Graduate Algorithms

CS673-2016F-16

Spanning Trees

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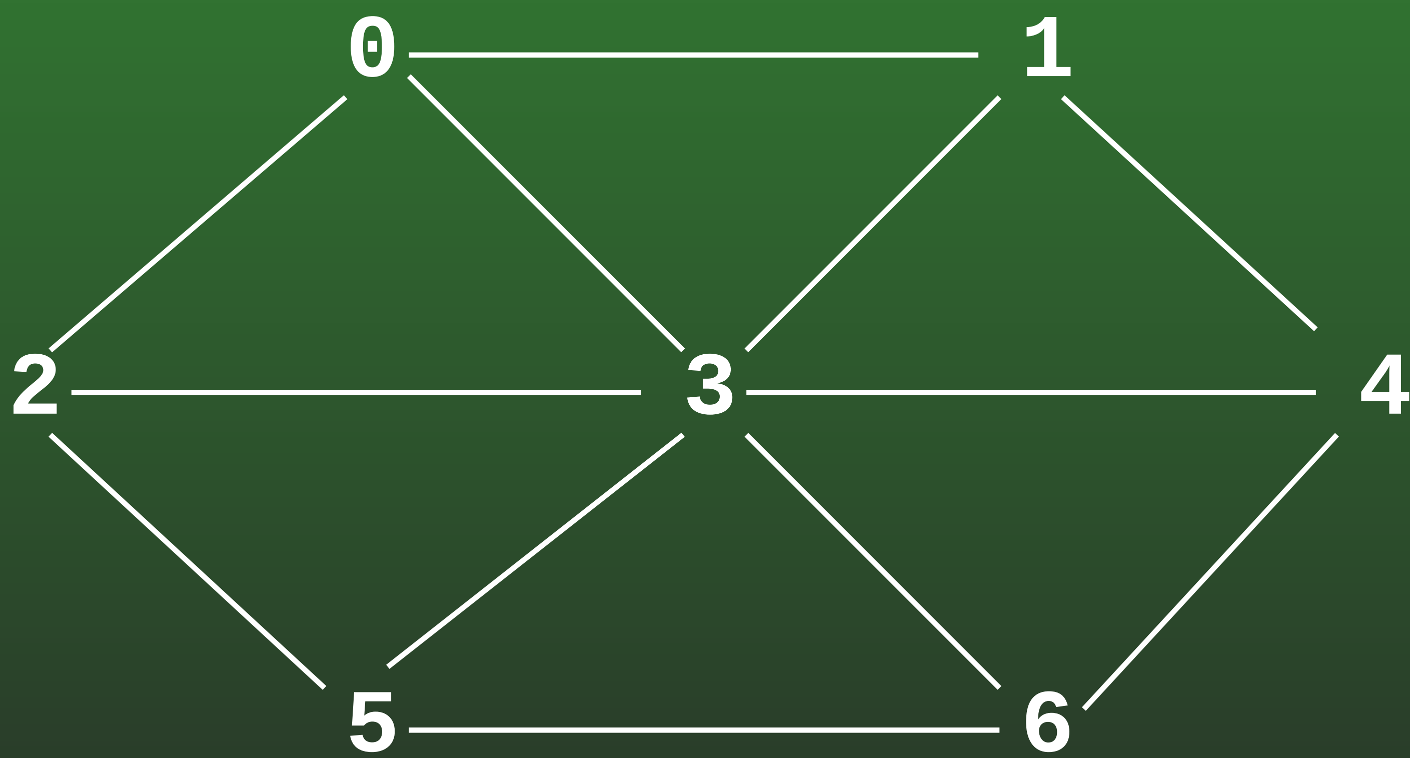
16-0: Spanning Trees

- Given a connected, undirected graph G
  - A *subgraph* of G contains a subset of the vertices and edges in G
  - A *Spanning Tree* T of G is:
    - subgraph of G
    - contains all vertices in G
    - connected
    - acyclic

