Graduate Algorithms

CS673-2016F-17

Shortest Path Algorithms

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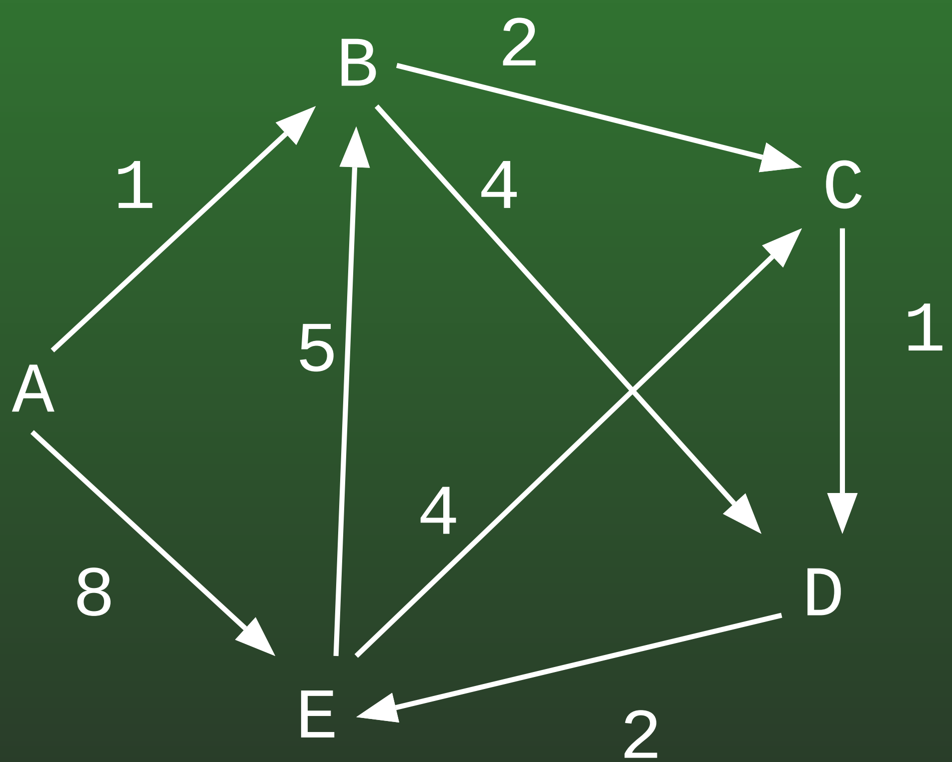
17-0: Computing Shortest Path

- Given a directed weighted graph G (all weights non-negative) and two vertices x and y, find the least-cost path from x to y in G.
  - Undirected graph is a special case of a directed graph, with symmetric edges
- Least-cost path may not be the path containing the fewest edges
  - “shortest path” == “least cost path”
  - “path containing fewest edges” = “path containing fewest edges”

17-1: Shortest Path Example

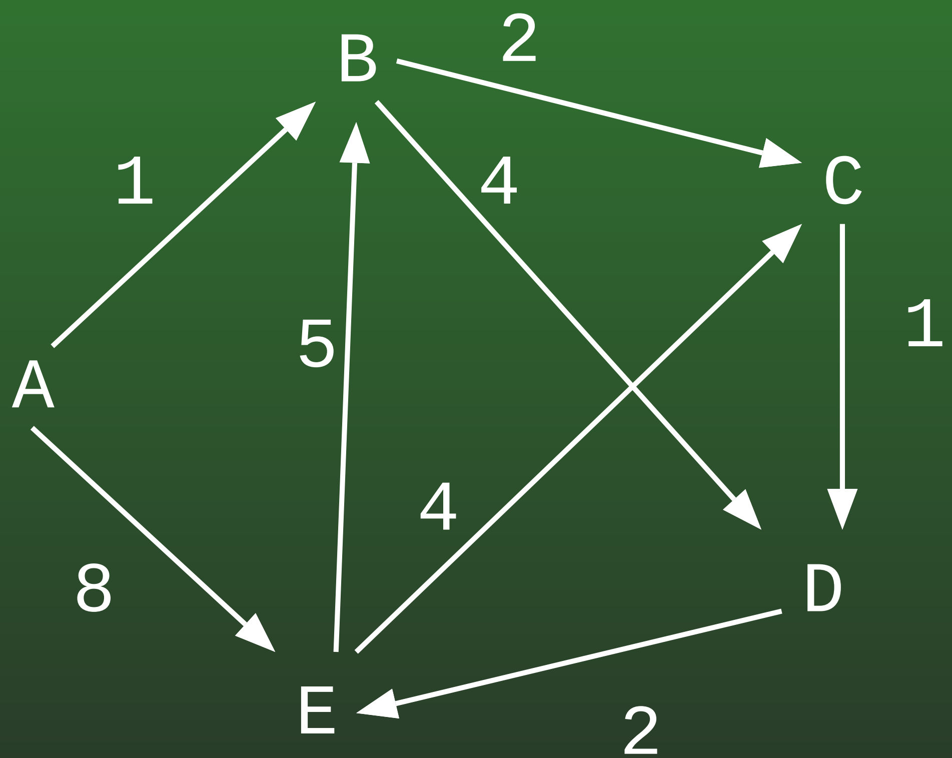
- Shortest path $\neq$ path containing fewest edges



- Shortest Path from A to E?

17-2: Shortest Path Example

- Shortest path $\neq$ path containing fewest edges



- Shortest Path from A to E:
  - A, B, C, D, E

17-3: Single Source Shortest Path

- To find the shortest path from vertex x to vertex y, we need (worst case) to find the shortest path from x to *all* other vertices in the graph
  - Why?

17-4: Single Source Shortest Path

- To find the shortest path from vertex x to vertex y, we need (worst case) to find the shortest path from x to *all* other vertices in the graph
  - To find the shortest path from x to y, we need to find the shortest path from x to all nodes on the path from x to y
  - Worst case, *all* nodes will be on the path

17-5: Single Source Shortest Path

- If all edges have unit weight ...

17-6: Single Source Shortest Path

- If all edges have unit weight,
- We can use Breadth First Search to compute the shortest path
- BFS Spanning Tree contains shortest path to each node in the graph
    - Need to do some more work to create & save BFS spanning tree
- When edges have differing weights, this obviously will not work

17-7: Single Source Shortest Path

- General Idea for finding Single Source Shortest Path
  - Start with the distance estimate to each node (except the source) as ∞
  - Repeatedly relax distance estimate until you can relax no more
  - To relax and edge (u, v)
    - dist(v) $>$ dist(u) + cost((u, v))
    - Set dist(v) $\leftarrow$ dist(u) + cost((u, v))

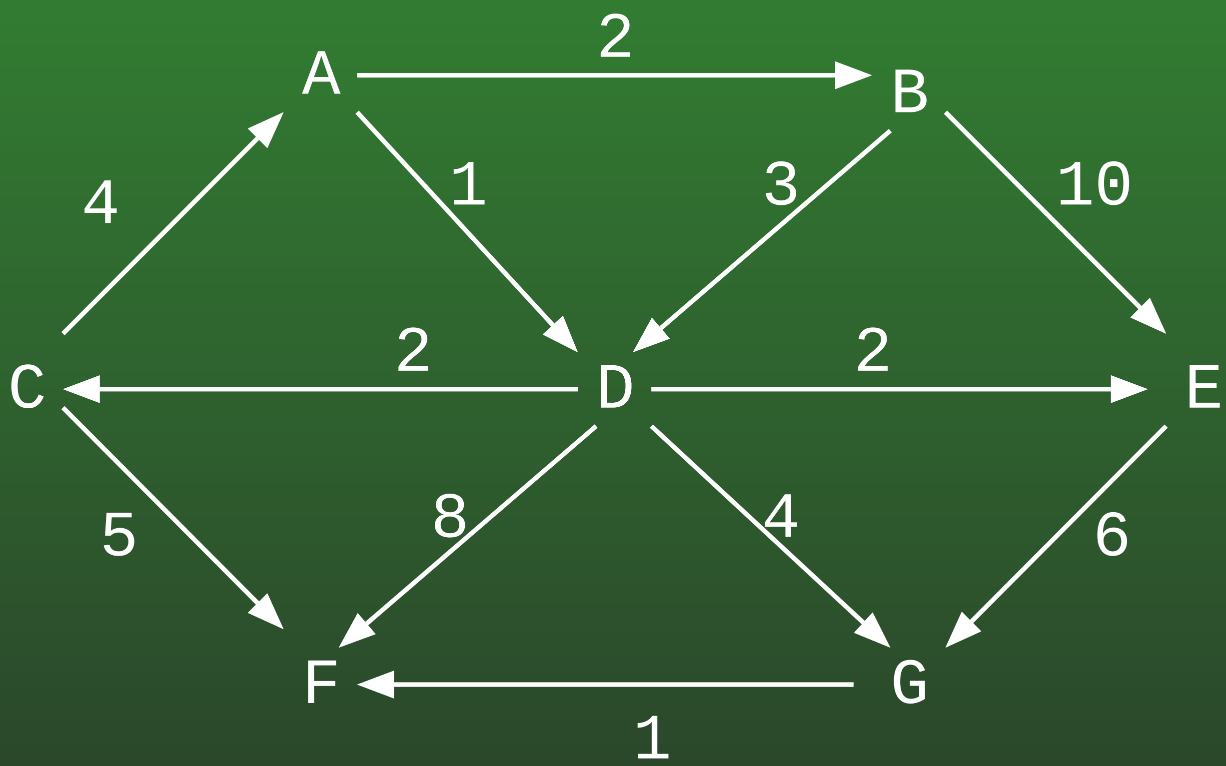
17-8: Single Source Shortest Path

- Dijkstra's algorithm
  - Relax edges from source
- *Remarkably* similar to Prim's MST algorith
  - Pretty neat – algorithms are doing different things, but code is almost identical

17-9: Single Source Shortest Path

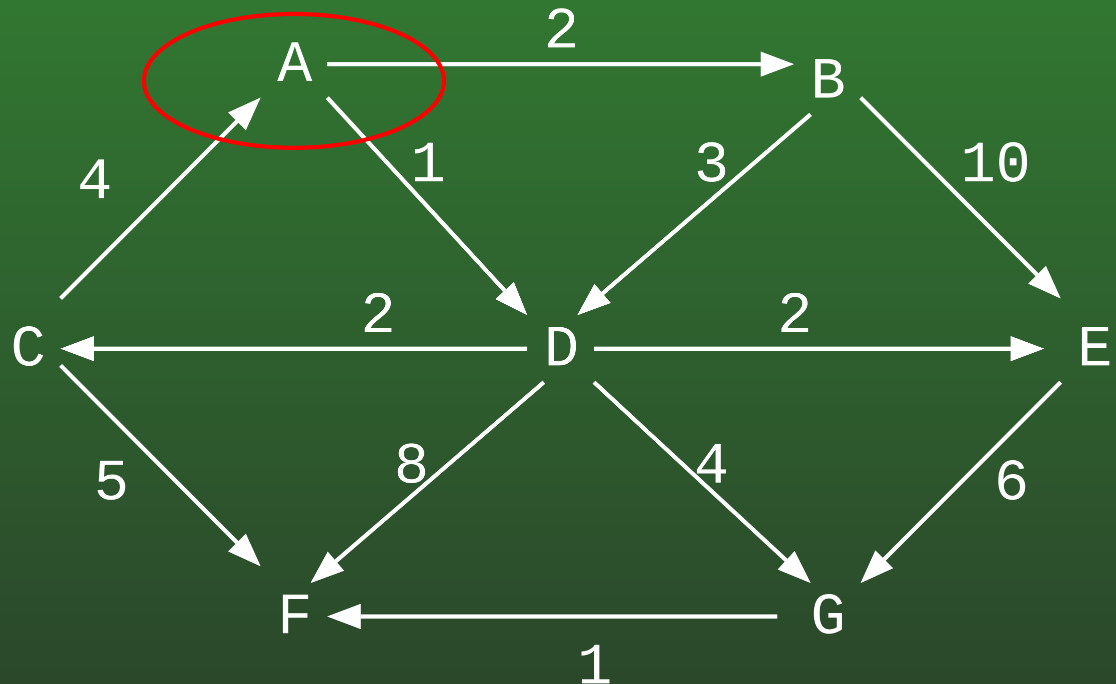
- Divide the vertices into two sets:
  - Vertices whose shortest path from the initial vertex is known
  - Vertices whose shortest path from the initial vertex is not known
- Initially, only the initial vertex is known
- Move vertices one at a time from the unknown set to the known set, until all vertices are known

17-10: Single Source Shortest Path



- Start with the vertex A

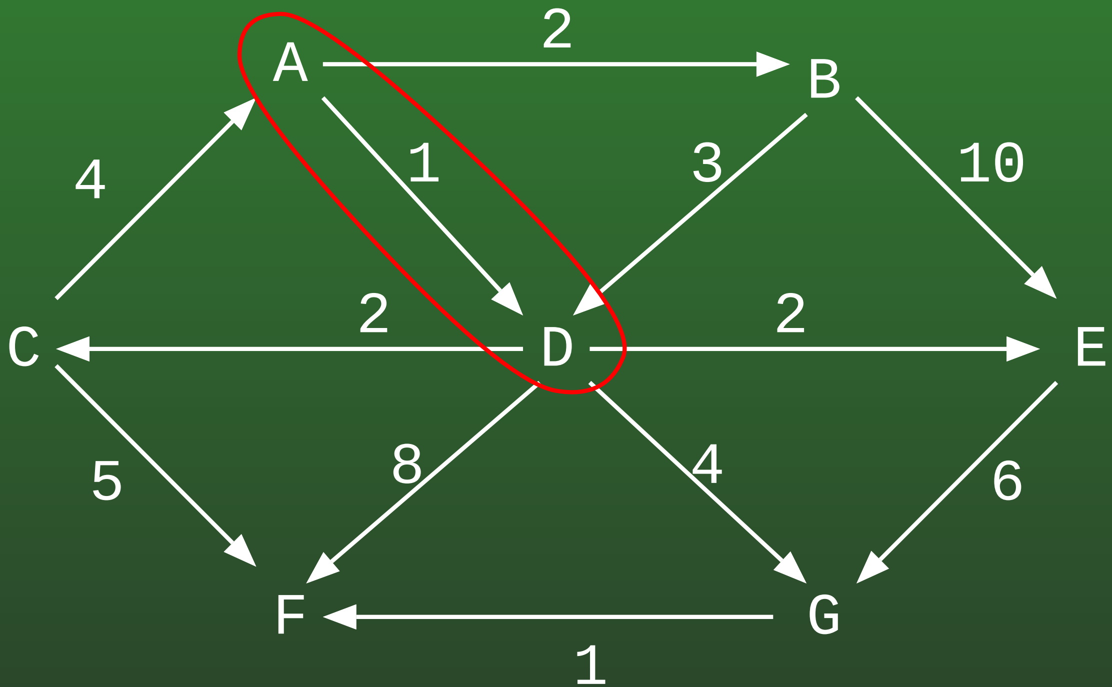
17-11: Single Source Shortest Path



Node	Distance
A	0
B	
C	
D	
E	
F	
G	

- Known vertices are circled in red
- We can now extend the known set by 1 vertex

