=0}^{\log_4 n} \left(\frac{3}{4}\right)^i cn^2 + n^{\log_4 3} \\
&< \sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i cn^2 + n^{\log_4 3} \\
&= \frac{1}{1-3/4} cn^2 + n^{\log_4 3} \\
&= 4cn^2 + n^{\log_4 3} \\
&\in O(n^2)
\end{aligned}
$$

01-68: Recursion Trees

$$T(n) = T(n/3) + T(2n/3) + cn$$

01-69: Recursion Trees

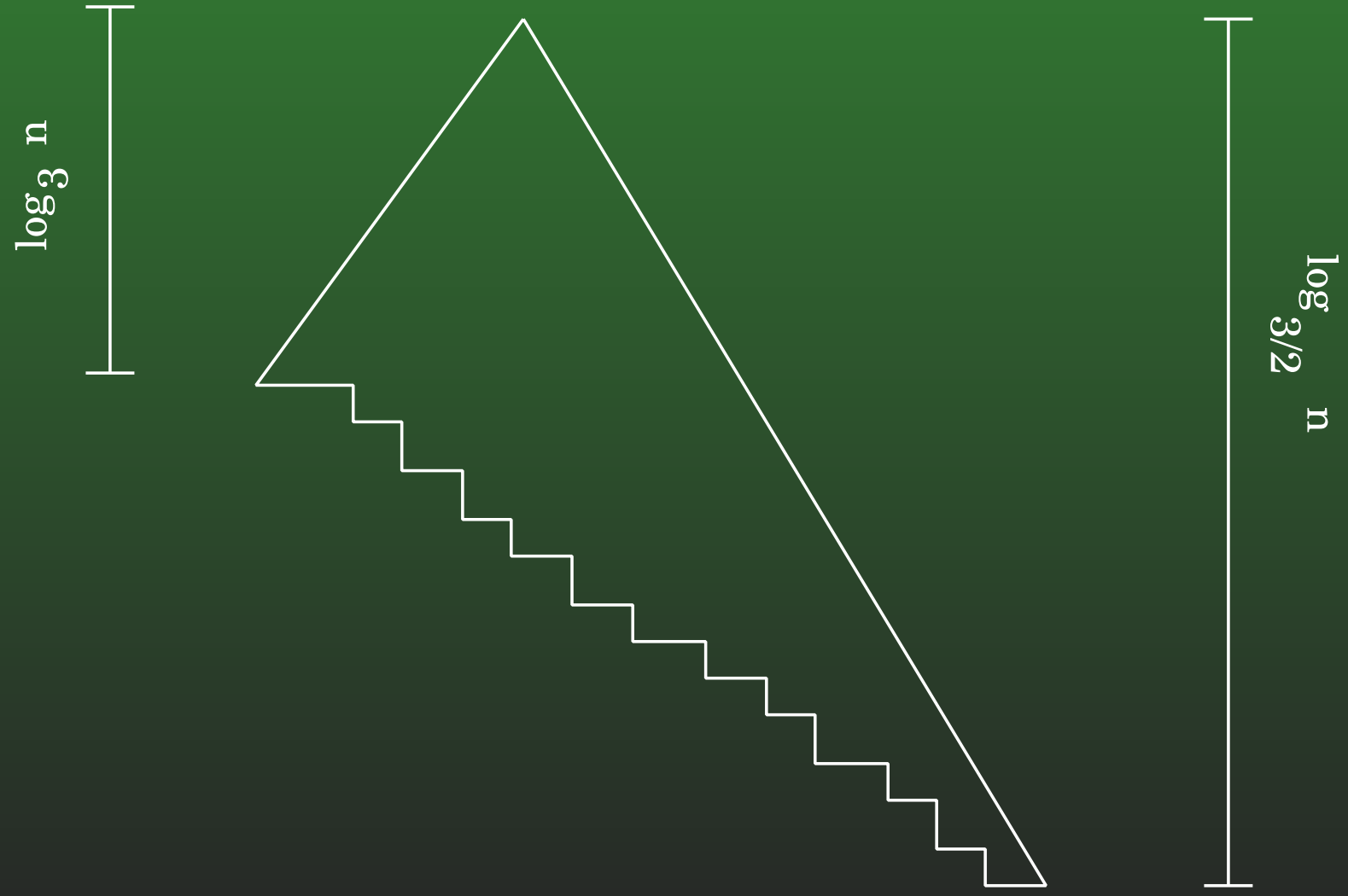


01-70: Recursion Trees

- There is a small problem – this tree is actually irregular in shape!