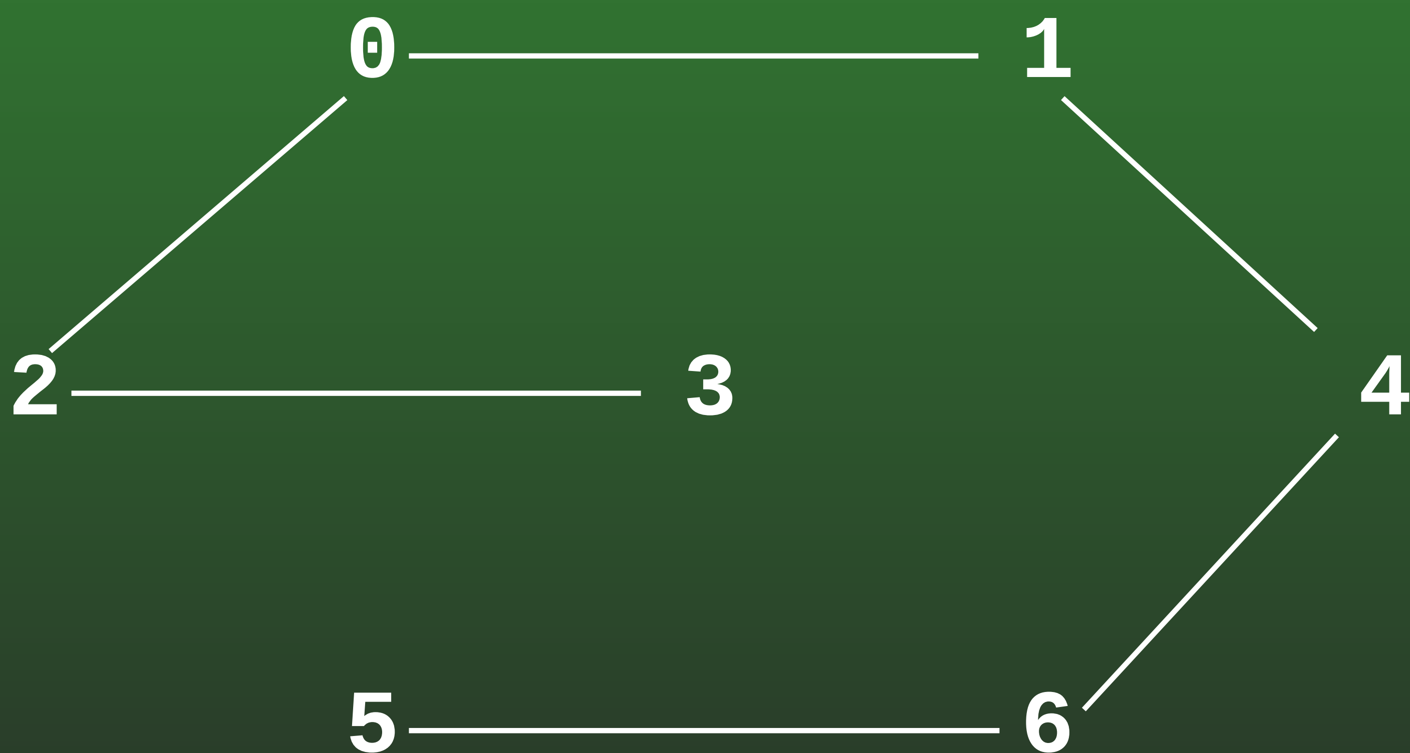
16-1: Spanning Tree Examples

- Graph



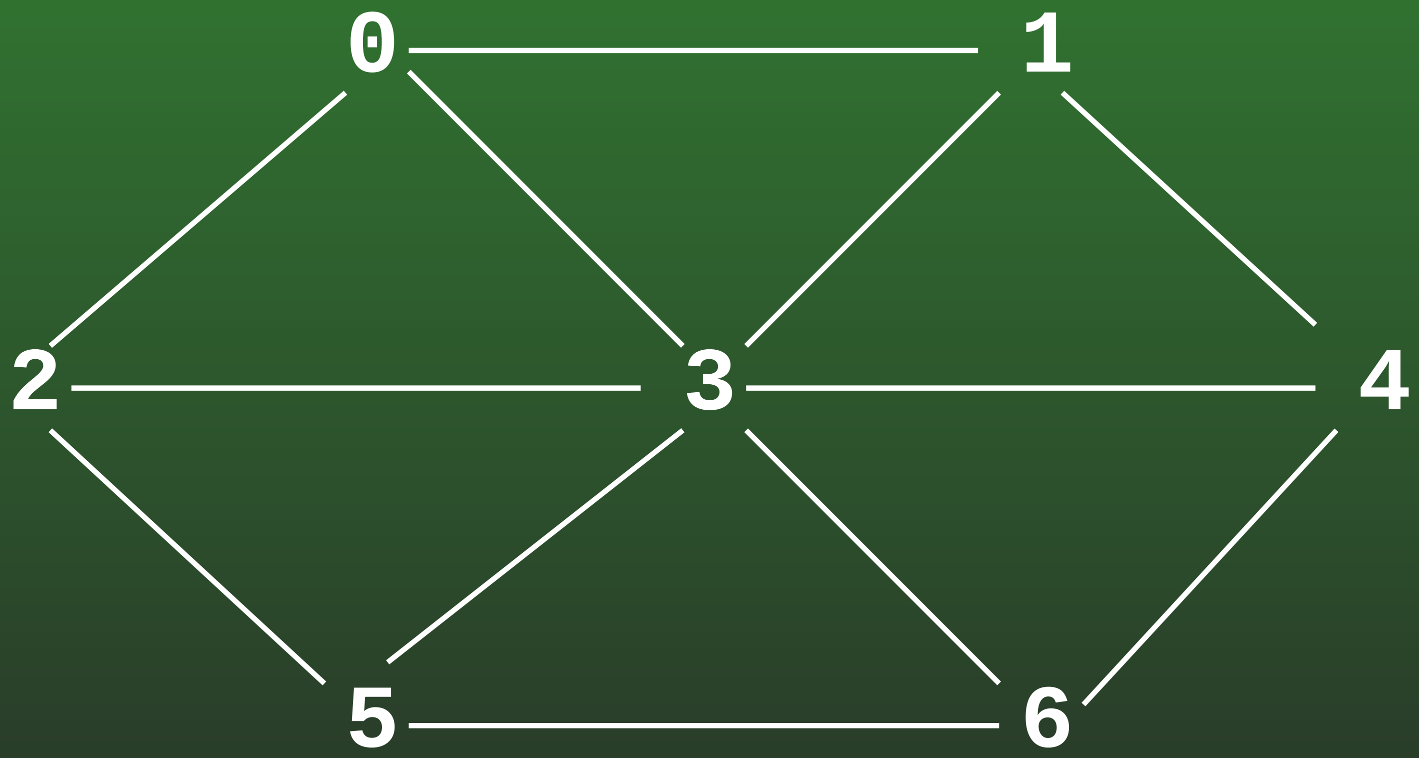
16-2: Spanning Tree Examples

- Spanning Tree



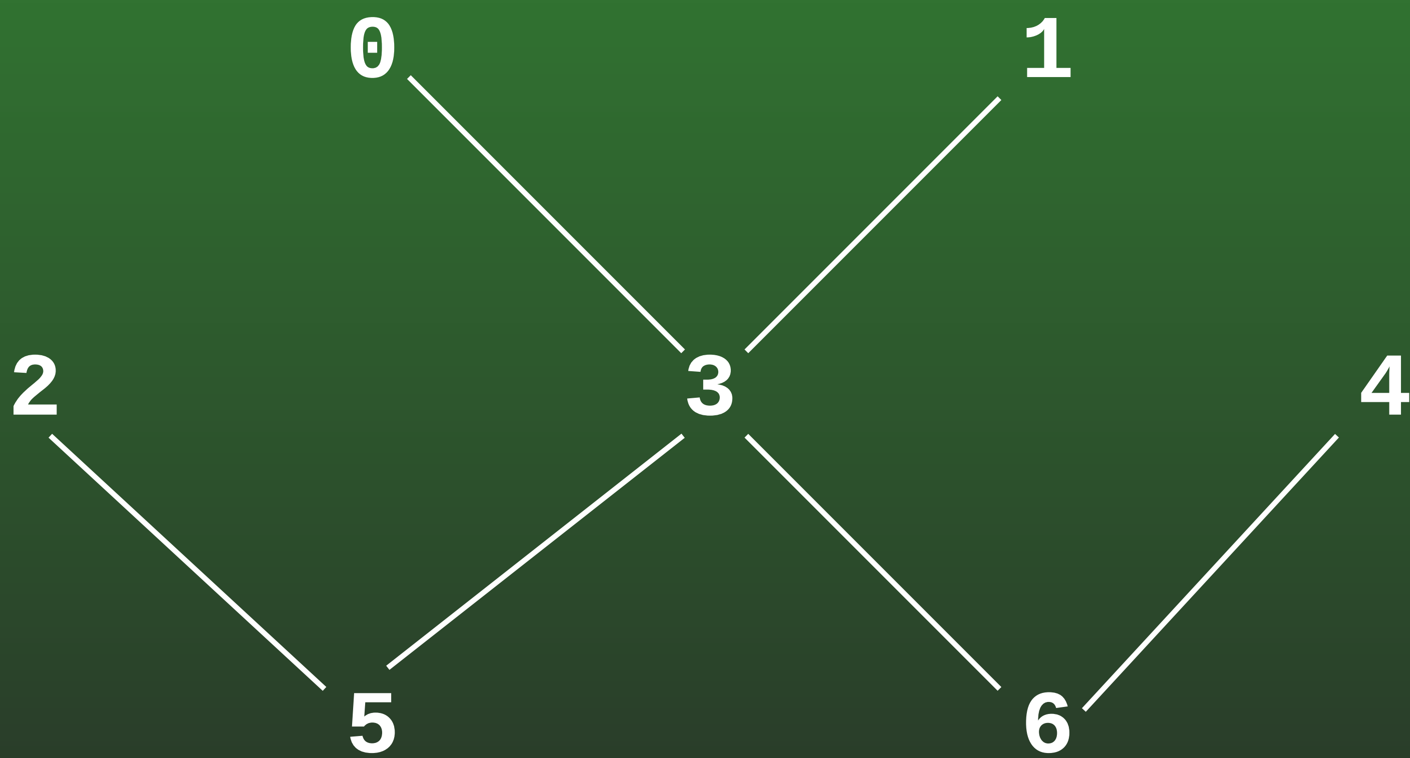
16-3: Spanning Tree Examples

- Graph



16-4: Spanning Tree Examples

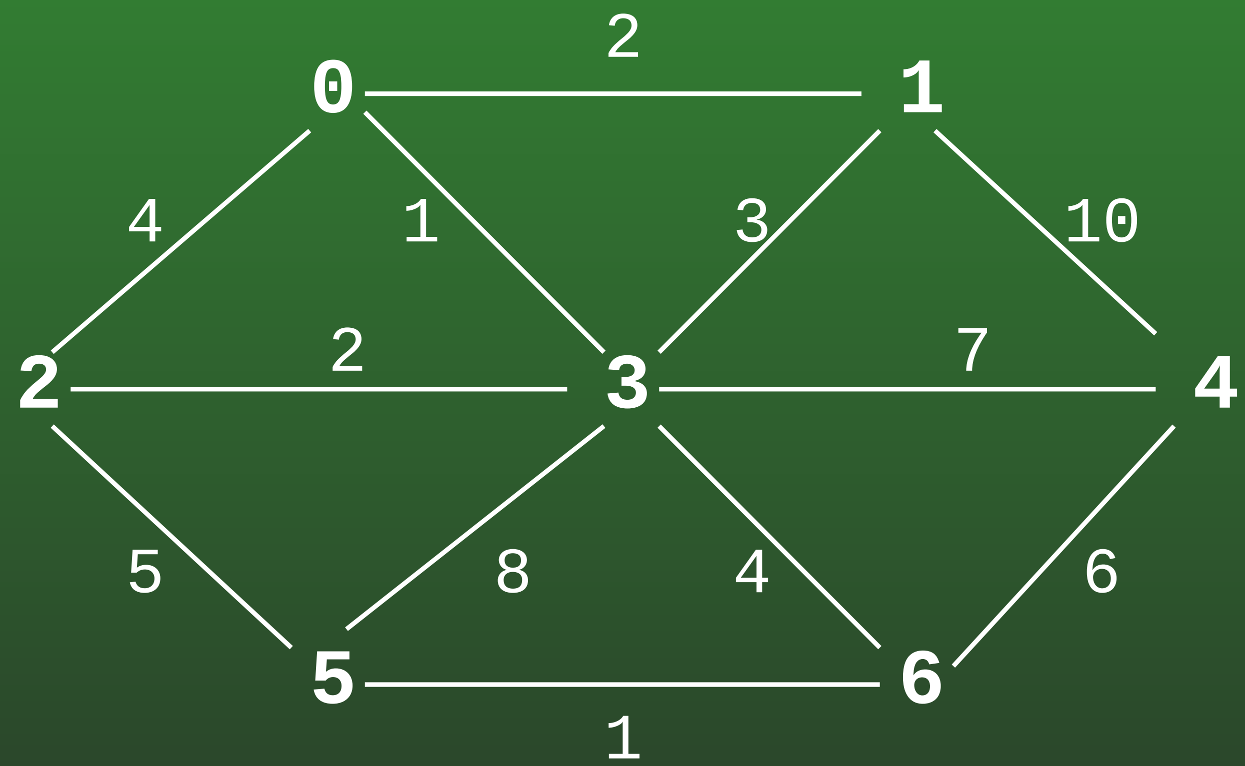
- Spanning Tree



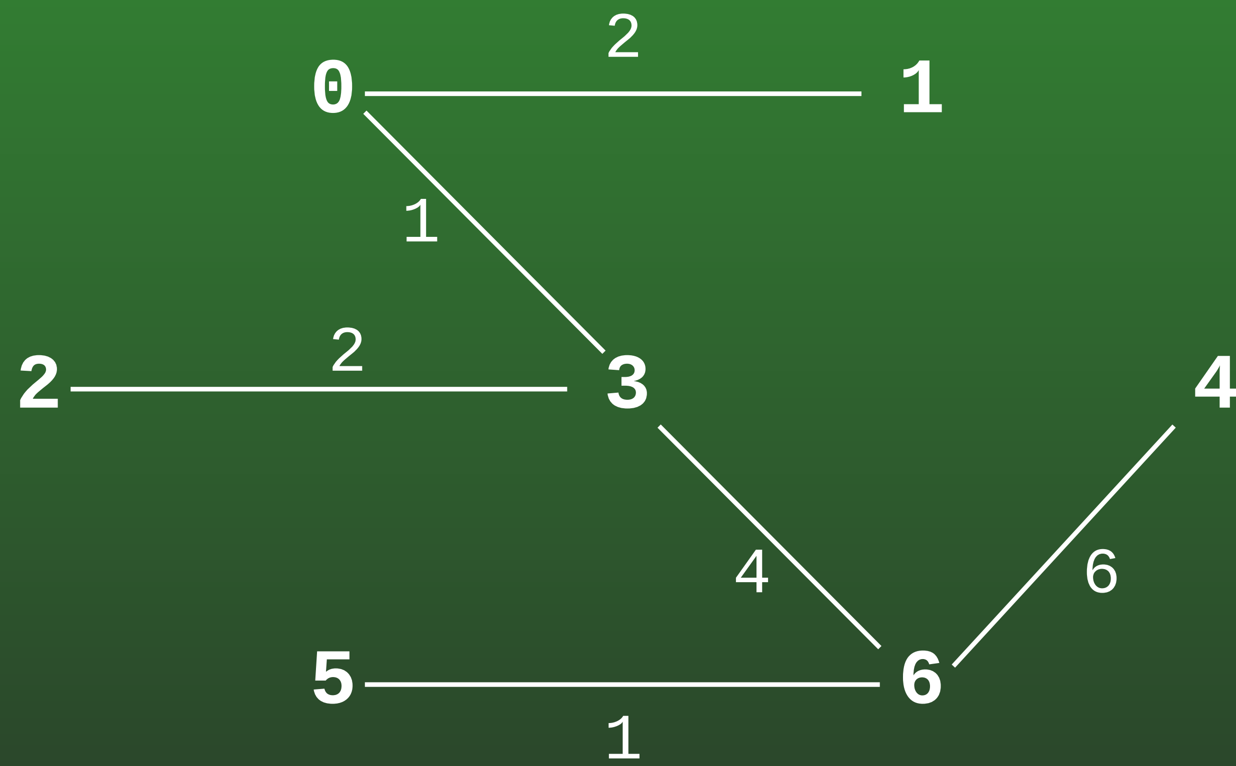
16-5: Minimal Cost Spanning Tree

- Minimal Cost Spanning Tree
