Graduate Algorithms

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Amortized Analysis

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12-0: Amortized Analysis

- Standard Stack
  - Push(S,elem)
  - Pop(S)
- How much time for each operation?

12-1: Amortized Analysis

- Standard Stack
  - Push(S,elem) $O(1)$
  - Pop(S) $O(1)$
  - Multipop(S,k)

    for $i \leftarrow$ 1 to k do

        Pop(S)
- How much time for multipop?

12-2: Amortized Analysis

- Standard Stack
  - Push(S,elem) $O(1)$
  - Pop(S) $O(1)$
  - Multipop(S,k) $O(k)$

12-3: Amortized Analysis

- Do n operations, each of which could be either a Push, Pop, or Multipop
- How long will each operation take?
  - Push
  - Pop
  - Multipop

12-4: Amortized Analysis

- Do n operations, each of which could be either a Push, Pop, or Multipop
- How long will each operation take?
  - Push $O(1)$
  - Pop $O(1)$
  - Multipop $O(n)$
- What if we were to do n operations in a row, each of which is either a push/pop/multipop – how long would those n operations take?

12-5: **Amortized Analysis**

- n operations in a row, each is either a push/pop/multipop. How long will it take?
  - Naive Method: n operations, each takes time $O(n)$ – total time: $O(n^2)$
- Looking closer:
  - How many times can Pop be called (even Pop in Multipop)?
    - Once for each push!
    - Total number of Pushes $\in O(n)$
    - Total number of Pops (including pops in multipop) $\in O(n)$
    - Total time for n operations: $O(n)$

12-6: Amortized Analysis

- n operations in a row, each is either a push/pop/multipop.
- Total time for n operations is $O(n)$
- Amortized cost for a Push, Pop, Multipop is $O(1)$

12-7: Aggregate Method

- Aggregate method
  - Total cost for n operations is $g(n)$
  - Amortized cost for 1 operation is $\frac{g(n)}{n}$
- Previous analysis of push/pop/Multipop used aggregate method

12-8: Aggregate Method

- Ripple counter, width k
  - Examples on board
- How long does an increment take?

Aggregate Method

- Ripple counter, width k
- How long does an increment take?
  - $O(k)$
  - But ...
    - Least sig. bit flips every time
    - 2nd least sig. bit flips every other time
    - 3rd least sig. bit flips every 4th time
    - kth least sig. bit flips every 2^kth time
- For n increments (if no overflow):

$$\sum_{i=1}^{\lg n} \left\lfloor \frac{n}{2^i} \right\rfloor < n * \sum_{i=0}^{\infty} \frac{1}{2^i} = 2n$$

12-10: Aggregate Method

- Ripple counter, width k
- Worst case time for a sequence of n increment operations, if counter starts at 0:
  - O(n)
- Amortized cost for a single increment
  - $O(1)$

12-11: Accounting Method

- Accounting Method
  - Assign a cost for each operation
    - Called “amortized cost”
  - When amortized cost $>$ actual cost, create a “credit” which can be used when actual cost $>$ amortized cost
  - Must design costs so that all sequences of operations always leave a “positive account”

12-12: Accounting Method

	actual cost	amortized cost
Push	1	
Pop	1	
Multipop	min(k,s)	

- What amortized costs should I give, so that any valid sequence of push/pop/multipop will never have a debt?

Accounting Method

	actual cost	amortized cost
Push	1	2
Pop	1	0
Multipop	min(k,s)	0

Accounting Method

- Binary Counter
  - Actual Cost of setting a bit to 1 is 1
  - Actual Cost of setting a bit to 0 is 1
  - Actual Cost of an increment = # of bits flipped from 1 to 0 + 1
- What should our amortized costs be, and why?

12-15: Accounting Method

- Binary Counter
  - Amortized cost of setting a bit to 1 is: 2
  - Amortized cost of setting a bit to 0 is: ?
  - Amortized cost of an increment is: ?