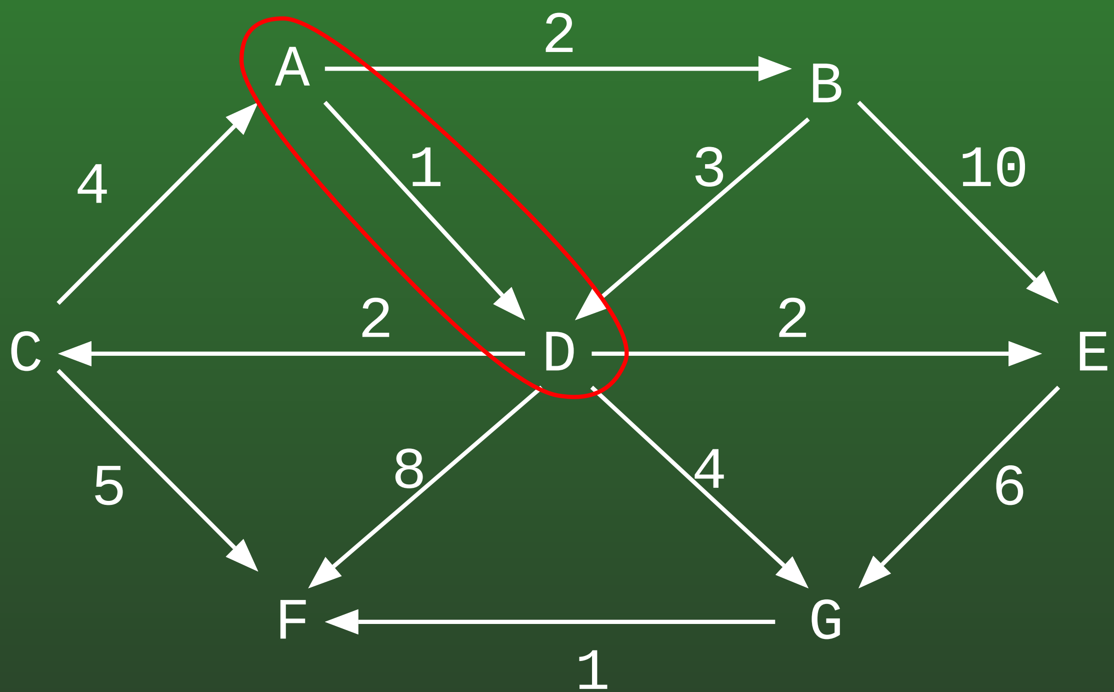
17-12: Single Source Shortest Path



Node	Distance
A	0
B	
C	
D	1
E	
F	
G	

- Why is it safe to add D, with cost 1?

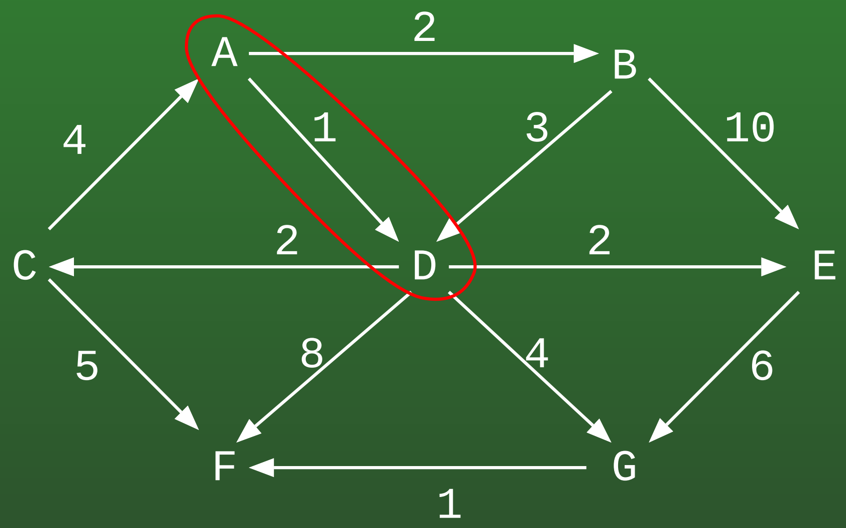
17-13: Single Source Shortest Path



Node	Distance
A	0
B	
C	
D	1
E	
F	
G	

- Why is it safe to add D, with cost 1?
  - Could we do better with a more roundabout path?

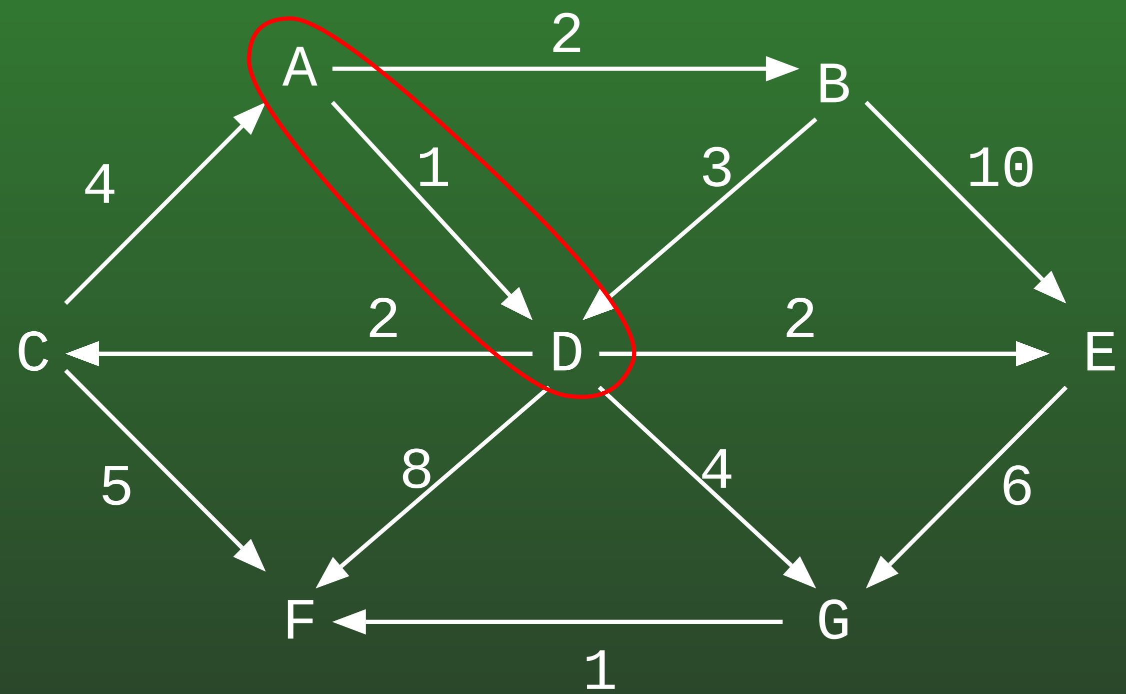
17-14: Single Source Shortest Path



Node	Distance
A	0
B	
C	
D	1
E	
F	
G	

- Why is it safe to add D, with cost 1?
  - Could we do better with a more roundabout path?
  - No – to get to any other node will cost at least 1
  - No negative edge weights, can't do better than 1

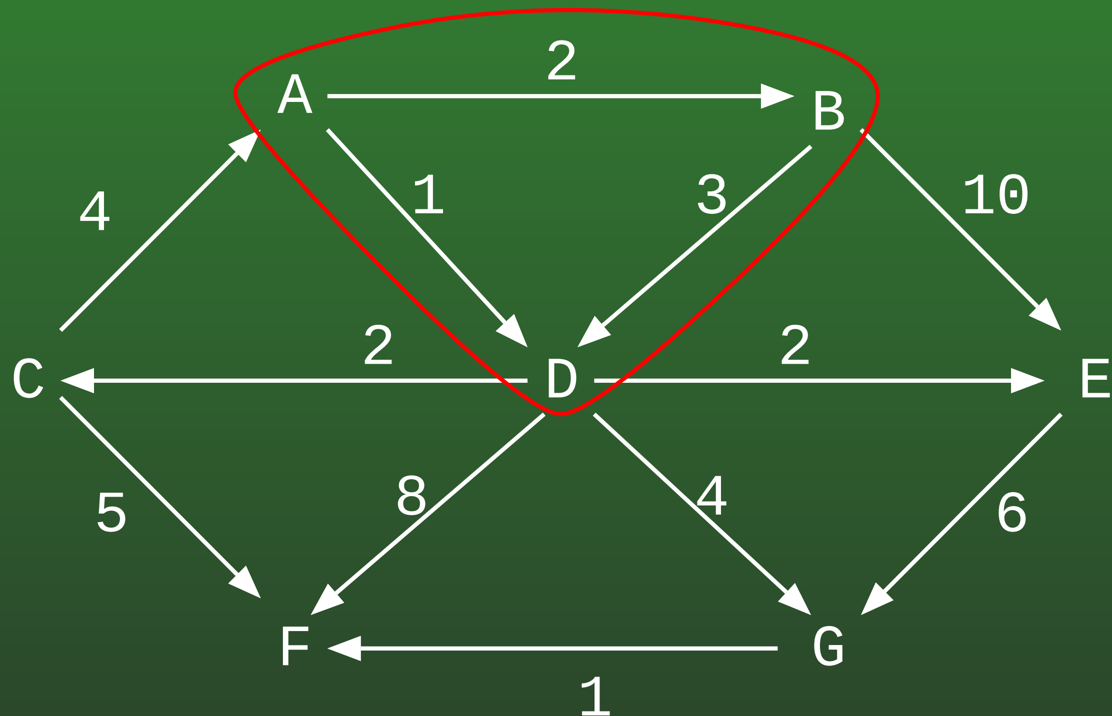
17-15: Single Source Shortest Path



Node	Distance
A	0
B	
C	
D	1
E	
F	
G	

- We can now add another vertex to our known list ...

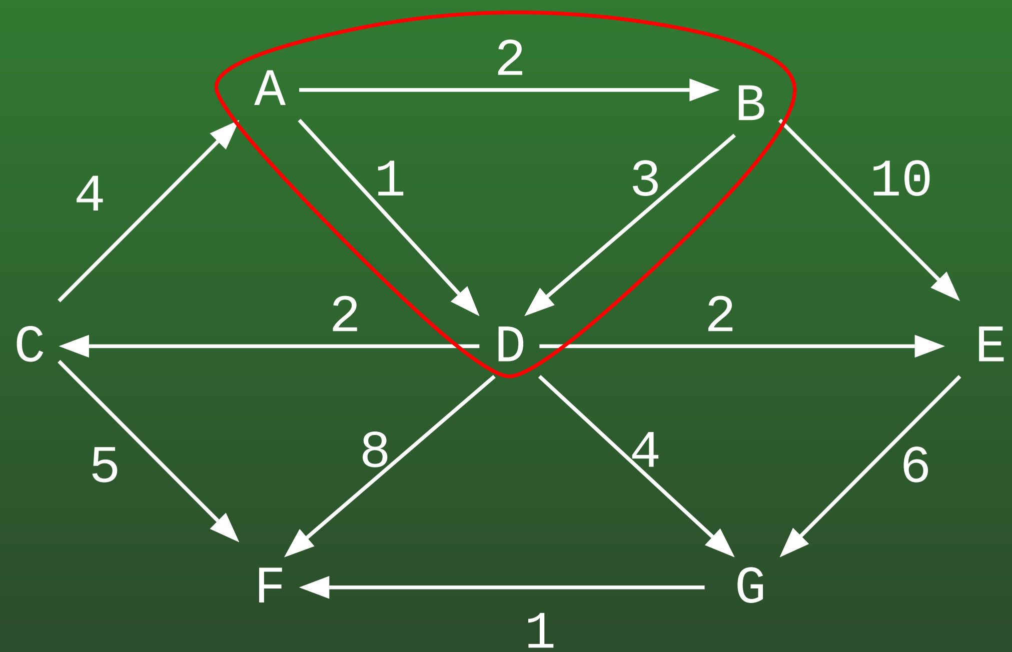
17-16: Single Source Shortest Path



Node	Distance
A	0
B	2
C	
D	1
E	
F	
G	

- How do we know that we could not get to B cheaper by going through D?