  - Given a weighted, undirected graph G
  - Spanning tree of G which minimizes the sum of all weights on edges of spanning tree

16-6: MST Example



MST Example



16-8: Minimal Cost Spanning Trees

- Can there be more than one minimal cost spanning tree for a particular graph?

16-9: Minimal Cost Spanning Trees

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- YES!
  - What happens when all edges have unit cost?

16-10: Minimal Cost Spanning Trees

- Can there be more than one minimal cost spanning tree for a particular graph?
- YES!
  - What happens when all edges have unit cost?
  - All spanning trees are MSTs

16-11: Calculating MST

- Generic MST algorithm:

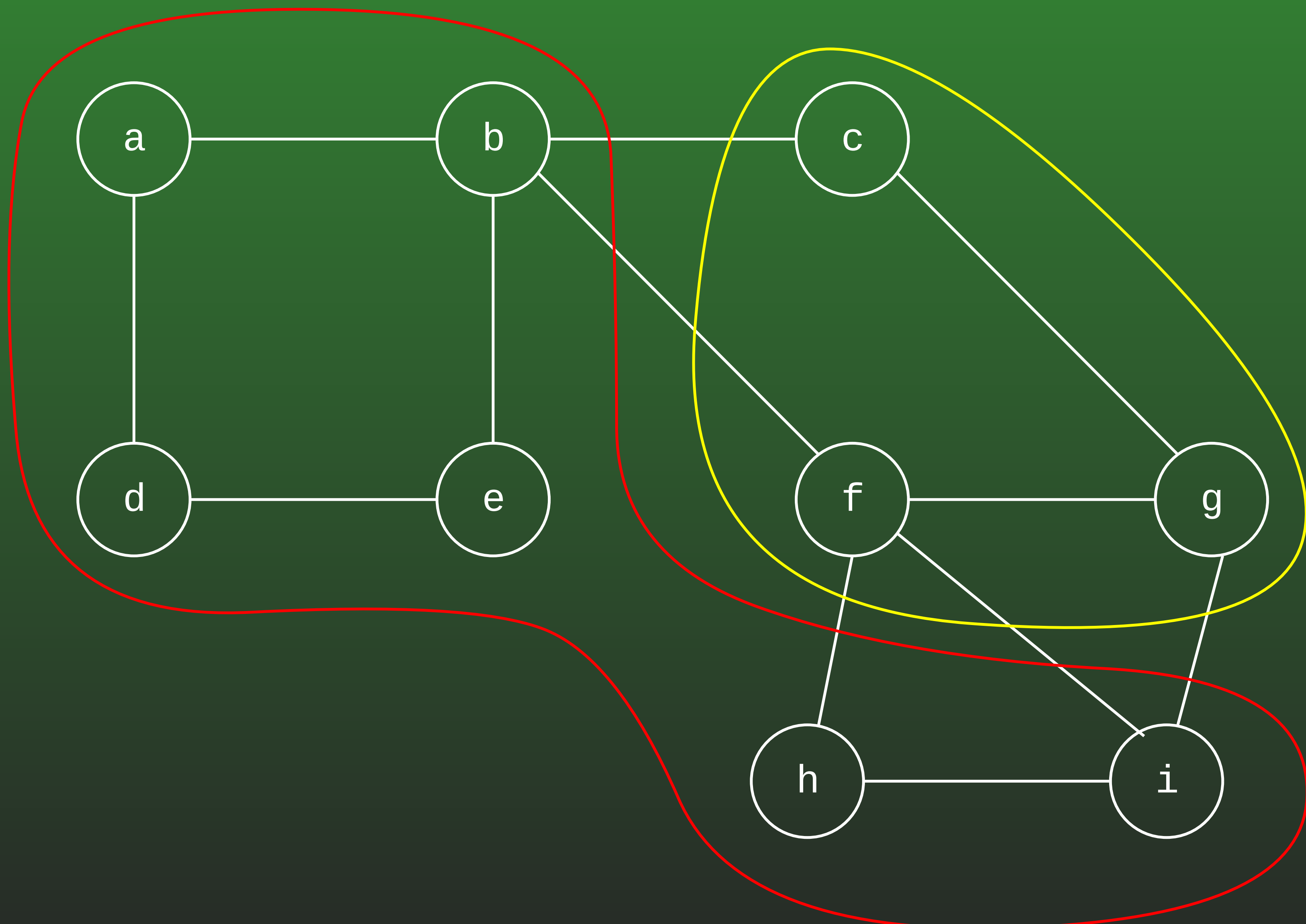
$A \leftarrow \{\}$
while A does not form a spanning tree
  find an edge (u, v) that is safe for A
  $A \leftarrow A \cup \{(u, v)\}$