12-16: Accounting Method

- Binary Counter
  - Amortized cost of setting a bit to 1 is: 2
  - Amortized cost of setting a bit to 0 is: 0 (!)
  - Amortized cost of an increment is: 2

For n increments, the total amortized cost is $O(n)$, which is also a bound on the actual cost

12-17: Potential Method

- Definte a “potential” for data structures that your algorithm uses
  - Kind of like potential energy
- When the amortized cost is greater than the actual cost, increase the potential of the data structure
- When the amortize cost is less than the actual cost, decrease the potential of the data structure
  - Potential can never be negative

Potential Method

- $\Phi(D)$ = potential of the data structure
- Amortized cost of operation c_i is am(c_i)
  - am(c_i) = $c_i + (\Phi(D_i) - \Phi(D_{i-1}))$
- Total amortized cost for a sequence of n operations:

$$\begin{aligned}\sum_{i=1}^{n} am(c_i) &= \sum_{i=1}^{n} (c_i\Phi(D_i) - \Phi(D_{i-1})) \\ &= (\sum_{i=1}^{n} c_i) + \Phi(D_n) - \Phi(D_0)\end{aligned}$$

- As long as the potential starts at 0, and never goes

12-19: Potential Method

- The potential function is on the *Data Structure*, not the operations
- Don't talk about the potential of a push or a pop
- Instead, talk about the potential of the stack
  - Define a potential function on the data structure
  - Use the potential function and actual cost to determine amortized cost

12-20: Potential Method

- Potential Method Examples:
  - Stack, with push/pop/multipop
    - What should the potential be?
    - What are the resulting amortized costs?

12-21: Potential Method

- Potential Method Examples:
  - Stack, with push/pop/multipop
    - Potential = # of elements in the stack
    - amortized cost = actual cost + change in potential
    - amortized cost of push = 1 + 1 = 2
    - amortized cost for pop = 1 + (-1) = 0
    - amortzied cost for multipop = k + (-k) = 0

12-22: Potential Method

- Potential Method Examples:
  - Binary Ripple-Carry Counter
    - What is the potential?
    - What are the resulting amortized costs

12-23: Potential Method

- Potential Method Examples:
  - Binary Ripple-Carry Counter
    - Potential = # of 1's in the counter
    - amortized cost = actual cost + change in potential
    - actual cost = # of bits flipped
    - Change in potential = # of bits fliped from 1 to 0 - # of bits flipped from 1 to 0

      = - # of bits flipped, if counter reset to 0

      = 2 - # of bits flipped, otherwise
    - Amortized cost ≤ 2

12-24: Dynamic Hash Tables

- Standard Hash Table
  - Insert/find in time $O(1)$ (no delete for now)
  - Need to know an upper bound on the table size beforehand
- If we don't know the table size beforehand?
  - Pick a size to start with
  - If table fills, double the table size, and add everything from old table to new table
- What is the time for an insert if the table can grow?

Dynamic Hash Tables

- Standard Hash Table
  - Insert/find in time $O(1)$ (no delete for now)
  - Need to know an upper bound on the table size beforehand
- If we don't know the table size beforehand?
  - Pick a size to start with
  - If table fills, double the table size, and add everything from old table to new table
- What is the time for an insert if the table can grow?
  - $O(n)$

12-26: Dynamic Hash Tables

- Any single insert into a Dynamic Hash Table can take time $O(n)$
- What is the amortized cost for an insert?
  - Aggregate method

$$c_i = \begin{cases} i & \text{if } i-1 \text{ is a power of 2} \\ 1 & \text{otherwise} \end{cases}$$

12-27: Dynamic Hash Tables

- Aggregate Method
  - Total cost for n inserts:

$$\begin{aligned}\sum_{i=1}^{n} c_i &\leq n + \sum_{i=1}^{\lg n} 2^i \\ &\leq n + 2n \\ &\leq 3n\end{aligned}$$