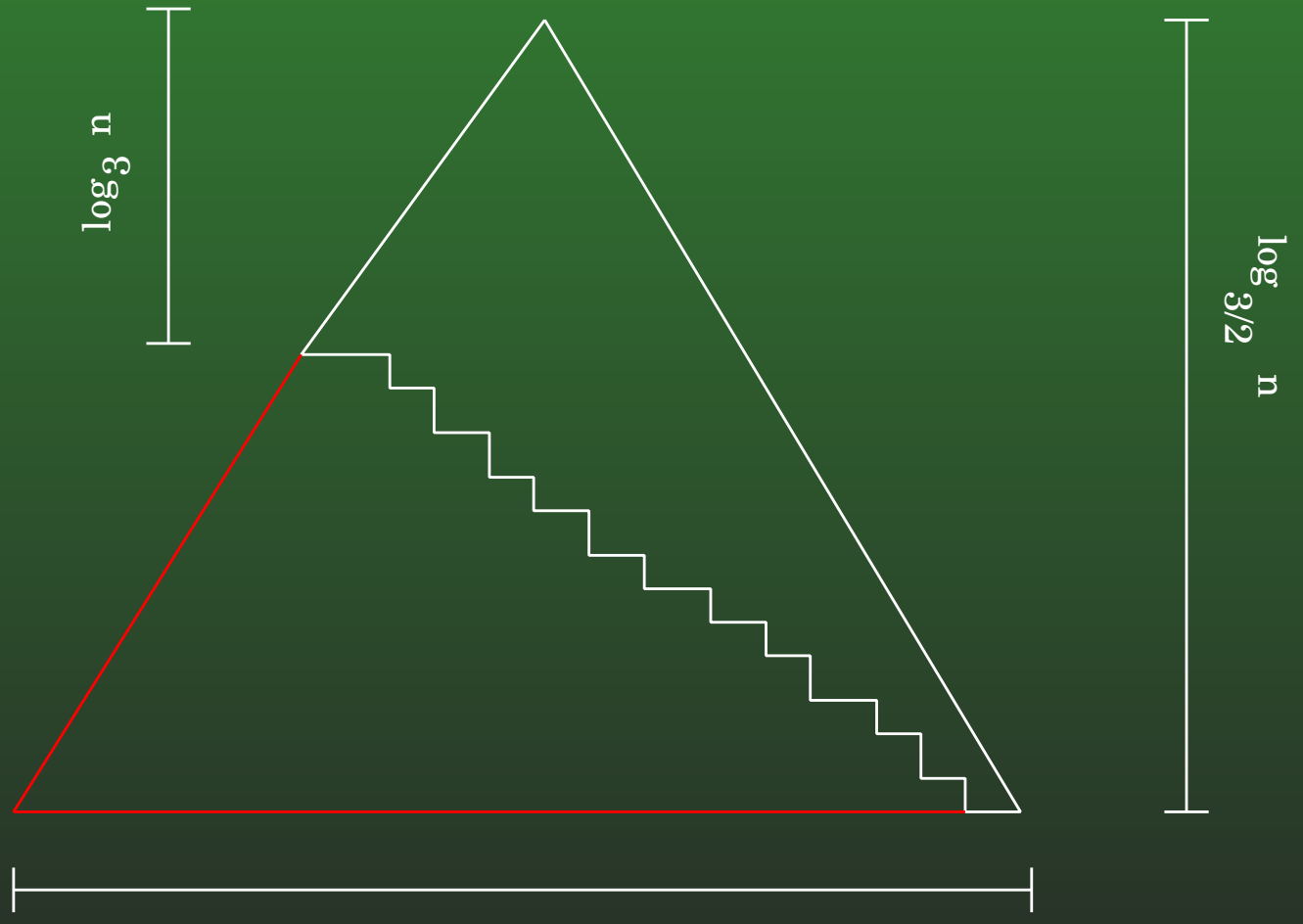
01-71: Recursion Trees

$$T(n) = T(n/3) + T(2n/3) + cn$$



01-72: Recursion Trees

$T(n) = T(n/3) + T(2n/3) + cn$



$$2^{\log_{3/2} n} = n^{\log_{3/2} 2} = \omega(n \lg n)$$

01-73: Recursion Trees

- If we are only using recursion trees to create a guess (that we will later verify using substitution method), then we can be a little sloppy.
- Show $T(n) = T(n/3) + T(2n/3) + cn \in O(n \lg n)$

01-74: Renaming Variables

- Consider:

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 2T\left(\sqrt{n}\right) + \lg n \end{aligned}$$

- The $\sqrt{}$ is pretty ugly – how can we make it go away?
- Rename variables!

01-75: Renaming Variables

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 2T\left(\sqrt{n}\right) + \lg n \end{aligned}$$

Let $m = \lg n$, (and so $n = 2^m$)

$$\begin{aligned} T(2^m) &= 2T\left(\sqrt{2^m}\right) + \lg 2^m \\ &= 2T\left(2^{m/2}\right) + m \end{aligned}$$

01-76: Renaming Variables

$$T(2^m) = 2T\left(2^{m/2}\right) + m$$

Now let $S(m) = T(2^m)$

$$\begin{aligned} S(m) &= T(2^m) \\ &= 2T\left(2^{m/2}\right) + m \\ &= 2S\left(\frac{m}{2}\right) + m \end{aligned}$$

Renaming Variables

$$\begin{aligned} S(m) &= 2S\left(\frac{m}{2}\right) + m \\ &\leq cm \lg m \end{aligned}$$