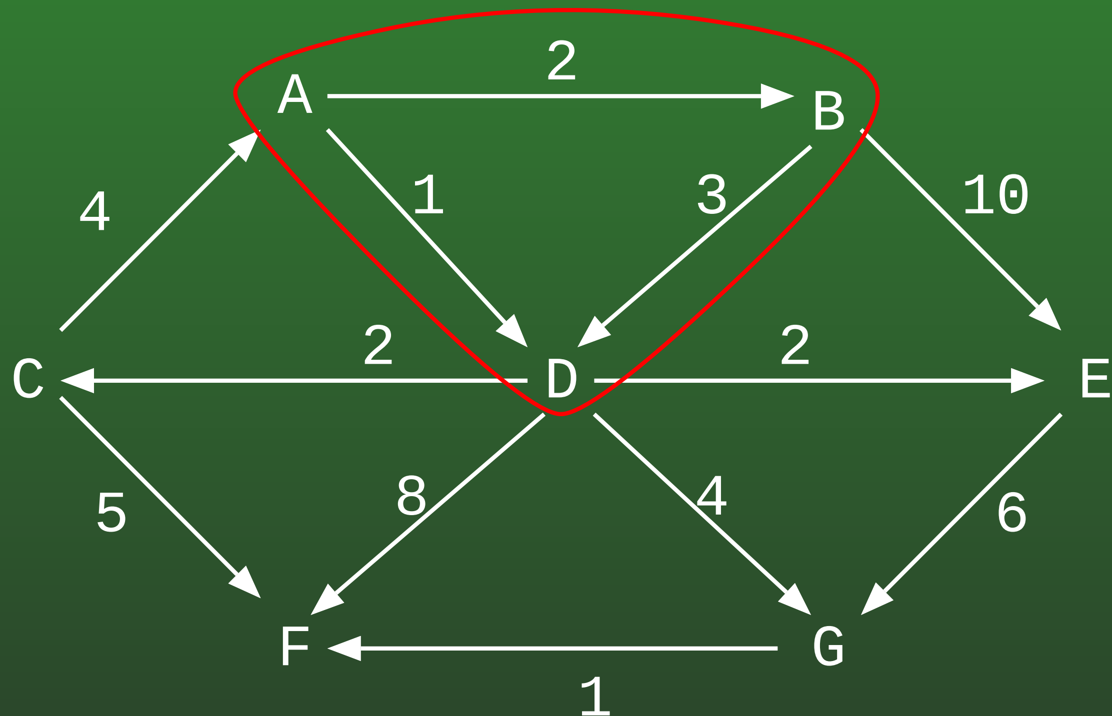
17-17: Single Source Shortest Path



Node	Distance
A	0
B	2
C	
D	1
E	
F	
G	

- How do we know that we could not get to B cheaper by going through D?
  - Costs 1 to get to D
  - Costs at least 2 to get anywhere from D
    - Cost *at least* (1+2 = 3) to get to B through D

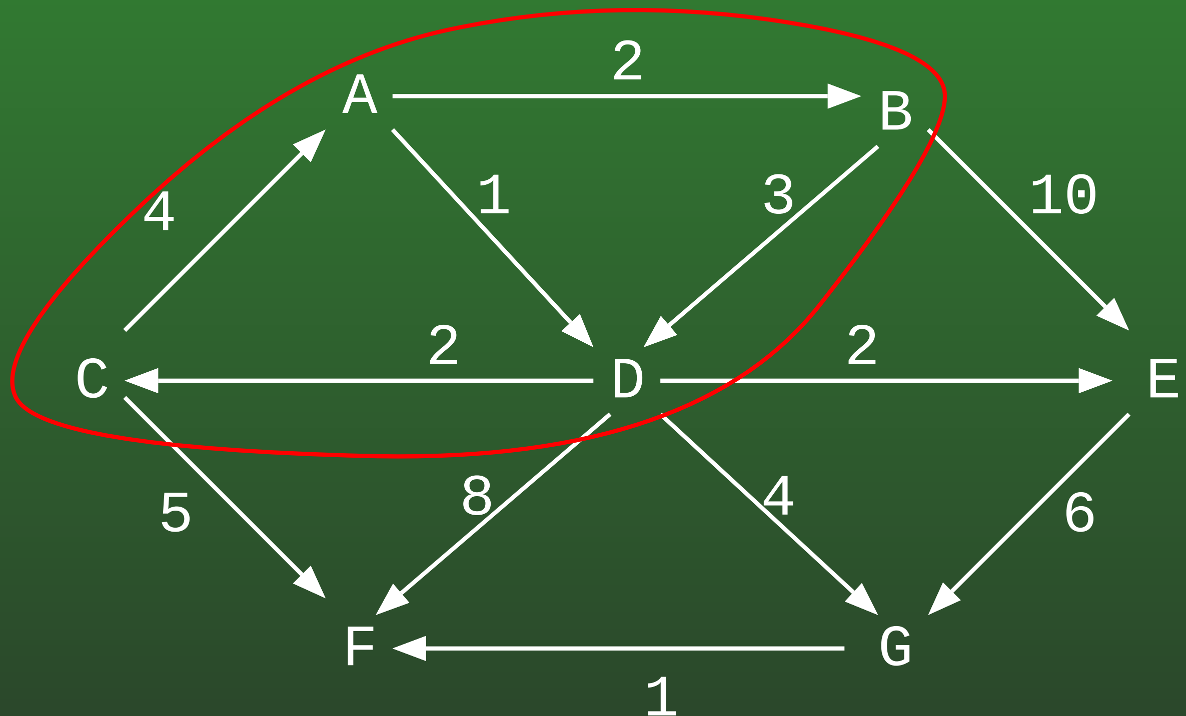
17-18: Single Source Shortest Path



Node	Distance
A	0
B	2
C	
D	1
E	
F	
G	

- Next node we can add ...

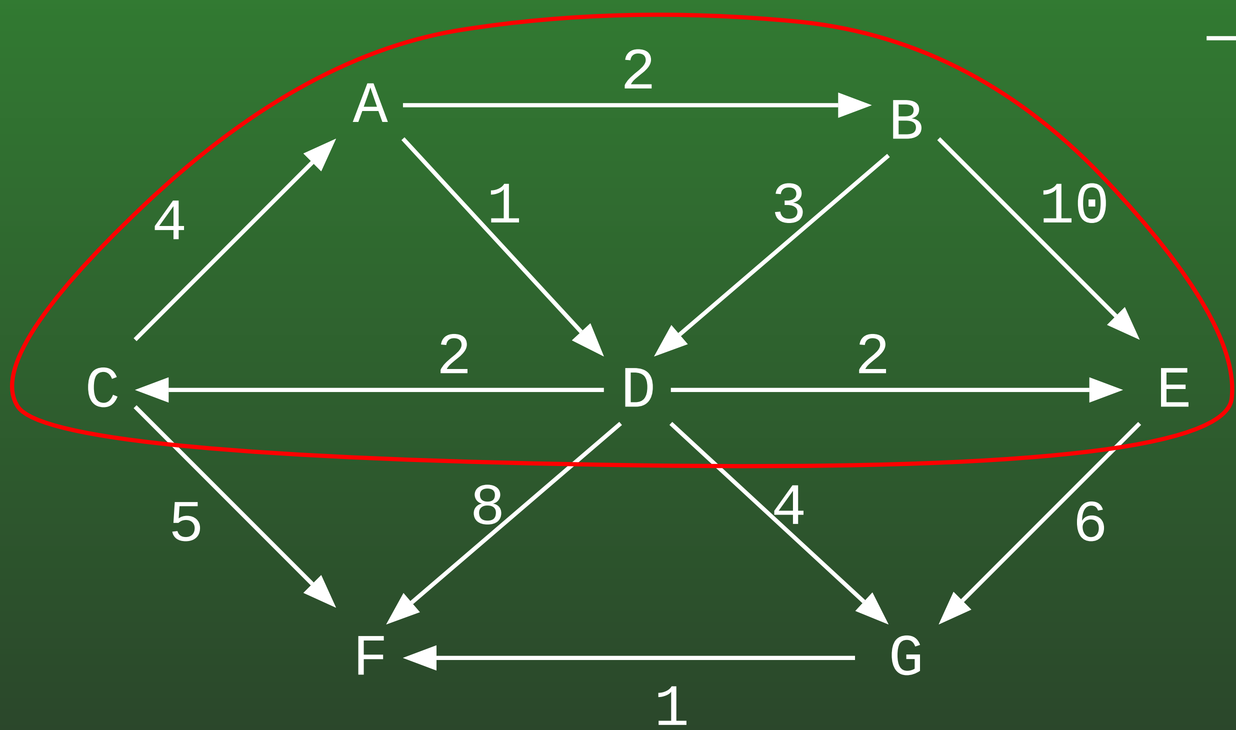
17-19: Single Source Shortest Path



Node	Distance
A	0
B	2
C	3
D	1
E	
F	
G	

- (We also could have added E for this step)
- Next vertex to add to Known ...

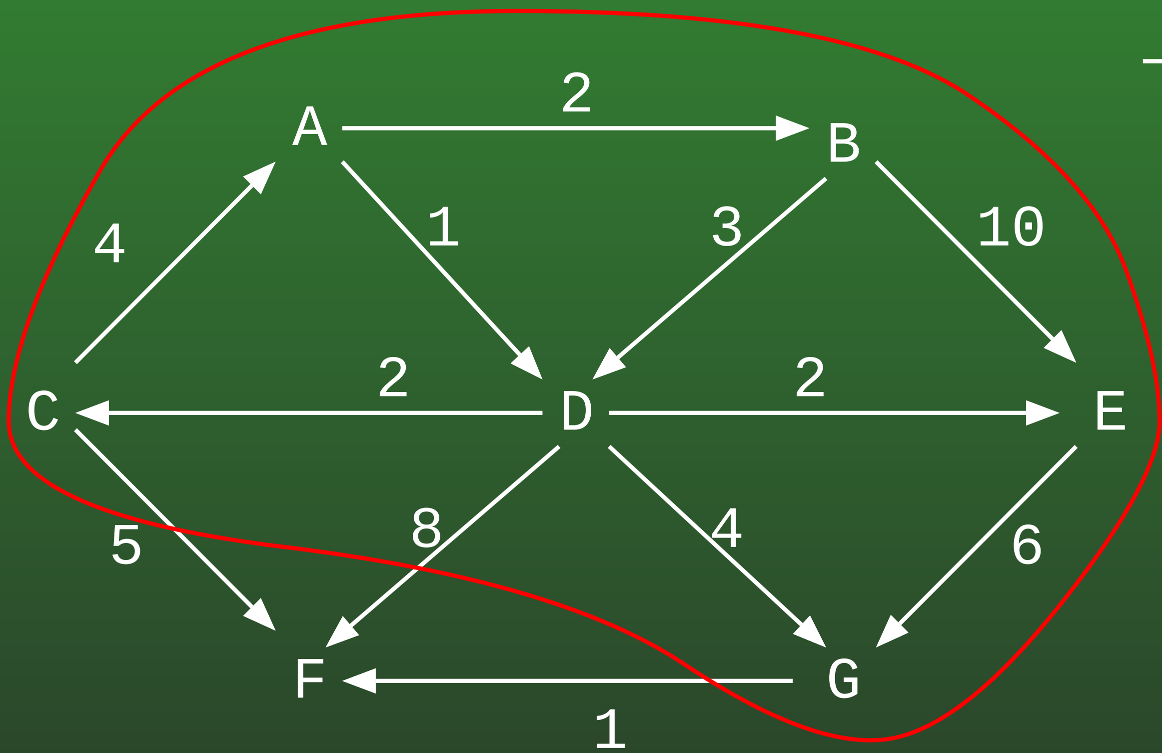
17-20: Single Source Shortest Path



Node	Distance
A	0
B	2
C	3
D	1
E	3
F	
G	

- Cost to add F is 8 (through C)
- Cost to add G is 5 (through D)

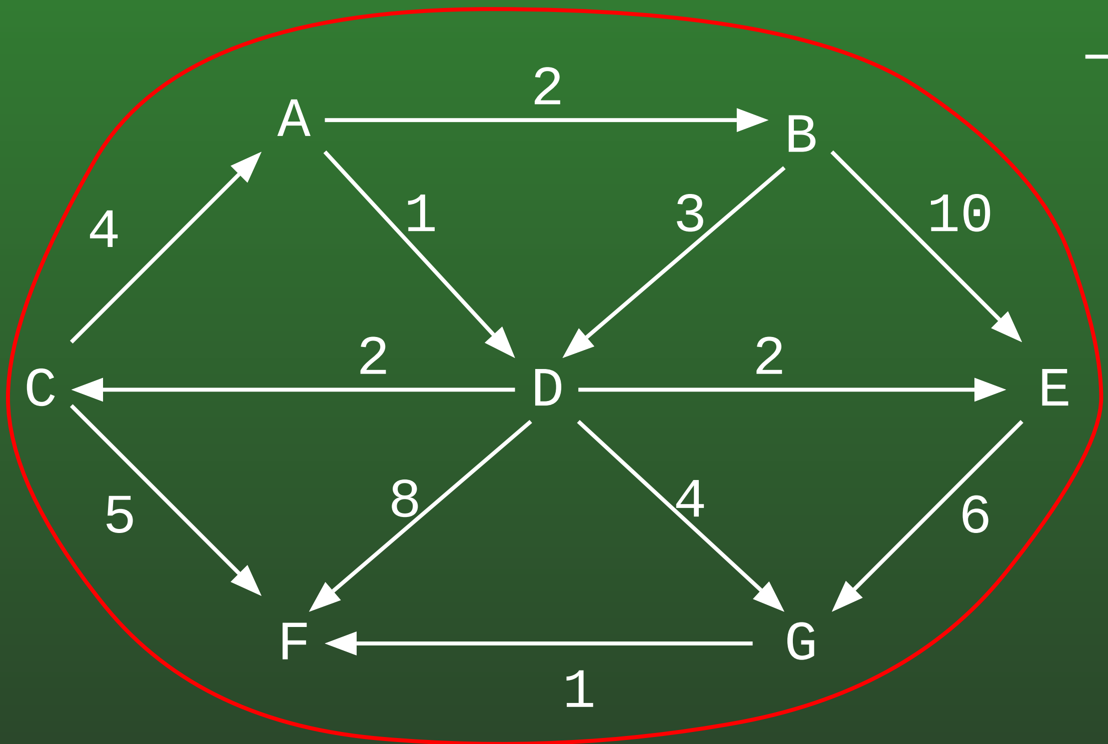
17-21: Single Source Shortest Path



Node	Distance
A	0
B	2
C	3
D	1
E	3
F	5
G	

- Last node ...