  - Amortized cost per insert is thus $O(1)$

12-28: Dynamic Hash Tables

- Accounting Method
  - Amortized cost for insert = 3
    - Cost to insert the element
    - Cost to move element when the table is expanded next time
    - Cost to move one other element when the table is expanded next time
      (Examples)

Dynamic Hash Tables

- Potential Method
  - Potential starts at 0, grows as we insert elements
  - When the table size increases, potential drops back to 0
    - Extra potential is used to grow the table

12-30: Dynamic Hash Tables

- Potential Method
  - Potential starts at 0, grows as we insert elements
  - When the table size increases, potential drops back to 0
  - $\Phi(T) = 2 * num[T] - size[T]$
    - $num[T]$ = number of elements in the table
    - $size[T]$ = size of table
  - Always positive (assuming we start with a table size of 0, when first element is added we go to a table size of 2 containing 1 element)

Dynamic Hash Tables

- Potential Method
  - Amortized cost for an insert = actual cost + change in potential
  - If ith insert did $\mathbf{not}$ cause the table to grow:

$$\begin{aligned} am(c_i) &= 1 + (2 * num_i - size_i) - (2 * num_{i-1} - size_{i-1}) \\ &= 1 + 2 * i - size_i - 2 * (i-1) + size_i \\ &= 3 \end{aligned}$$

  - If ith insert $\mathbf{did}$ cause the table to grow:

$$\begin{aligned} am(c_i) &= 1 + num_{i-1} + (2 * num_i - size_i) - (2 * num_{i-1} - size_{i-1}) \\ &= 1 + (i-1) + (2 * i - 2 * (i-1)) - (2 * (i-1) - (i-1)) \\ &= 3 \end{aligned}$$

12-32: Dynamic Hash Tables

- Add in deletes
- Want to keep the table from being too big
- Shrink the table when it gets too large (freeing space)
- First try:
  - When table gets full, double the size of the table, copying elements
  - When table gets less than half full, cut the size of the table in half, copying elements
- Will this still give us $O(1)$ amortized cost for an insert/delete?

Dynamic Hash Tables

- Consider a table that is full
- What happens when we do the following operations:
  - Insert, Delete, Delete, Insert, Insert Delete, Delete, . . .

12-34: Dynamic Hash Tables

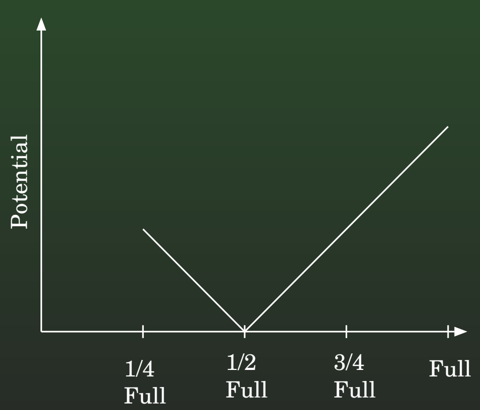
- Consider a table that is full
- What happens when we do the following operations:
  - Insert, Delete, Delete, Insert, Insert Delete, Delete, . . .
  - Every other operation takes time $O(n)$!
  - Amortized cost per operation is $O(n)$, $\mathbf{not}$ $O(1)$!
- What can we do?

12-35: Dynamic Hash Tables

- When table gets full, double the size of the table, copying elements
- When table gets less than 1/4 full, halve the size of the table, copying elements

12-36: Dynamic Hash Tables

- Potential Function Φ:
  - 0 when list is exactly half full
  - Increase as # of elements in the list increases, so that the potential = # of elements in the list when the list is full
  - Increase as # of elements decreases (below 1/2 full) so that the potential = # of elements in the list when the list is 1/4 full



12-37: Dynamic Hash Tables

- Potential Function Φ:
  - α = load of the table: Size of table / # of elements

$$\Phi(T) = \begin{cases} 2 * num[T] - size[T] & \text{if } \alpha(T) \geq 1/2 \\ size[T]/2 - num[T] & \text{if} \alpha(T) < 1/2 \end{cases}$$