- (u, v) is safe to for A when $A \cup \{(u, v)\}$ is a subset of some MST

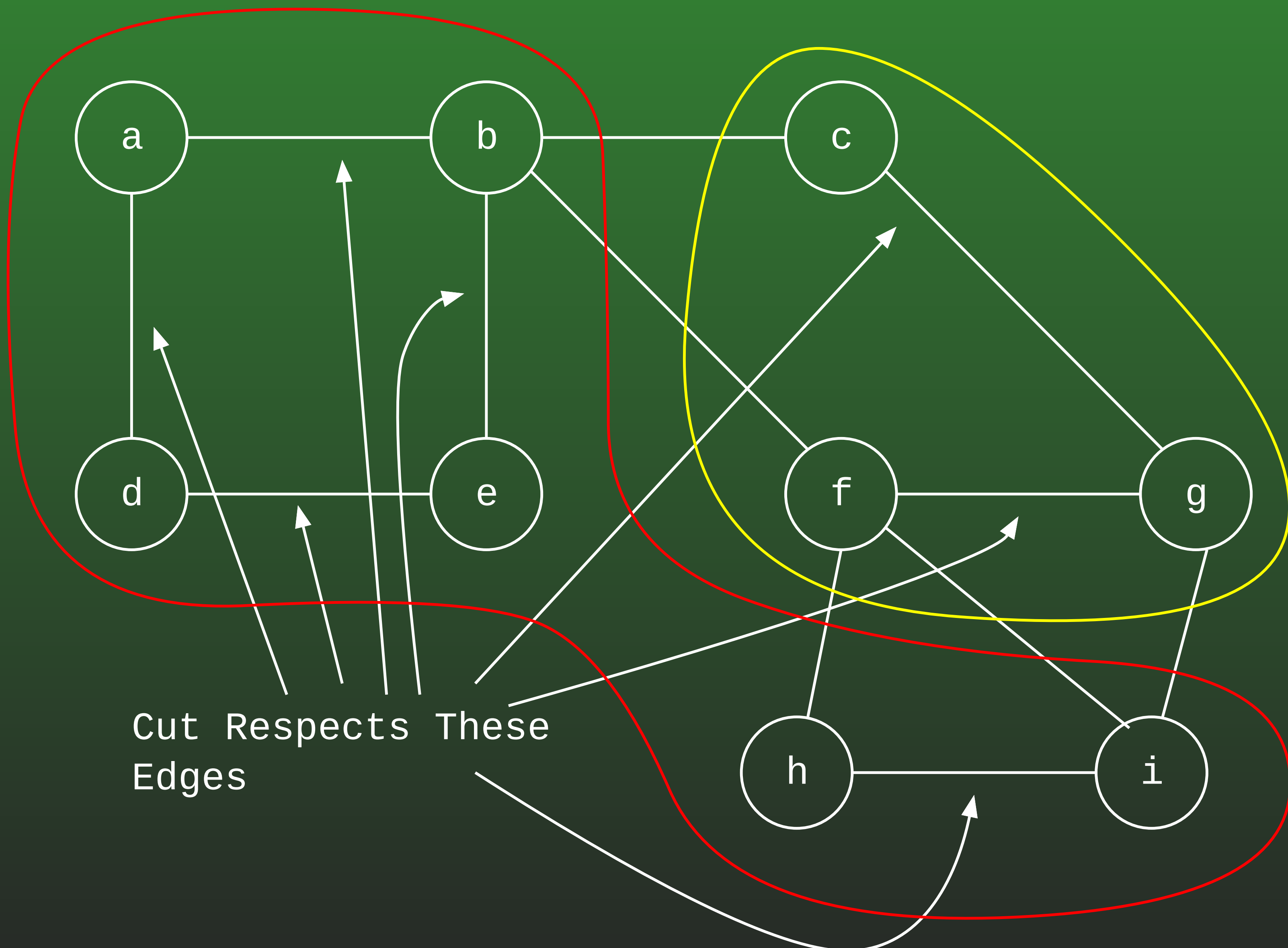
16-12: Graph Cut

- “Cut” of a undirected graph is a partition of the vertices in the graph
  - An edge crosses a cut if the vertices are in different sets of the partition
  - A cut respects a series of edges of no edge crosses the cut
  - light edge is an edge that crosses the cut that has minimum cost