17-22: Single Source Shortest Path



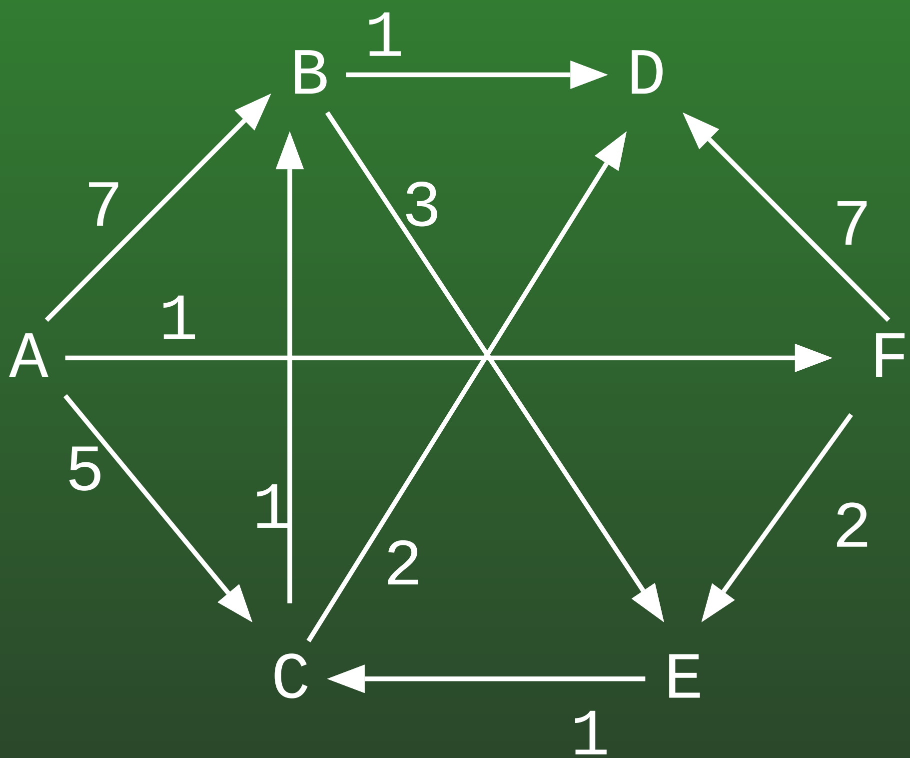
Node	Distance
A	0
B	2
C	3
D	1
E	3
F	5
G	6

- We now know the length of the shortest path from A to all other vertices in the graph

Dijkstra's Algorithm

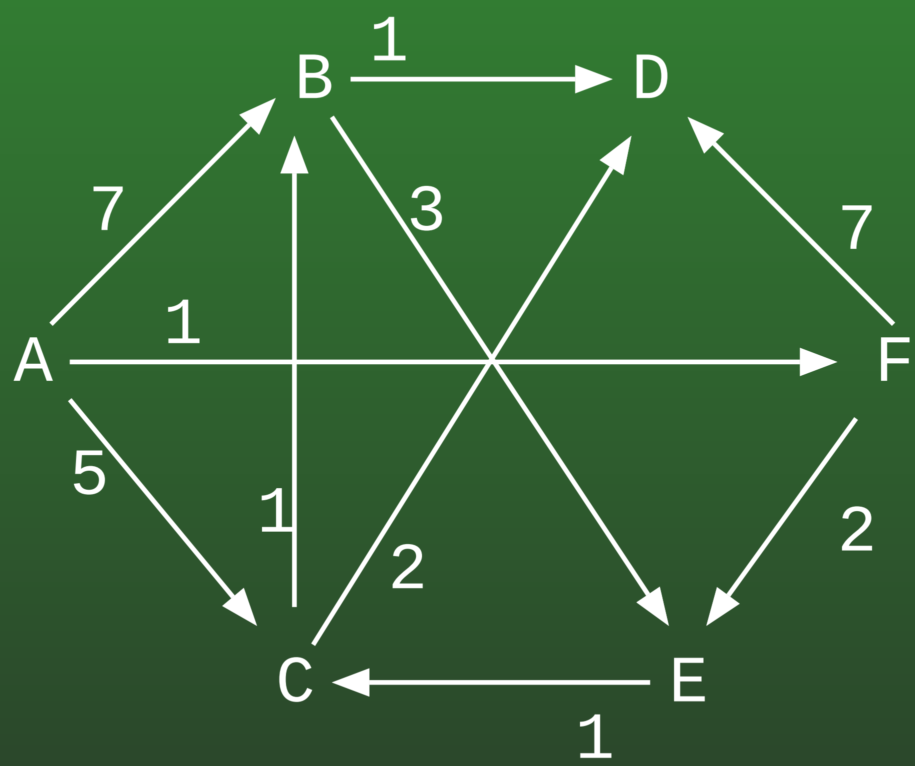
- Keep a table that contains, for each vertex
  - Is the distance to that vertex known?
  - What is the best distance we've found so far?
- Repeat:
  - Pick the smallest unknown distance
  - mark it as known
  - update the distance of all unknown neighbors of that node
- Until all vertices are known

17-24: Dijkstra's Algorithm Example



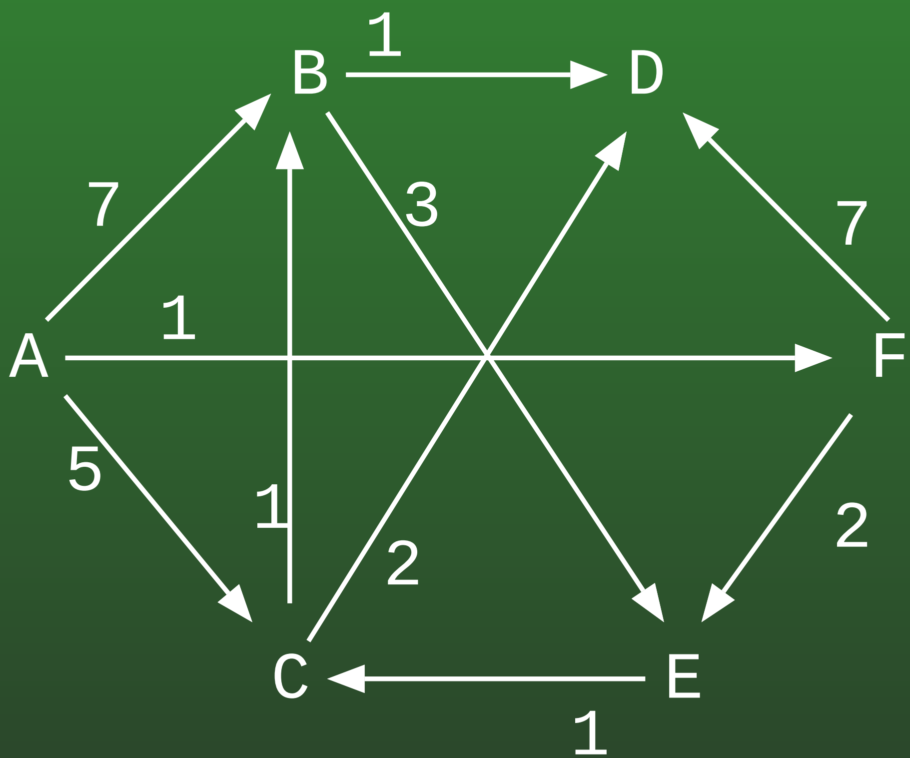
Node	Known	Distance
A	false	0
B	false	∞
C	false	∞
D	false	∞
E	false	∞
F	false	∞

17-25: Dijkstra's Algorithm Example



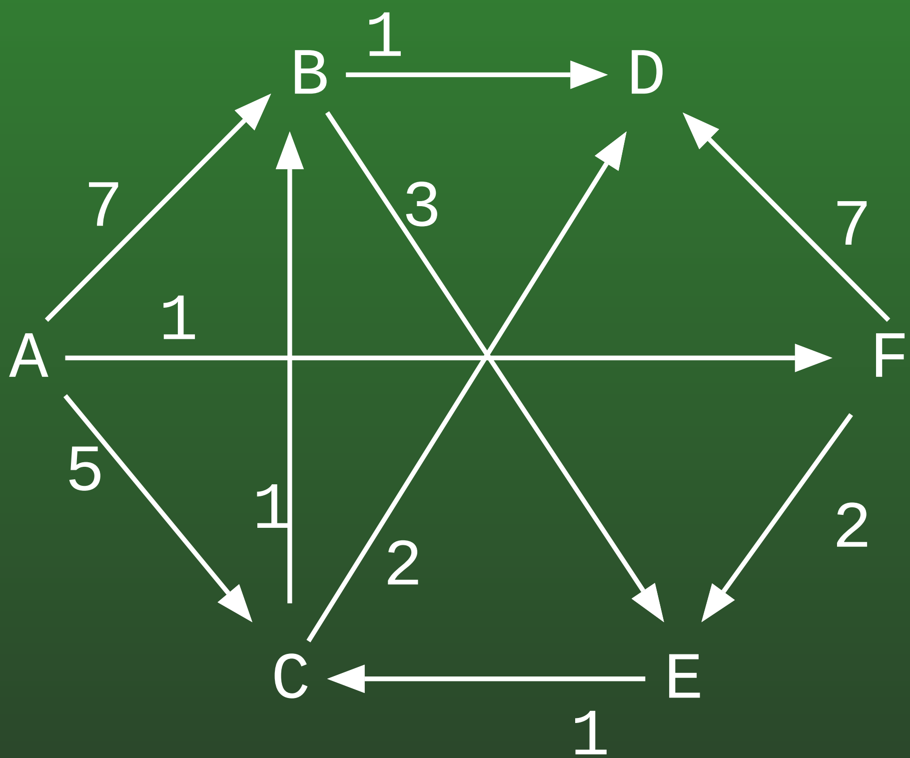
Node	Known	Distance
A	true	0
B	false	7
C	false	5
D	false	∞
E	false	∞
F	false	1

Dijkstra's Algorithm Example



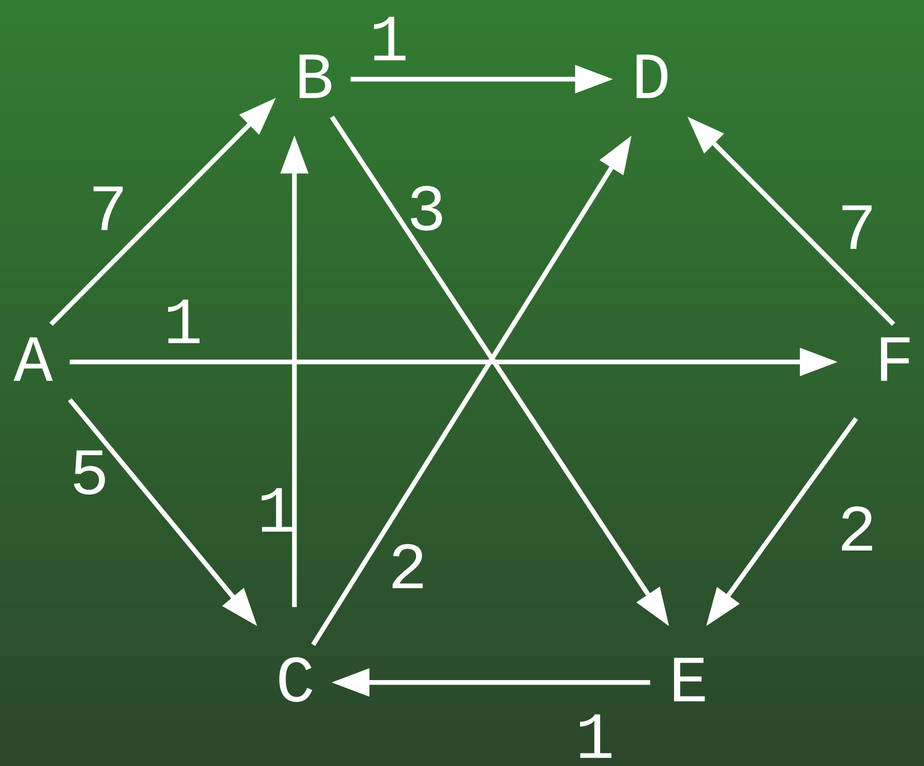
Node	Known	Distance
A	true	0
B	false	7
C	false	5
D	false	8
E	false	3
F	true	1

Dijkstra's Algorithm Example



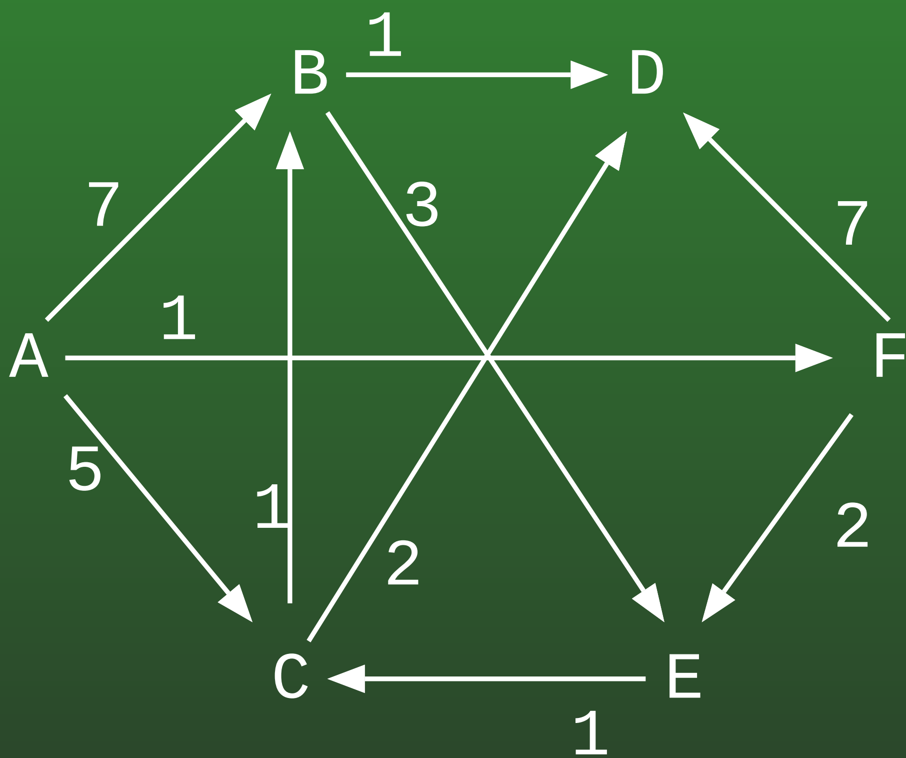
Node	Known	Distance
A	true	0
B	false	7
C	false	4
D	false	8
E	true	3
F	true	1