Dynamic Hash Tables

- Amortized cost for insert:
  - Amortized cost = actual cost + growth in potential
  - $am(c_i) = c_i + \Phi(T_i) - \Phi(T_{i-1})$
  - Several cases:
    - $1/4 < \alpha < 1/2$
    - $1/2 \leq \alpha < 1$
    - $\alpha = 1$

Dynamic Hash Tables

- Amortized cost for insert, $1/2 \leq \alpha < 1$
  - $am(c_i) = c_i + \Phi(T_i) - \Phi(T_{i-1})$

$$\begin{aligned} am(c_i) &= 1 + (2 * num_i - size_i) - (2 * num_{i-1} - size_{i-1}) \\ &= 1 + 2 * i - size_i - 2 * (i - 1) + size_i \\ &= 3 \end{aligned}$$

12-40: Dynamic Hash Tables

- Amortized cost for insert, $\alpha = 1$
  - $am(c_i) = c_i + \Phi(T_i) - \Phi(T_{i-1})$

$$
\begin{aligned}
am(c_i) &= 1 + num_{i-1} + (2 * num_i - size_i) - (2 * num_{i-1} - size_{i-1}) \\
&= 1 + (i-1) + (2 * i - 2 * (i-1)) - (2 * (i-1) - (i-1)) \\
&= 3
\end{aligned}
$$

Dynamic Hash Tables

- Amortized cost for insert, $1/4 < \alpha < 1/2$
  - $am(c_i) = c_i + \Phi(T_i) - \Phi(T_{i-1})$

$$
\begin{aligned}
am(c_i) &= 1 + (size_i/2 - num_i) - (size_{i-1}/2 - num_{i-1}) \\
&= 1 + (size_i/2 - size_i/2) - num_i + (num_i) - 1 \\
&= 0
\end{aligned}
$$

Dynamic Hash Tables

- Amortized cost for delete:
  - Amortized cost = actual cost + growth in potential
  - $am(c_i) = c_i + \Phi(T_i) - \Phi(T_{i-1})$
  - Several cases:
    - $1/4 < \alpha < 1/2$
    - $1/2 \leq \alpha \leq 1$
    - $\alpha = 1/4$

12-43: Dynamic Hash Tables

- Amortized cost for delete, $1/2 \leq \alpha \leq 1$
  - $am(c_i) = c_i + \Phi(T_i) - \Phi(T_{i-1})$

$$
\begin{aligned}
am(c_i) &= 1 + (2 * num_i - size_i) - (2 * num_{i-1} - size_{i-1}) \\
&= 1 + 2 * num_i - 2 * (num_i + 1) + size_i - size_i \\
&= -1
\end{aligned}
$$

12-44: Dynamic Hash Tables

- Amortized cost for delete, $1/4 < \alpha < 1/2$
  - $am(c_i) = c_i + \Phi(T_i) - \Phi(T_{i-1})$

$$\begin{aligned} am(c_i) &= 1 + (size_i/2 - num_i) - (size_{i-1}/2 - num_{i-1}) \\ &= 1 + (size_i/2 - size_i/2) + num_{i-1} - numi \\ &= 2 \end{aligned}$$

12-45: Dynamic Hash Tables

- Amortized cost for delete, $\alpha = 1/4$
  - $am(c_i) = c_i + \Phi(T_i) - \Phi(T_{i-1})$

$$size_i/2 = size_{i-1}/4 = num_{i-1} = num_i + 1$$

$$\begin{aligned}
am(c_i) &= 1 + num_i + (size_i/2 - num_i) - (size_{i-1}/2 - num_{i-1}) \\
&= 1 + num_i + (num_i + 1 - num_i) - ((2 * num_i + 2) \\
&\quad -(num_i + 1) \\
&= 1
\end{aligned}$$