So:

$$\begin{aligned} T(n) &= T(2^m) \\ &= S(m) \\ &\leq cm \lg m \\ &= c \lg n \lg \lg n \end{aligned}$$

01-78: Master Method

$$T(n) = aT(n/b) + f(n)$$

1. if $f(n) \in O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$, then $T(n) \in \Theta(n^{\log_b a})$
2. if $f(n) \in \Theta(n^{\log_b a})$ then $T(n) \in \Theta(n^{\log_b a} * \lg n)$
3. if $f(n) \in \Omega(n^{\log_b a + \epsilon})$ for some $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some $c < 1$ and large n, then $T(n) \in \Theta(f(n))$

01-79: Master Method

$$T(n) = 9T(n/3) + n$$

01-80: Master Method

$$T(n) = 9T(n/3) + n$$

- $a = 9, b = 3, f(n) = n$
- $n^{\log_b a} = n^{\log_3 9} = n^2$
- $n \in O(n^{2-\epsilon})$

$T(n) = \Theta(n^2)$

01-81: Master Method

$$T(n) = T(2n/3) + 1$$

01-82: Master Method

$$T(n) = T(2n/3) + 1$$

- $a = 1, b = 3/2, f(n) = 1$
- $n^{\log_b a} = n^{\log_{3/2} 1} = n^0 = 1$
- $1 \in O(1)$

$$T(n) = \Theta(1 * \lg n) = \Theta(\lg n)$$

Master Method

$$T(n) = 3T(n/4) + n \lg n$$

01-84: Master Method

$$T(n) = 3T(n/4) + n \lg n$$

- $a = 3, b = 4, f(n) = n \lg n$
- $n^{\log_b a} = n^{\log_4 3} = n^{0.792}$
- $n \lg n \in \Omega(n^{0.792+\epsilon})$
- $3(n/4) \lg(n/4) \leq c * n \lg n$