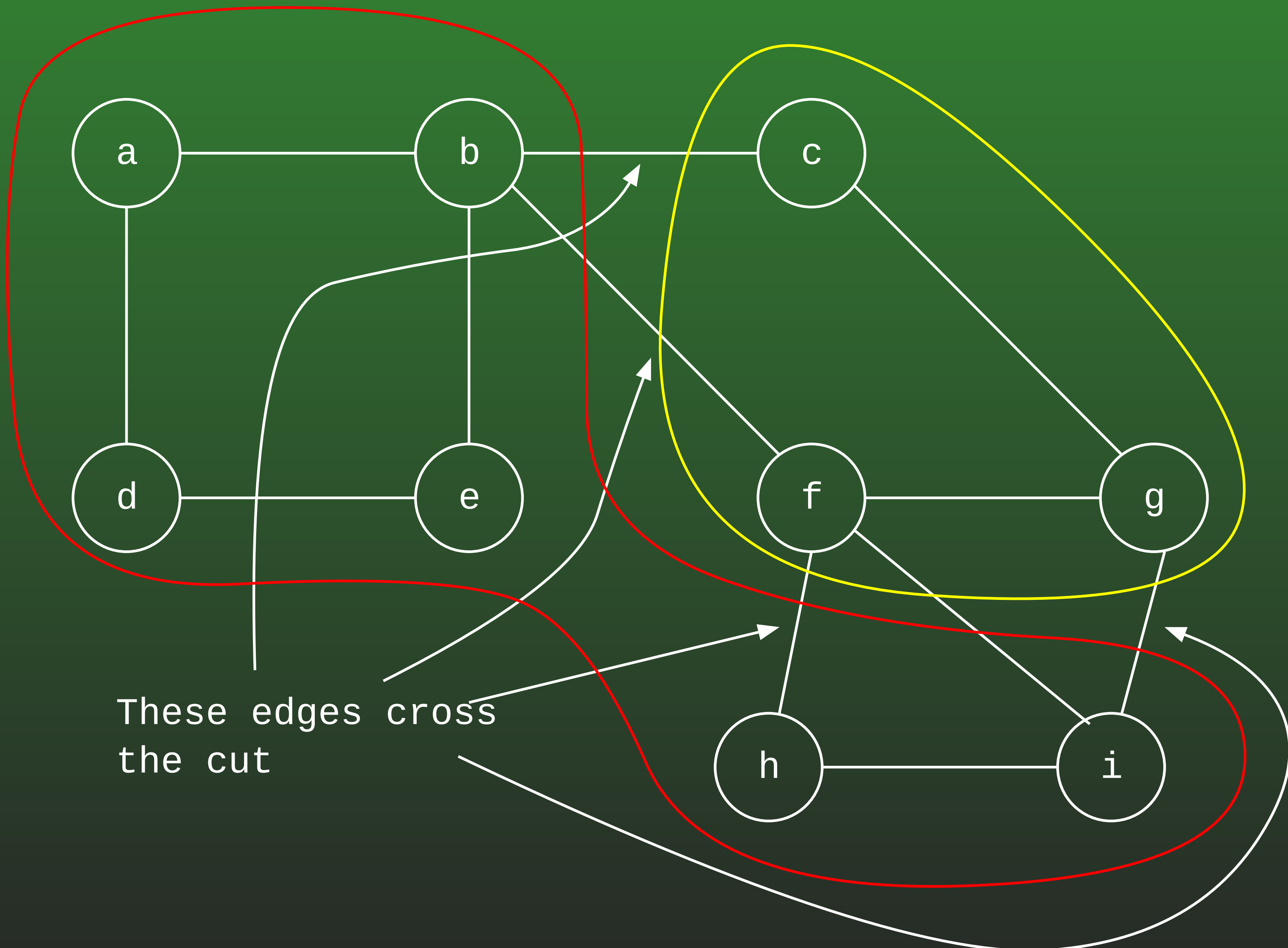
16-13: Graph Cut



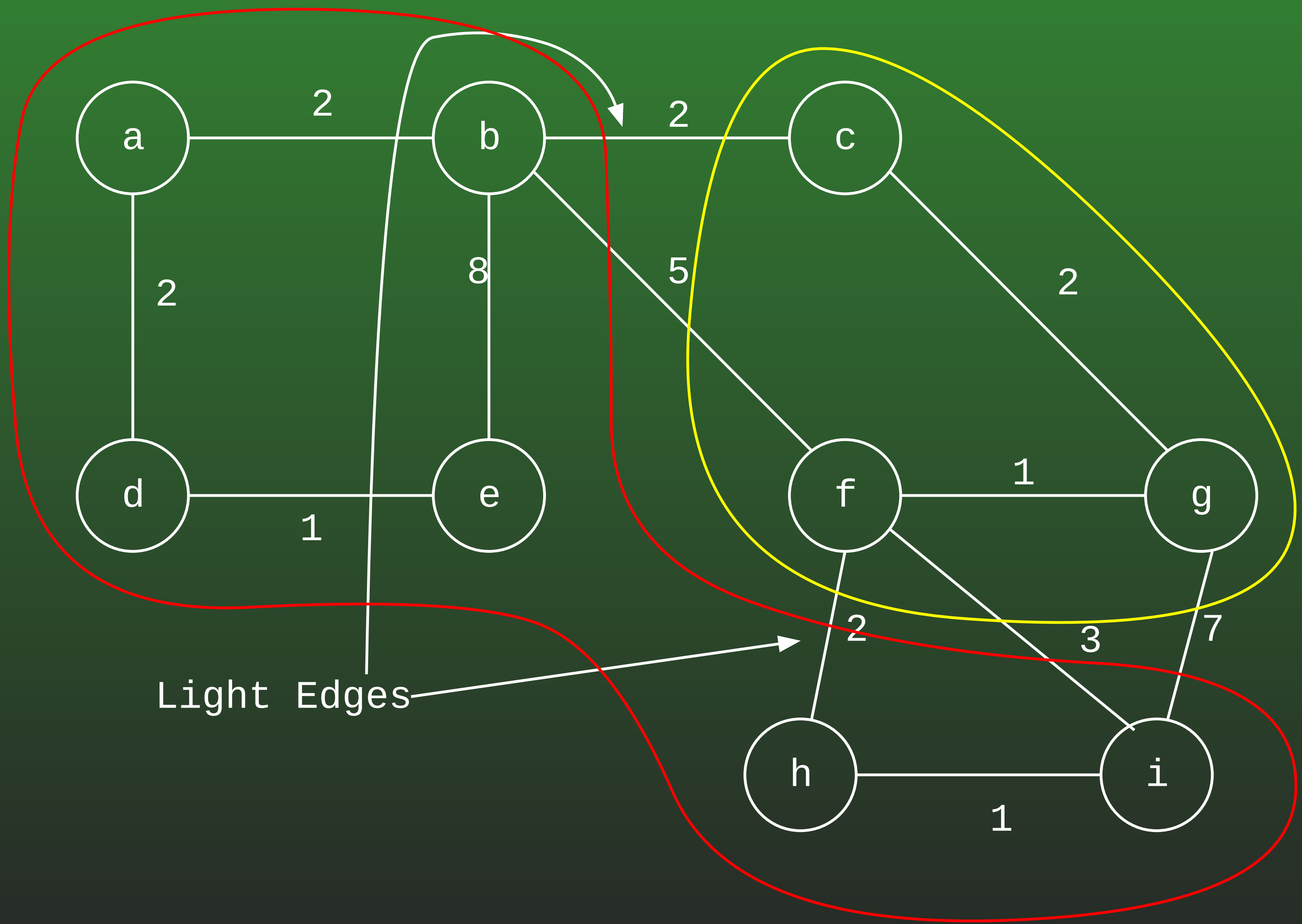
16-14: Graph Cut



16-15: Graph Cut



16-16: **Graph Cut**



16-17: Safe Edges

- A is a set of edges, which is a subset of some MST
- Cut $\{S, V - S\}$ which respects A
- Any light edge (with respect to the cut $\{S, V - S\}$) is safe
  - That is, $A \cup \{(u, v)\}$ is a subset of some MST if $\{(u, v)\}$ is a light edge in a cut that respects A