Dijkstra's Algorithm Example



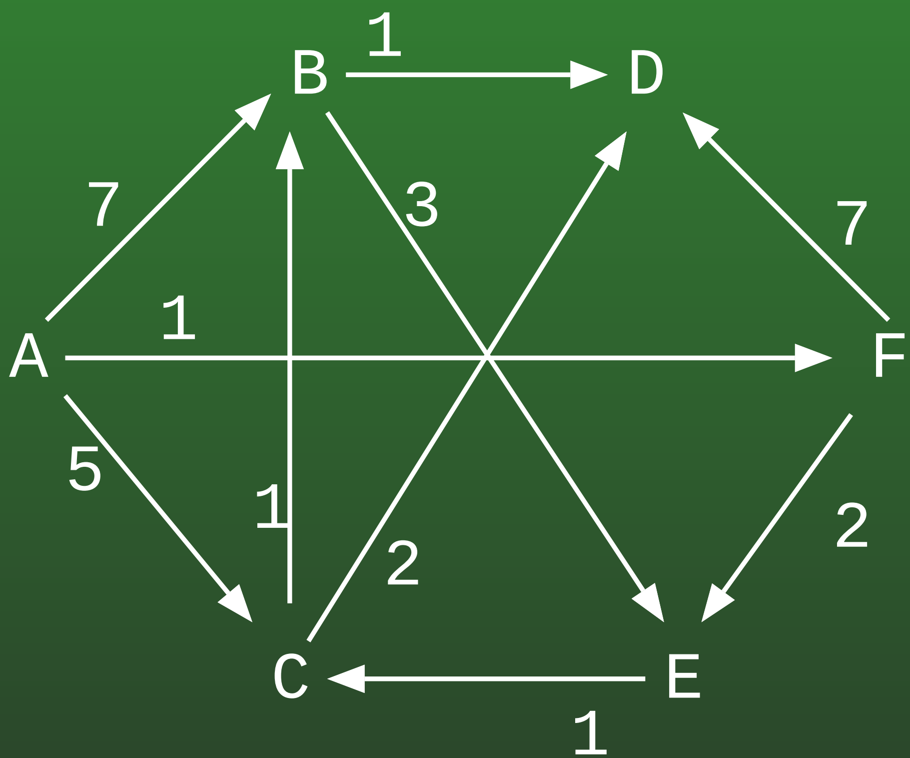
Node	Known	Distance
A	true	0
B	false	5
C	true	4
D	false	6
E	true	3
F	true	1

17-29: Dijkstra's Algorithm Example



Node	Known	Distance
A	true	0
B	true	5
C	true	4
D	false	6
E	true	3
F	true	1

Dijkstra's Algorithm Example



Node	Known	Distance
A	true	0
B	true	5
C	true	4
D	true	6
E	true	3
F	true	1

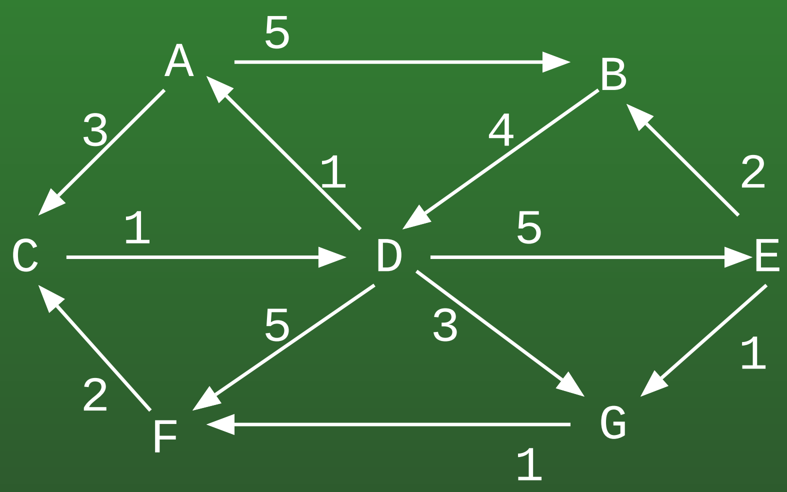
17-31: Dijkstra's Algorithm

- After Dijkstra's algorithm is complete:
  - We know the *length* of the shortest path
  - We do not know *what* the shortest path is
- How can we modify Dijstra's algorithm to compute the path?

17-32: Dijkstra’s Algorithm

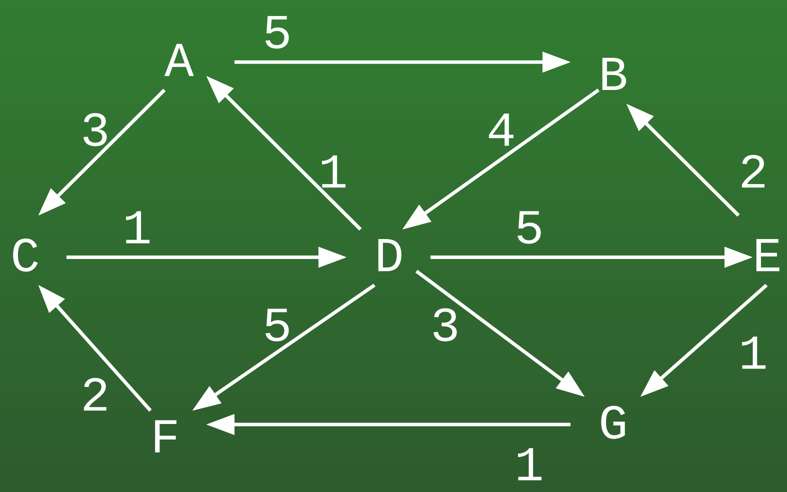
- After Dijkstra’s algorithm is complete:
  - We know the *length* of the shortest path
  - We do not know *what* the shortest path is
- How can we modify Dijstra’s algorithm to compute the path?
  - Store not only the distance, but the immediate parent that led to this distance

17-33: Dijkstra's Algorithm Example



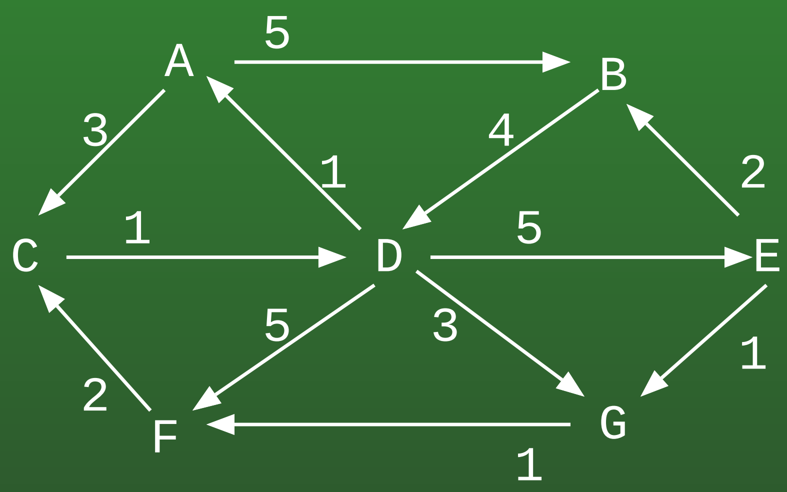
Node	Known	Dist	Path
A	false	0	
B	false	∞	
C	false	∞	
D	false	∞	
E	false	∞	
F	false	∞	
G	false	∞	

17-34: Dijkstra's Algorithm Example



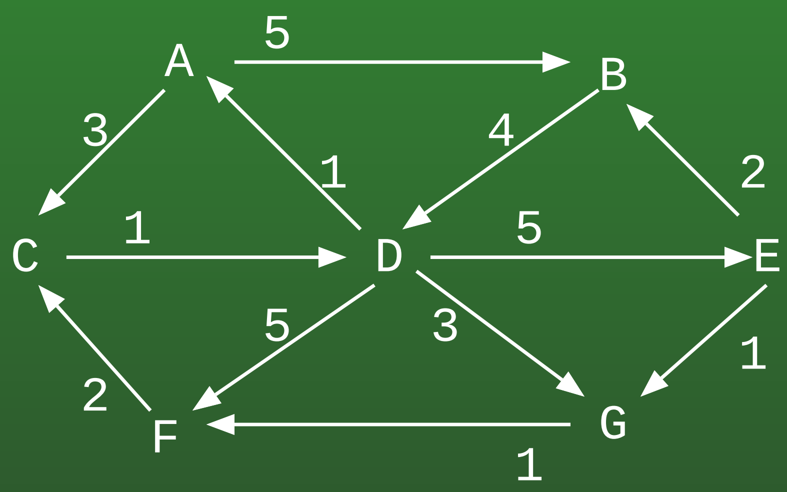
Node	Known	Dist	Path
A	true	0	
B	false	5	A
C	false	3	A
D	false	∞	
E	false	∞	
F	false	∞	
G	false	∞	

17-35: Dijkstra's Algorithm Example



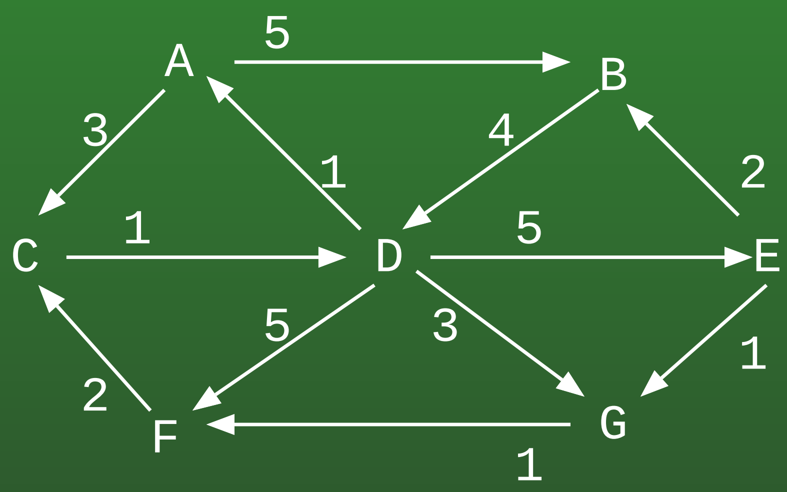
Node	Known	Dist	Path
A	true	0	
B	false	5	A
C	true	3	A
D	false	4	C
E	false	∞	
F	false	∞	
G	false	∞	

17-36: Dijkstra's Algorithm Example



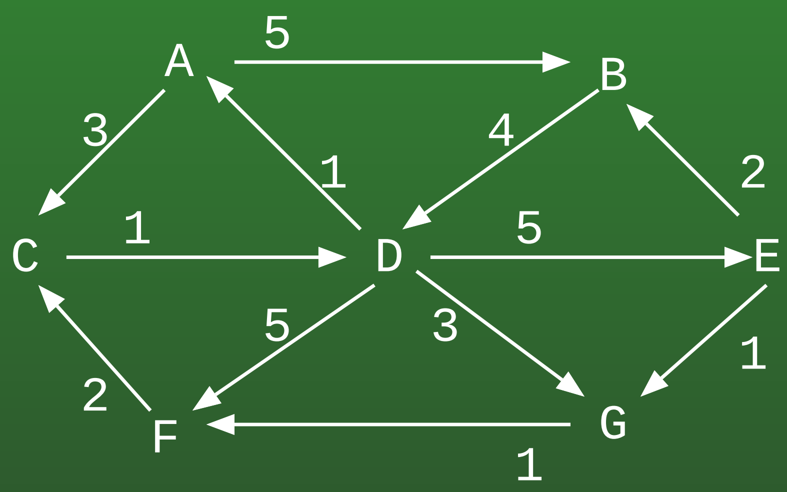
Node	Known	Dist	Path
A	true	0	
B	false	5	A
C	true	3	A
D	true	4	C
E	false	9	D
F	false	9	D
G	false	7	D

Dijkstra's Algorithm Example



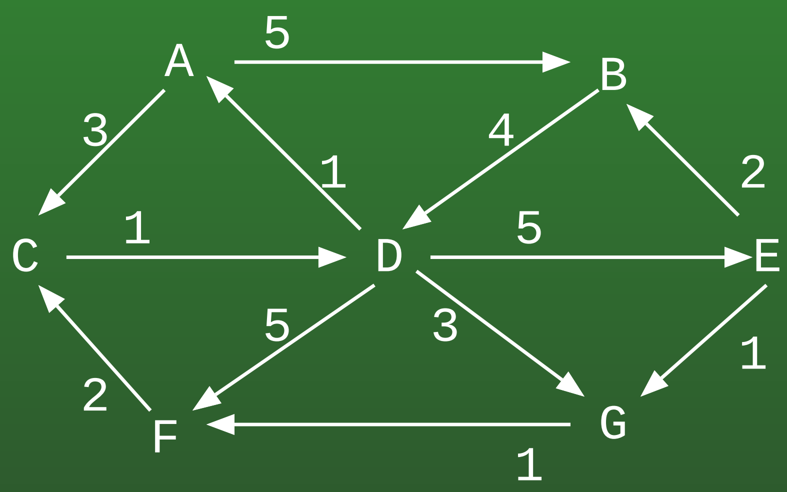
Node	Known	Dist	Path
A	true	0	
B	true	5	A
C	true	3	A
D	true	4	C
E	false	9	D
F	false	9	D
G	false	7	D

17-38: Dijkstra's Algorithm Example



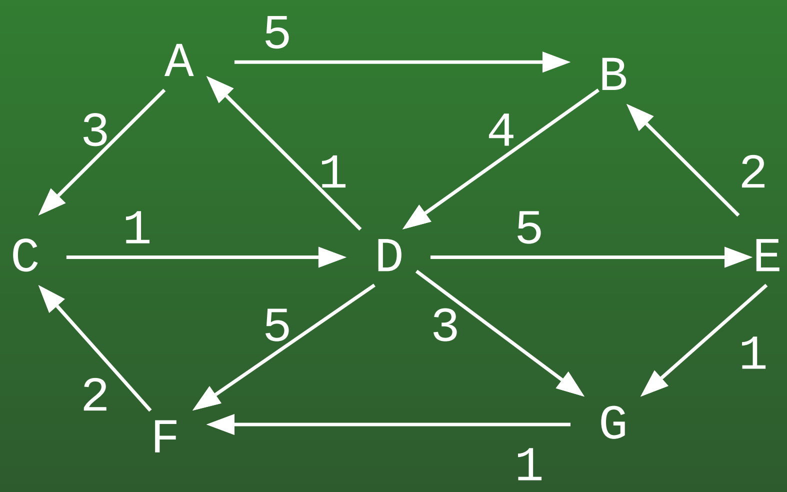
Node	Known	Dist	Path
A	true	0	
B	true	5	A
C	true	3	A
D	true	4	C
E	false	9	D
F	false	8	G
G	true	7	D