$T(n) \in \Theta(n \lg n)$

Master Method

$$T(n) = 2T(n/2) + n \lg n$$

01-86: Master Method

$$T(n) = 2T(n/2) + n \lg n$$

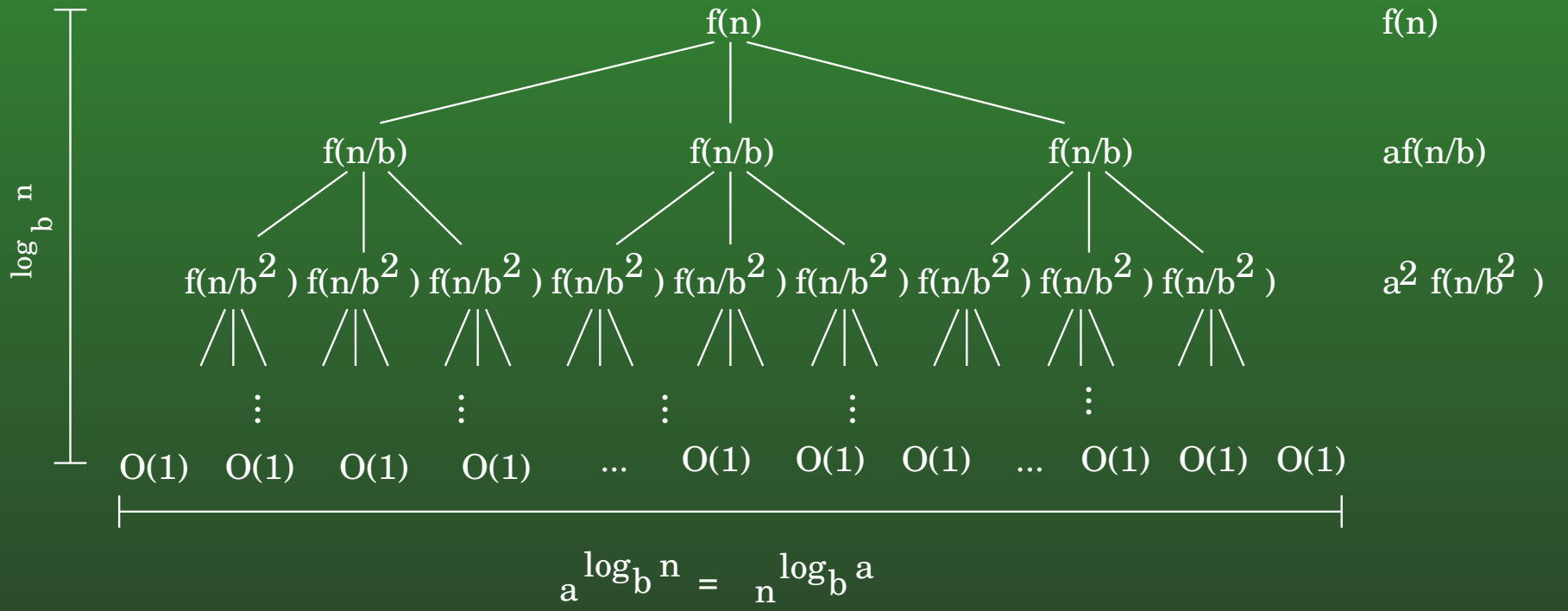
- $a = 2, b = 2, f(n) = n \lg n$
- $n^{\log_b a} = n^{\log_2 2} = n^1$

Master method does not apply!
$n^{1+\epsilon}$ grows faster than $n \lg n$ for *any* $\epsilon > 0$
Logs grow *incredibly* slowly! $\lg n \in o(n^\epsilon)$ for any $\epsilon > 0$

Master Method

- Proof Sketch (not all formal, see textbook for details)
- We will consider the recursion tree for $T(n) = aT(n/b) + f(n)$
  (We'll assume that n is an exact power of b, to simplify the math. See the textbook for a complete proof)

01-88: Master Method



$$a^{\log_b n} = n^{\log_b a}$$

01-89: Master Method

$$T(n) = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) + cn^{\log_b a}$$

- Case 1: Leaves of recursion tree dominate cost
- Case 2: Cost is evenly divided amoung all levels in the tree
- Case 3: Root dominates the cost

(see the textbook for the algebra)