Safe Edges

- Proof by contradiction:
  - Assume there is:
    - a subset of a MST A
    - a Cut $\{S, V - S\}$ that respects A
    - a light edge (u, v)
  - such that $A \cup \{(u, v)\}$ is not a subset of any MST
- We will show that this leads to a contradiction

16-19: Safe Edges

- Let A' be a MST that is a superset of A
- Add (u, v) to A' to get A'' – now have a cycle
- This cycle must cross the cut at least twice
  - (u, v) is one crossing
  - Must be another crossing (u', v') back across the cut
- remove (u', v') from A'' to get A'''
- A''' is a spanning tree
- cost(A''') = cost(A') - cost((u', v')) + cost((u, v))
- cost((u, v)) $\leq$ cost((u', v') $\Rightarrow$ cost(A''') $\leq$ cost(A')

Safe Edges

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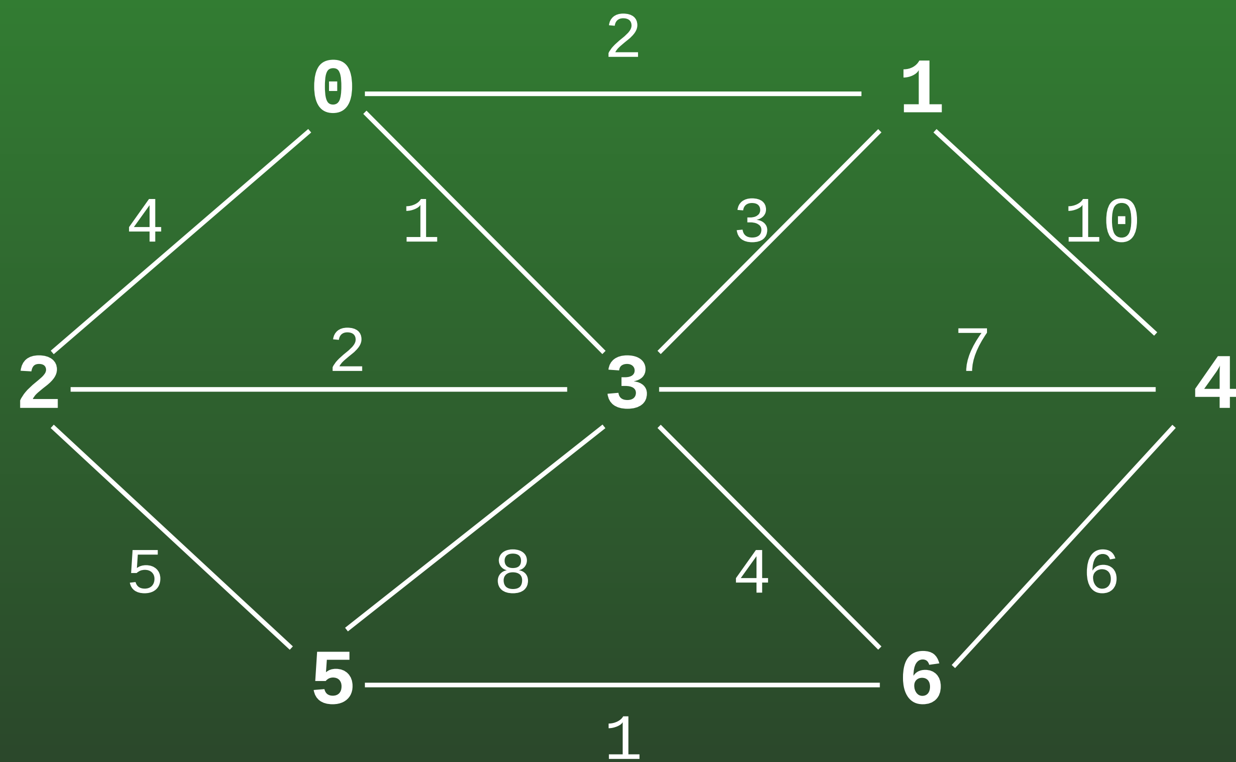
- A''' is a spanning tree
- cost(A''') = cost(A') - cost((u', v')) + cost((u, v))
- cost((u, v)) $\leq$ cost((u', v')) $\Rightarrow$ cost(A''') $\leq$ cost(A')
- Thus A''' must be a MST that contains A and $\{(u, v)\}$, a contradiction

16-21: Kruskal's Algorithm

- Start with an empty graph (no edges)
- Sort the edges by cost
- For each edge e (in increasing order of cost)
  - Add e to G if it would not cause a cycle

Kruskal's Algorithm Examples



Kruskal's Algorithm

- Correctness proof:
  - Kruskal's algorithm always selects a light edge, with according to some cut that respects all edges added so far.
    - Let (u, v) be the cheapest edge that does not cause a cycle
    - Let S be the connected component that contains u.
    - $\{S, V - S\}$ respects edges chosen so far
    - (u, v) crosses the cut, and is the edge with the smallest cost that crosses the cut $\Rightarrow$ (u, v) is a light edge
  - Thus, Kruskal's algorithm always selects a safe edge, and produces a MST

16-24: Kruskal's Algorithm

- Coding Kruskal's Algorithm:
  - Place all edges into a list
  - Sort list of edges by cost
  - For each edge in the list
    - Select the edge if it does not form a cycle with previously selected edges
    - How can we do this?

16-25: Kruskal's Algorithm

- Determining of adding an edge will cause a cycle
  - Start with a forest of V trees (each containing one node)
  - Each added edge merges two trees into one tree
  - An edge causes a cycle if both vertices are in the same tree
    - (examples)

Kruskal's Algorithm

- We need to:
  - Put each vertex in its own tree
  - Given any two vertices v_1 and v_2, determine if they are in the same tree
  - Given any two vertices v_1 and v_2, merge the tree containing v_1 and the tree containing v_2
- ... sound familiar?

16-27: Kruskal's Algorithm

- Disjoint sets!
- Create a list of all edges
- Sort list of edges
- For each edge $e = (v_1, v_2)$ in the list
  - if FIND(v_1) != FIND(v_2)
    - Add e to spanning tree
    - UNION(v_1, v_2)

Kruskal's Algorithm

- Running time?