17-39: Dijkstra's Algorithm Example



Node	Known	Dist	Path
A	true	0	
B	true	5	A
C	true	3	A
D	true	4	C
E	false	9	D
F	true	8	G
G	true	7	D

Dijkstra's Algorithm Example



Node	Known	Dist	Path
A	true	0	
B	true	5	A
C	true	3	A
D	true	4	C
E	true	9	D
F	true	8	G
G	true	7	D

Dijkstra’s Algorithm

- Given the “path” field, we can construct the shortest path
  - Work backward from the end of the path
  - Follow the “path” pointers until the start node is reached
    - We can use a sentinel value in the “path” field of the initial node, so we know when to stop

17-42: Dijkstra Code

```
void Dijkstra(Edge G[], int s, tableEntry T[]) {
  int i, v;
  Edge e;
  for(i=0; i<G.length; i++) {
    T[i].distance = Integer.MAX_VALUE;
    T[i].path = -1;
    T[i].known = false;
  }
  T[s].distance = 0;
  for (i=0; i < G.length; i++) {
    v = minUnknownVertex(T);
    T[v].known = true;
    for (e = G[v]; e != null; e = e.next) {
      if (T[e.neighbor].distance >
            T[v].distance + e.cost) {
        T[e.neighbor].distance = T[v].distance + e.cost;
        T[e.neighbor].path = v;
      }
    }
  }
}
```

17-43: Prim Code

```
void Prim(Edge G[], int s, tableEntry T[]) {
  int i, v;
  Edge e;
  for(i=0; i<G.length; i++) {
    T[i].distance = Integer.MAX_VALUE;
    T[i].path = -1;
    T[i].known = false;
  }
  T[s].distance = 0;
  for (i=0; i < G.length; i++) {
    v = minUnknownVertex(T);
    T[v].known = true;
    for (e = G[v]; e != null; e = e.next) {
      if (T[e.neighbor].distance > e.cost) {
        T[e.neighbor].distance = e.cost;
        T[e.neighbor].path = v;
      }
    }
  }
}
```

Dijkstra Running Time

- If minUnknownVertex(T) is calculated by doing a linear search through the table:
  - Each minUnknownVertex call takes time $\Theta(|V|)$
    - Called $|V|$ times – total time for all calls to minUnkownVertex: $\Theta(|V|^2)$
  - If statement is executed $|E|$ times, each time takes time $O(1)$
  - Total time: $O(|V|^2 + |E|) = O(|V|^2)$.

17-45: Dijkstra Running Time

- If minUnknownVertex(T) is calculated by inserting all vertices into a min-heap (using distances as key) updating the heap as the distances are changed
  - Each minUnknownVertex call tatkes time $\Theta(\lg |V|)$
    - Called $|V|$ times – total time for all calls to minUnknownVertex: $\Theta(|V| \lg |V|)$
  - If statement is executed $|E|$ times – each time takes time $O(\lg |V|)$, since we need to update (decrement) keys in heap
  - Total time: $O(|V| \lg |V| + |E| \lg |V|) \in O(|E| \lg |V|)$

17-46: Dijkstra Running Time

- If minUnknownVertex(T) is calculated by inserting all vertices into a Fibonacci heap (using distances as key) updating the heap as the distances are changed
  - Each minUnknownVertex call takes amortized time $\Theta(\lg |V|)$
    - Called $|V|$ times – total amortized time for all calls to minUnknownVertex: $\Theta(|V| \lg |V|)$
  - If statement is executed $|E|$ times – each time takes amortized time $O(1)$, since decrementing keys takes time $O(1)$.
  - Total time: $O(|V| \lg |V| + |E|)$

17-47: Negative Edges

- Does Dijkstra's algorithm work when edge costs can be negative?
  - Give a counterexample!
- What happens if there is a negative-weight cycle in the graph?

Bellman-Ford

- Bellman-Ford allows us to calculate shortest paths in graphs with negative edge weights, as long as there are no negative-weight cycles
- As a bonus, we will also be able to detect negative-weight cycles

17-49: Bellman-Ford

- For each node v, maintiain:
  - A “distance estimate” from source to v, $d[v]$
  - Parent of v, $\pi[v]$, that gives this distance estimate
- Start with $d[v] = \infty$, $\pi[v]$ = nil for all nodes
- Set $d[\text{source}] = 0$
- udpate estimates by “relaxing” edges

17-50: Bellman-Ford

- Relaxing an edge (u, v)
  - See if we can get a better distance estimate for v by going thorugh u

Relax(u,v,w)
    if $d[v] > d[u] + w(u, v)$
        $d[v] \leftarrow d[u] + w(u, v)$
        $\pi[v] \leftarrow u$

17-51: Bellman-Ford

- Relax all edges edges in the graph (in any order)
- Repeat until relax steps cause no change
  - After first relaxing, all optimal paths from source of length 1 are computed
  - After second relaxing, all optimal paths from source of length 2 are computed
  - after $|V| - 1$ relaxing, all optimal paths of length $|V| - 1$ are computed
  - If some path of length $|V|$ is cheaper than a path of length $|V| - 1$ that means ...

17-52: Bellman-Ford

- Relax all edges edges in the graph (in any order)
- Repeat until relax steps cause no change
  - After first relaxing, all optimal paths from source of length 1 are computed
  - After second relaxing, all optimal paths from source of length 2 are computed
  - after $|V| - 1$ relaxing, all optimal paths of length $|V| - 1$ are computed
  - If some path of length $|V|$ is cheaper than a path of length $|V| - 1$ that means ...
    - Negative weight cycle

Bellman-Ford

```
BellamanFord(G, s)
    Initialize d[], π[]
    for i ← 1 to |V| − 1 do
        for each edge (u, v) ∈ G do
            if d[v] > d[u] + w(u, v)
                d[v] ← d[u] + w(u, v)
                π[v] ← u
    for each edge (u, v) ∈ G do
        if d[v] > d[u] + w(u, v)
            return false
    return true
```

17-54: Bellman-Ford

- Running time:
  - Each iteration requires us to relax all $|E|$ edges
  - Each single relaxation takes time $O(1)$
  - $|V| - 1$ iterations ($|V|$ if we are checking for negative weight cycles)
  - Total running time $O(|V| * |E|)$

17-55: Shortest Path/DAGs

- Finding Single Source Shorest path in a Directed, Acyclic graph
- Very easy! How can we do this quickly?

17-56: Shortest Path/DAGs

- Finding Single Source Shorest path in a Directed, Acyclic graph
- Very easy!
- How can we do this quickly?
  - Do a topological sort
  - Relax edges in topological order
  - We're done!

17-57: All-Source Shortest Path

- What if we want to find the shortest path from all vertices to all other vertices?
- How can we do it?

17-58: All-Source Shortest Path

- What if we want to find the shortest path from all vertices to all other vertices?
- How can we do it?
  - Run Dijktra's Algorithm V times
  - How long will this take?

All-Source Shortest Path

- What if we want to find the shortest path from all vertices to all other vertices?
- How can we do it?
  - Run Dijktra's Algorithm V times
  - How long will this take?
  - $\Theta(V^2 \lg V + VE)$ (using Fibonacci heaps)
    - Doesn't work if there are negative edges! Running Bellman-Ford V times (which does work with negative edges) takes time $O(V^2 E)$ – which is $\Theta(V^4)$ for dense graphs

Multi-Source Shortest Path

- Let $L^{(m)}[i,j]$ (in text, $l_{i,j}^{(m)}$) be cost of the shortest path from i to j that contains at most m edges
- If $m = 0$, there is a shortest path from i to j with no edges iff $i = j$

$$L^{(0)}[i,j] = \begin{cases} 0 & \text{if } i = j \\ \infty & \text{otherwise} \end{cases}$$

- How can we calculate $L^m[i,j]$ recursively?

17-61: Multi-Source Shortest Path

- Let $L^{(m)}[i,j]$ (in text, $l_{i,j}^{(m)}$) be cost of the shortest path from i to j that contains at most m edges

$$L^{(0)}[i,j] = \begin{cases} 0 & \text{if } i = j \\ \infty & \text{otherwise} \end{cases}$$

- How can we calculate $L^m[i,j]$ recursively?

$$\begin{aligned} L^{(m)}[i,j] &= \min\left(L^{(m-1)}[i,j], \min_{1 \le k \le n}(L^{(m-1)}[i,k] + w_{kj})\right) \\ &= \min_{1 \le k \le n}(L^{(m-1)}[i,k] + w_{kj}) \end{aligned}$$

Multi-Source Shortest Path

- Create $L^{(m+1)}$ from $L^{(m)}$:

Extend-Shortest-Paths(L, W)

$n \leftarrow$ rows[L]
$L' \leftarrow$ new $n \times n$ matrix
for $i \leftarrow 1$ to n do
    for $j \leftarrow 1$ to n do
        $L'[i,j] \leftarrow \infty$
        for $k \leftarrow 1$ to n do
            $L'[i,j] \leftarrow \min(L'[i,j], L[i,k] + W[k,j])$
return L'

Multi-Source Shortest Path

- Need to calculate $L^{(n-1)}$
  - Why $L^{(n-1)}$, and not $L^{(n)}$ or $L^{(n+1)}$?

All-Pairs-Shortest-Paths(W)
  $n \leftarrow$ rows[W]
  $L^{(1)} \leftarrow W$
  for $m \leftarrow 2$ to $n - 1$ do
    $L^{(m)} \leftarrow$Extend-Shortest-Path($L^{(m-1)}, W$)
  return $L^{(n-1)}$

Multi-Source Shortest Path

- We really don't care about any of the L matrices except $L^{(n-1)}$
- We can save some time by not calculating all of the intermediate matrices $L^{(1)} \ldots L^{(n-2)}$
- Note that Extend-Shortest-Path looks a lot like matrix multiplication

Multi-Source Shortest Path

Square-Matrix-Multiply(A, B)

   $n \leftarrow$ rows$[A]$

   $C \leftarrow$ new $n \times n$ matrix

   for $i \leftarrow 1$ to n do

      for $j \leftarrow 1$ to n do

         $C[i, j] \leftarrow 0$

         for $k \leftarrow 1$ to n do

            $C[i, j] \leftarrow C[i, j] + A[i, k] * B[k, j])$

   return L'

- Replace min with +, + with $*$

17-66: Multi-Source Shortest Path

- Using our "Extend-Multiplication"
  - Replace + with min, $*$ with +

$$
\begin{aligned}
L^{(1)} = L^{(0)} * W &= W \\
L^{(1)} = L^{(1)} * W &= W^2 \\
L^{(2)} = L^{(2)} * W &= W^3 \\
L^{(3)} = L^{(3)} * W &= W^4 \\
&\vdots \\
L^{(n-1)} = L^{(n-2)} * W &= W^{n-1}
\end{aligned}
$$

17-67: Multi-Source Shortest Path

$$
\begin{aligned}
L^{(1)} &= W \\
L^{(2)} &= W^2 &= \quad & W * W \\
L^{(4)} &= W^4 &= \quad & W^2 * W^2 \\
L^{(8)} &= W^8 &= \quad & W^4 * W^4 \\
&\vdots \\
L^{2^{\lceil \lg(n-1) \rceil}} = L^{2^{\lceil \lg(n-1) \rceil}} &&= \quad & L^{2^{\lceil \lg(n-1) \rceil - 1}} * L^{2^{\lceil \lg(n-1) \rceil - 1}}
\end{aligned}
$$

- Since $L^{(n-1)} = L^{(n)} = L^{(n+1)} = \ldots$, it doesn't matter if n is an exact power of 2 – we just need to get to at least $L^{(n-1)}$, not hit it exactly

Multi-Source Shortest Path

All-Pairs-Shortest-Paths(W)

$n \leftarrow$ rows[W]

$L^{(1)} \leftarrow W$

$m \leftarrow 1$

while $m < n - 1$ do

$L^{(2m)} \leftarrow$Extend-Shortest-Path($L^{(m)}, L^{(m)}$)

$m \rightarrow m * 2$

return $L^{(m)}$

17-69: Multi-Source Shortest Path

- Each call to Extend-Shortest-Path takes time:
- # of calls to Extend-Shortest-Path:
- Total time:

17-70: Multi-Source Shortest Path

- Each call to Extend-Shortest-Path takes time $\Theta(|V|^3)$
- # of calls to Extend-Shortest-Path: $\Theta(\lg |V|)$
- Total time: $\Theta(|V|^3 \lg |V|)$

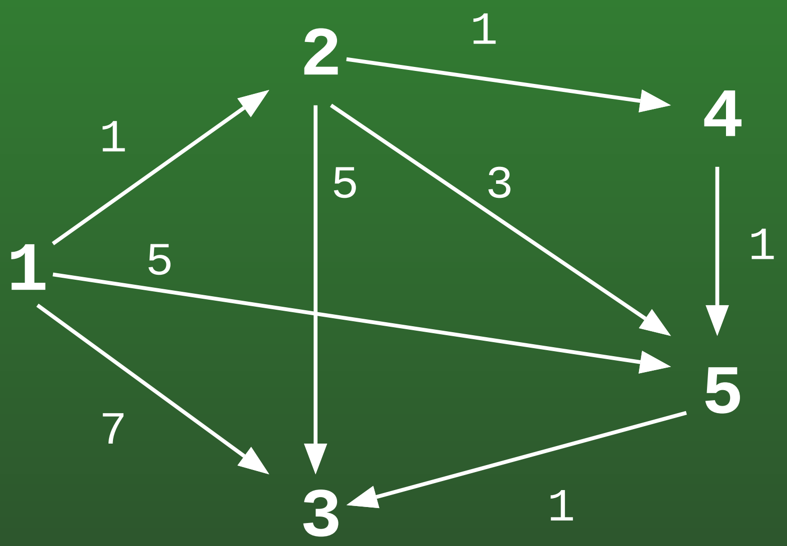
17-71: Floyd's Algorithm

- Alternate solution to all pairs shortest path
- Yields $\Theta(V^3)$ running time for all graphs

Floyd's Algorithm

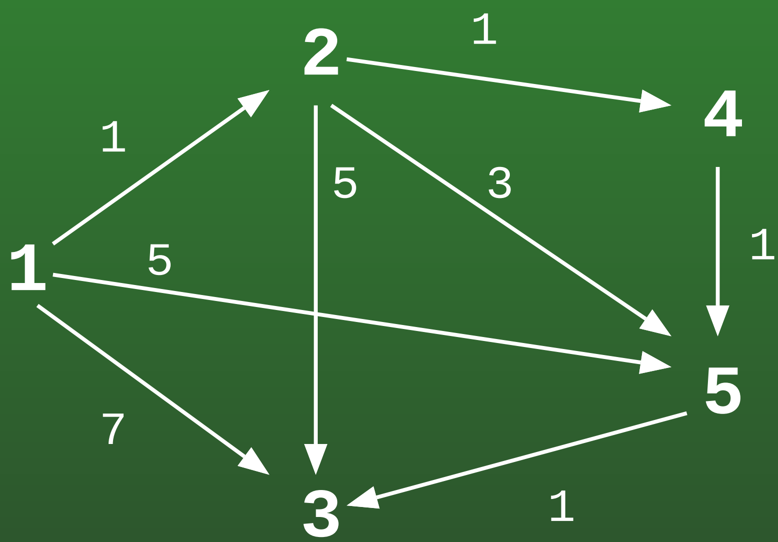
- Vertices numbered from 1..n
- k-path from vertex v to vertex u is a path whose intermediate vertices (other than v and u) contain only vertices numbered k or less
- 0-path is a direct link

17-73: k-path Examples



- Shortest 0-path from 1 to 5: 5
- Shortest 1-path from 1 to 5: 5
- Shortest 2-path from 1 to 5: 4
- Shortest 3-path from 1 to 5: 4
- Shortest 4-path from 1 to 5: 3

k-path Examples



- Shortest 0-path from 1 to 3: 7
- Shortest 1-path from 1 to 3: 7
- Shortest 2-path from 1 to 3: 6
- Shortest 3-path from 1 to 3: 6
- Shortest 4-path from 1 to 3: 6
- Shortest 5-path from 1 to 3: 4

Floyd's Algorithm

- Shortest n-path = Shortest path
- Shortest 0-path:
  - ∞ if there is no direct link
  - Cost of the direct link, otherwise

Floyd’s Algorithm

- Shortest n-path = Shortest path
- Shortest 0-path:
  - ∞ if there is no direct link
  - Cost of the direct link, otherwise
- If we could use the shortest k-path to find the shortest $(k+1)$ path, we would be set

Floyd's Algorithm

- Shortest k-path from v to u either goes through vertex k, or it does not
- If not:
  - Shortest k-path = shortest $(k-1)$-path
- If so:
  - Shortest k-path = shortest $k-1$ path from v to k, followed by the shortest $k-1$ path from k to w

Floyd's Algorithm

- If we had the shortest k-path for all pairs (v,w), we could obtain the shortest $k+1$-path for all pairs
  - For each pair v, w, compare:
    - length of the k-path from v to w
    - length of the k-path from v to k appended to the k-path from k to w
  - Set the $k+1$ path from v to w to be the minimum of the two paths above

17-79: Floyd's Algorithm

- Let $D_k[v, w]$ be the length of the shortest k-path from v to w.
- $D_0[v, w]$ = cost of arc from v to w (∞ if no direct link)
- $D_k[v, w]$ = MIN($D_{k-1}[v, w]$, $D_{k-1}[v, k] + D_{k-1}[k, w]$)
- Create D_0, use D_0 to create D_1, use D_1 to create D_2, and so on – until we have D_n

17-80: Floyd's Algorithm

- Use a doubly-nested loop to create D_k from D_{k-1}
  - Use the same array to store D_{k-1} and D_k – just overwrite with the new values
- Embed this loop in a loop from 1..k

17-81: Floyd's Algorithm

```
Floyd(Edge G[], int D[][]) {
  int i,j,k

  Initialize D, D[i][j] = cost from i to j

  for (k=0; k<G.length; k++;
    for(i=0; i<G.length; i++)
      for(j=0; j<G.length; j++)
        if ((D[i][k] != Integer.MAX_VALUE) &&
            (D[k][j] != Integer.MAX_VALUE) &&
            (D[i][j] > (D[i,k] + D[k,j])))
          D[i][j] = D[i][k] + D[k][j]
}
```

Floyd's Algorithm

- We've only calculated the *distance* of the shortest path, not the path itself
- We can use a similar strategy to the PATH field for Dijkstra to store the path
  - We will need a 2-D array to store the paths: P[i][j] = last vertex on shortest path from i to j

Johnson's Algorithm

- Yet another all-pairs shortest path algorithm
- Time $O(|V|^2 \lg |V| + |V| * |E|)$
  - If graph is dense ($|E| \in \Theta(|V|^2)$) , no better than Floyd
  - If graph is sparse, better than Floyd
- Basic Idea: Run Dijkstra $|V|$ times
  - Need to modify graph to remove negative edges

17-84: Johnson's Algorithm

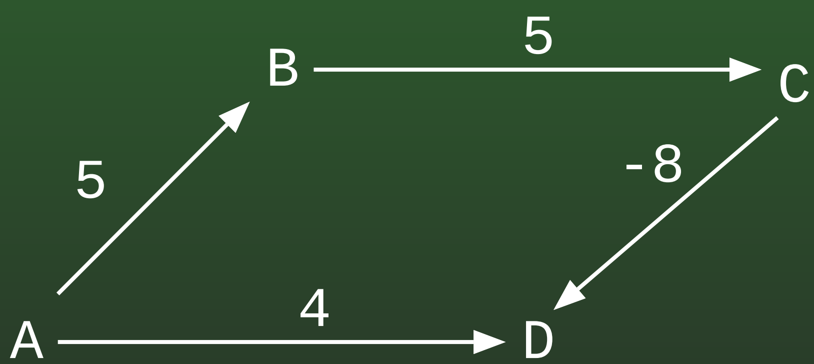
- Reweighing Graph
  - Create a new weight function $\hat{w}$, such that:
    - For all pairs of vertices $u, v \in V$, a path from u to v is a shortest path using w if and only if it is also a shortest path using $\hat{w}$.
    - For all edges (u, v), $\hat{w}(u, v)$ is non-negative

17-85: Johnson's Algorithm

- Reweighing Graph
  - First Try:
  - Smallest weight is $-w$, for some positive w
  - Add w to each edge in the graph
  - Is this a valid reweighing?

17-86: Johnson's Algorithm

- Reweighing Graph
  - First Try:
  - Smallest weight is $-w$, for some positive w
  - Add w to each edge in the graph
  - Is this a valid reweighing?



17-87: Johnson's Algorithm

- Reweighing Graph
  - Second Try:
  - Define some function on vertices $h(v)$
  - $\hat{w}(u, v) = w(u, v) + h(u) - h(v)$
  - Does this preserve shortest paths?

Johnson's Algorithm

- Let $p = v_0, v_1, v_2, \ldots, v_k$ be a path in G
- Cost of p under $\hat{w}$:

$$
\begin{aligned}
\hat{w}(p) &= \sum_{i=1}^{k} \hat{w}(v_{i-1}, v_i) \\
&= \sum_{i=1}^{k} (w(v_{i-1}, v_i) + h(v_{i-1}) - h(v_i)) \\
&= \left(\sum_{i=1}^{k} (w(v_{i-1}, v_i)) + h(v_0) - h(v_k) \right) \\
&= w(p) + h(v_0) - h(v_k)
\end{aligned}
$$

17-89: Johnson's Algorithm

- So, if we can come up with a function $h(V)$ such that $w(u,v) + h(u) - h(v)$ is positive for all edges (u,v) in the graph, we're set
  - Use the function h to reweigh the graph
  - Run Dijkstra's algorithm $|V|$ times, starting from each vertex on the new graph, calculating shortest paths
  - Shortest path in new graph = shortest path in old graph

Johnson's Algorithm

- Add a new vertex s to the graph
- Add an edge from s to every other vertex, with cost 0
- Find the shortest path from s to every other vertex in the graph
- $h(v) = \delta(s, v)$, the cost of the shortest path from s to v
    - Using this $h(V)$ function, all new weights are guaranteed to be non-negative

17-91: Johnson's Algorithm

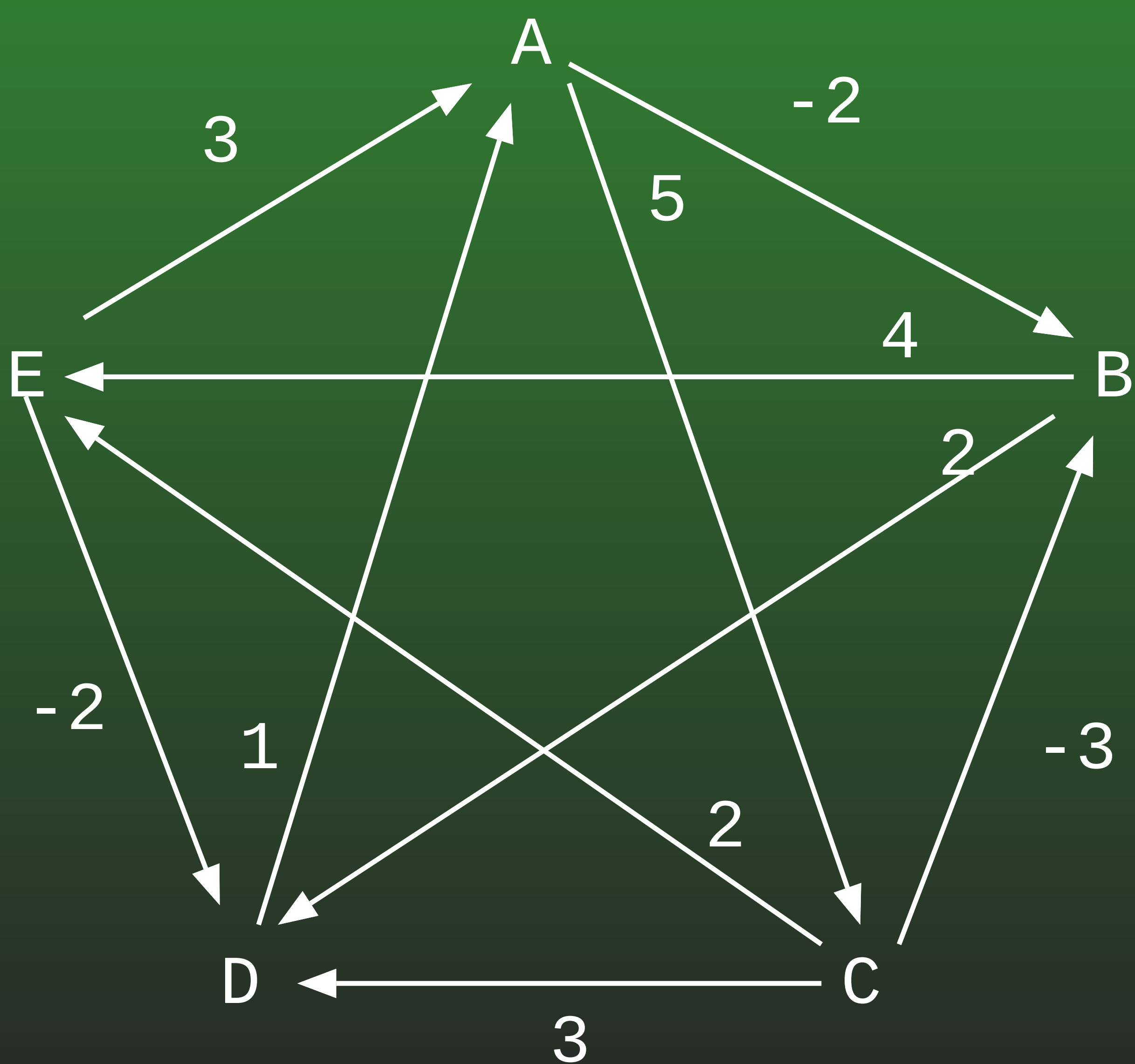
- $h(v) = \delta(s, v)$, the cost of the shortest path from s to v

$$\begin{aligned}\hat{w}(u, v) &= w(u, v) + h(u) - h(v) \\ &= w(u, v) + \delta(s, u) - \delta(s, v)\end{aligned}$$