16-29: Kruskal's Algorithm

- Running time?
  - Sort edges: $\Theta(|E| \lg |E|)$
  - Build tree: $O(E)$
- Total: $\Theta(|E| \lg |E|)$

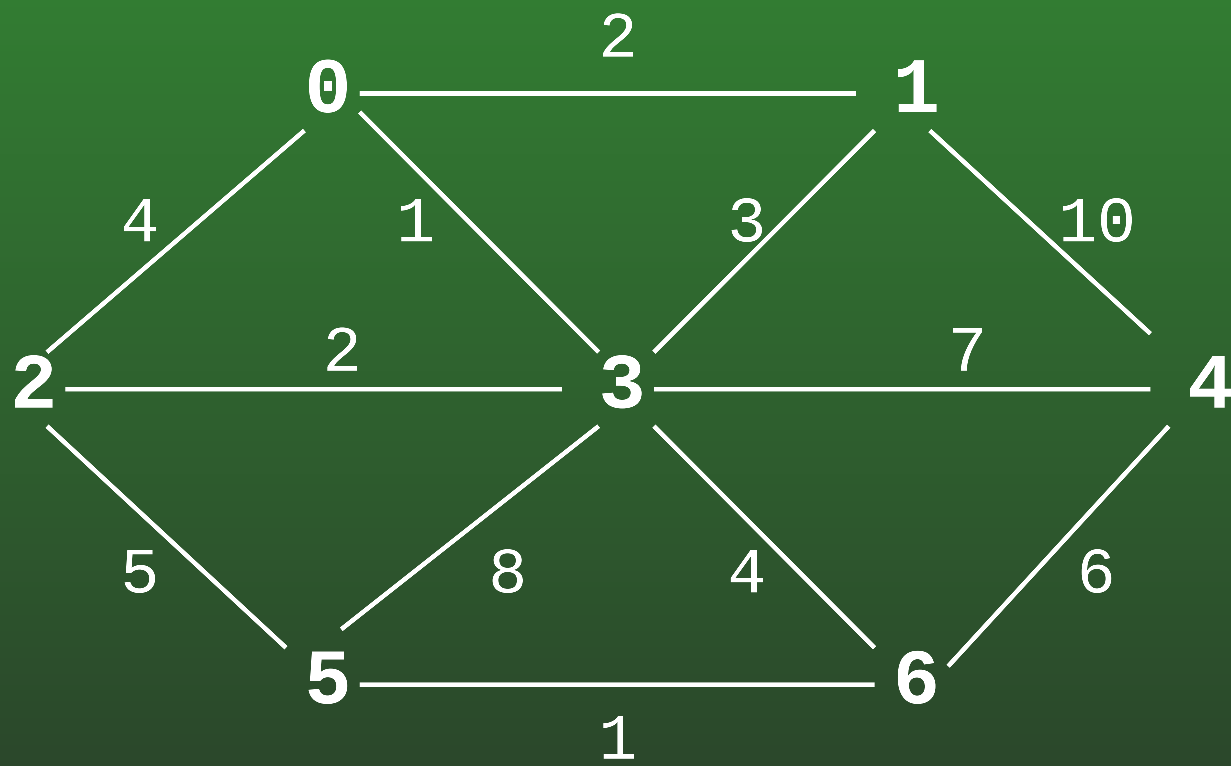
Prim's Algorithm

- Grow that spanning tree out from an initial vertex
- Divide the graph into two sets of vertices
  - vertices in the spanning tree
  - vertices *not* in the spanning tree
- Initially, Start vertex is in the spanning tree, all other vertices are not in the tree
  - Pick the initial vertex arbitrarily

16-31: Prim's Algorithm

- While there are vertices not in the spanning tree
  - Add the cheapest vertex to the spanning tree

16-32: Prims's Algorithm Examples



Prim's Algorithm

- Maintain a table, which keeps track of:
  - Whether or not the vertex has been added to the MST (Known)
  - Current cheapest cost to add the vertex to the MST (Cost)
  - Neighber to connect to, to get the cheapest cost (Path)

16-34: Prim Code

```
void Prim(Edge G[], int s, tableEntry T[]) {
  int i, v;
  Edge e;
  for(i=0; i<G.length; i++) {
    T[i].distance = Integer.MAX_VALUE;
    T[i].path = -1;
    T[i].known = false;
  }
  T[s].distance = 0;
  for (i=0; i < G.length; i++) {
    v = minUnknownVertex(T);
    T[v].known = true;
    for (e = G[v]; e != null; e = e.next) {
      if (T[e.neighbor].distance > e.cost) {
        T[e.neighbor].distance = e.cost;
        T[e.neighbor].path = v;
      }
    }
  }
}
```

Prim Running Time

- If minUnknownVertex(T) is calculated by doing a linear search through the table:
  - Each minUnknownVertex call takes time $\Theta(|V|)$
    - Called $|V|$ times – total time for all calls to minUnkownVertex: $\Theta(|V|^2)$
  - If statement is executed $|E|$ times, each time takes time $O(1)$
  - Total time: $O(|V|^2 + |E|) = O(|V|^2)$.

Prim Running Time

- If minUnknownVertex(T) is calculated by inserting all vertices into a min-heap (using distances as key) updating the heap as the distances are changed
  - Each minUnknownVertex call tatkes time $\Theta(\lg |V|)$
    - Called $|V|$ times – total time for all calls to minUnknownVertex: $\Theta(|V| \lg |V|)$
  - If statement is executed $|E|$ times – each time takes time $O(\lg |V|)$, since we need to update (decrement) keys in heap
  - Total time: $O(|V| \lg |V| + |E| \lg |V|) \in O(|E| \lg |V|)$
- Is this better or wose than the previous method?

Prim Running Time

- If minUnknownVertex(T) is calculated by inserting all vertices into a Fibonacci heap (using distances as key) updating the heap as the distances are changed
  - Each minUnknownVertex call takes amortized time $\Theta(\lg |V|)$
    - Called $|V|$ times – total amortized time for all calls to minUnknownVertex: $\Theta(|V| \lg |V|)$
  - If statement is executed $|E|$ times – each time takes amortized time $O(1)$, since decrementing keys takes time $O(1)$.
  - Total time: $O(|V| \lg |V| + |E|)$
- Is this better or wose than the previous methods? Explain!

16-38: Prim Correctness

- Every time we select a vertex as known, pick an edge to add to MST
- If the set of known vertices are K:
  - Create a partition $\{K, V - K\}$
  - Next vertex that we select will be connected to the known vertices by the cheapest possible edge
  - Thus, we're always picking a light edge, according to some partition that repsects all edges we've previously chosen