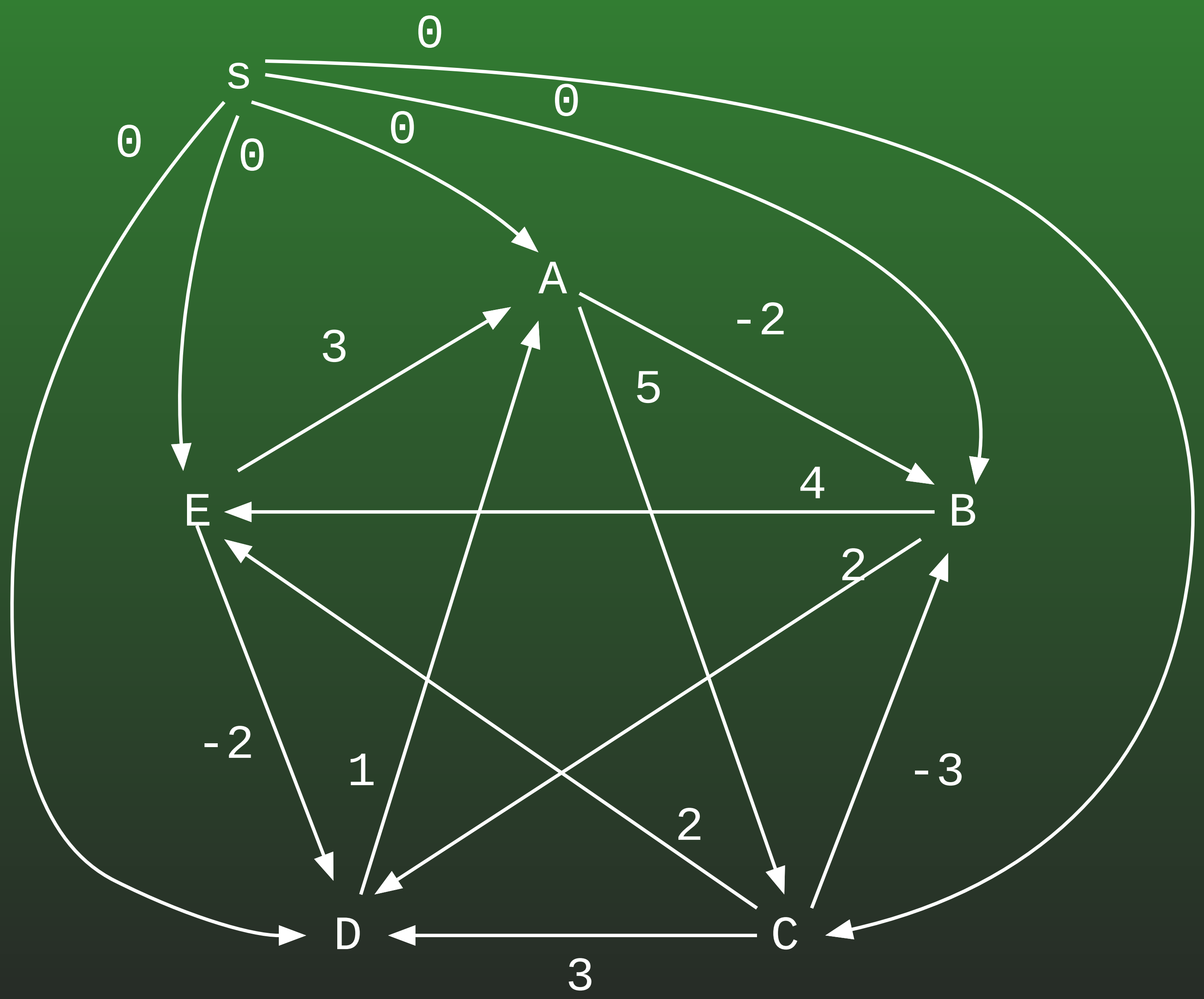
- Since δ is a shortest path,

$$\begin{aligned}\delta(s, v) &\leq \delta(s, u) + w(u, v) \\ 0 &\leq w(u, v) + \delta(s, u) - \delta(s, v)\end{aligned}$$

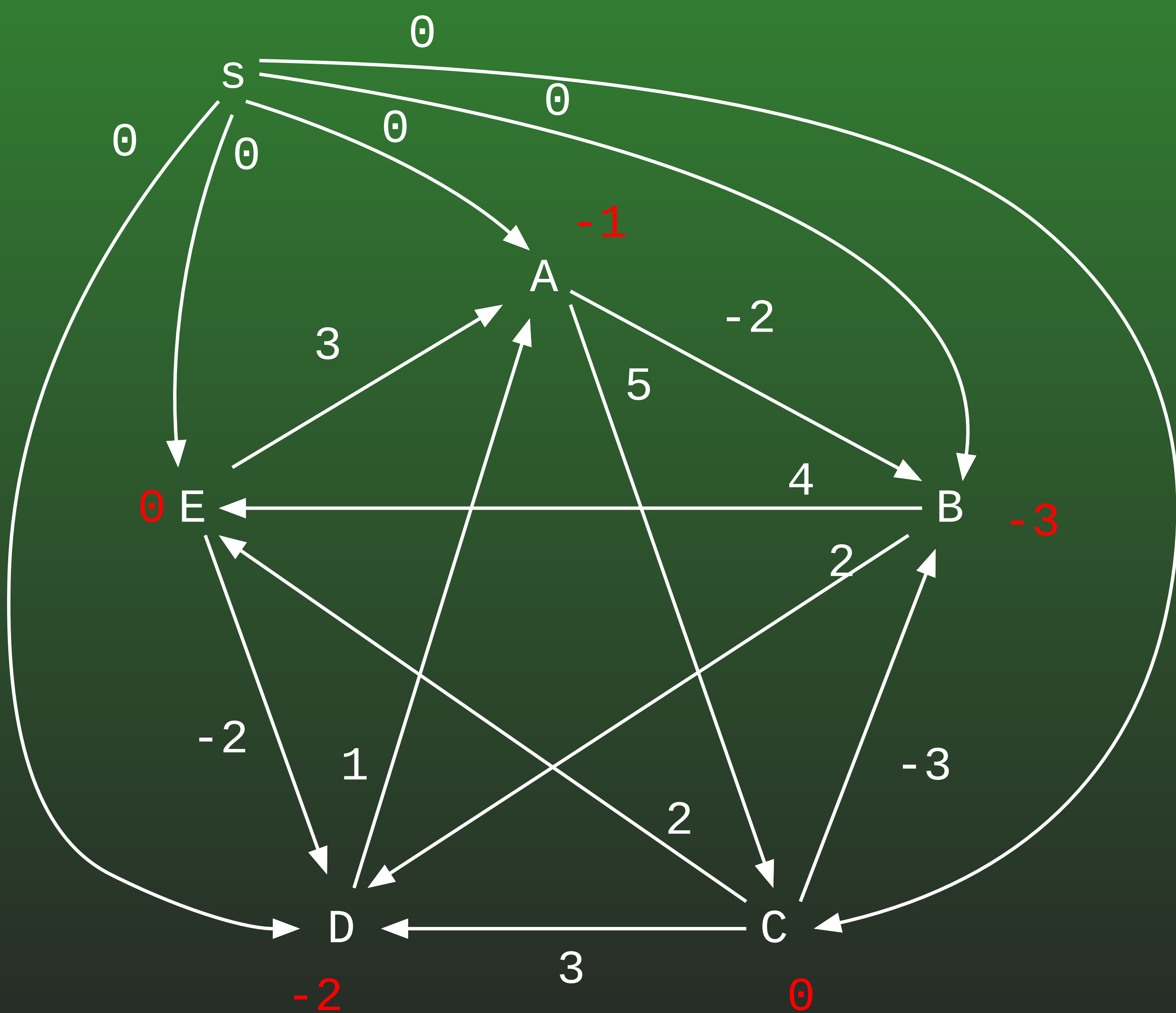
Johnson's Algorithm



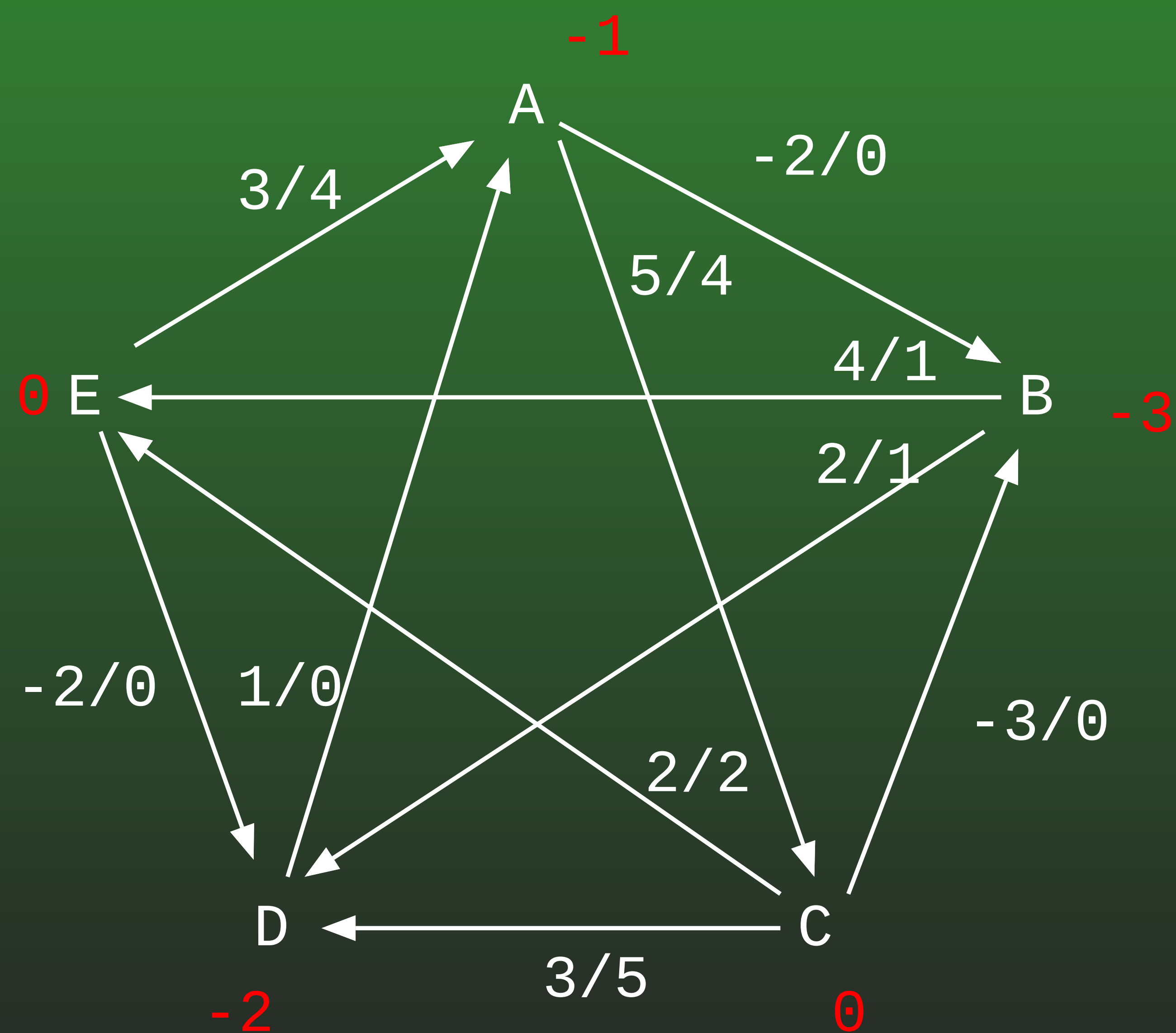
17-93: Johnson's Algorithm



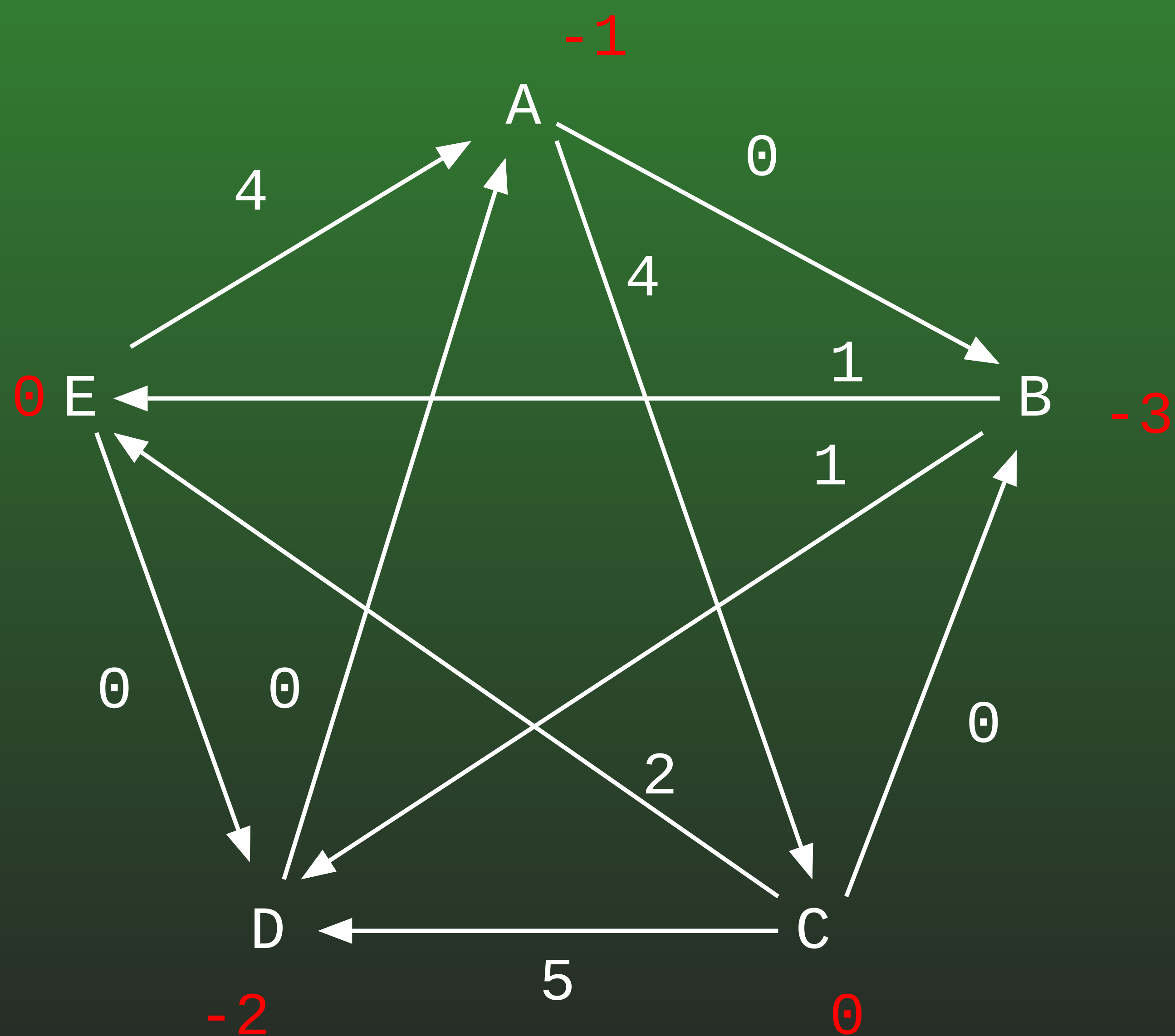
17-94: Johnson's Algorithm



Johnson's Algorithm



Johnson's Algorithm



Johnson's Algorithm

Johnson(G)
  Add s to G, with 0 weight edges to all vertices
  if Bellman-Ford(G, s) = FALSE
    There is a negative weight cycle, fail
  for each vertex $v \in G$
    set $h(v) \leftarrow \delta(s, v)$ from B-F
  for each edge $(u, v) \in G$
    $\hat{w}(u, v) = w(u, v) + h(u) - h(v)$
  for each vertex $u \in G$
    run Dijkstra($G, \hat{w}, u$) to compute $\hat{\delta}(u, v)$
    $\delta(u, v) = \hat{\delta}(u, v) + h(v) - h(u)$