Graduate Algorithms

CS673-20016F-02

Probabilistic Analysis

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02-0: Hiring Problem

- Need an office assistant
  - Employment Agency sends one candidate every day
  - Interview that person, either hire that person (and fire the old one), or keep old person
  - Always want the best person – always hire if interviewee is better than current person

02-1: Hiring Problem

```
HIRE-ASSISTANT(n)
  best <- 0
  for i <- 1 to n do
    if candidate[i] is better than candidate[best]
      best <- i
      hire candidate i
```

- Cost to interview candidate is C_i
- Cost to hire a candidate is C_h
- Assume C_i is much less than C_h
- Total cost: $O(C_i * n + C_h * m)$, where m = # of hirings

02-2: Hiring Problem

- Best case cost?
- Worst case cost?
- Average cost?

02-3: Hiring Problem

- Best case cost? $C_i * n + C_h$
- Worst case cost? $C_i * n + C_h * n$
- Average cost?
  - Assume applicants come in random order
  - Each permutation of applicants is equally likely

02-4: Probability Review

- Indicator variable associated with event A:

$$I\{A\} = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A \text{ does not occur} \end{cases}$$

- Example: Flip a coin: Y is a random variable representing the coin flip

$$X_H = I\{Y = H\} = \begin{cases} 1 & \text{if } Y = H \\ 0 & \text{otherwise} \end{cases}$$

02-5: Probability Review

- Expected value $E[]$ of a random variable
  - Value you "expect" a random variable to have
  - Average (mean) value of the variable over many trials
  - Does not have to equal the value of any particular trial
    - Bus example(s)

02-6: Probability Review

- Expected value $E[]$ of a random variable

$$E[X] = \sum_{\text{all values } x \text{ of } X} x * Pr\{X = x\}$$

- When we want the "average case" running time of an algorithm, we want the Expected Value of the running time

Probability Review

$X_H = I\{Y = H\}$

$$\begin{aligned} E[X_H] &= E[I\{Y = H\}] \\ &= 1 * Pr\{Y = H\} + 0 * Pr\{Y = T\} \\ &= 1 * 1/2 + 0 * 1/2 \\ &= 1/2 \end{aligned}$$

02-8: Probability Review

- Expected # of heads in n coin flips
  - X = # of heads in n flips
  - X_i = indicator variable: coin flip i is heads

02-9: Probability Review

- Expected # of heads in n coin flips
  - X = # of heads in n flips
  - X_i = indicator variable: coin flip i is heads

$$
\begin{aligned}
E[X] &= E\left[\sum_{i=1}^{n} X_i\right] \\
&= \sum_{i=1}^{n} E[X_i] \\
&= \sum_{i=1}^{n} \frac{1}{2} = \frac{n}{2}
\end{aligned}
$$

02-10: Probability Review

- For any event A, indicator variable $X_A = I\{A\}$
  $E[X_A] = Pr\{A\}$

$$
\begin{aligned}
E[X_A] &= 1 * Pr\{A\} + 0 * Pr\{\neg A\} \\
&= Pr\{A\}
\end{aligned}
$$

02-11: Hiring Problem

- Calculate the expected number of hirings
  - X = # of candidates hired
  - $X_i = I\{\text{Candidate } i \text{ is hired}\}$
  - $X = X_1 + X_2 + \ldots + X_n$

$$E[X] =$$

02-12: Hiring Problem

- Calculate the expected number of hirings
  - X = # of candidates hired
  - $X_i = I\{\text{Candidate } i \text{ is hired}\}$
  - $X = X_1 + X_2 + \ldots + X_n$

$$
\begin{aligned}
E[X] &= E\left[\sum_{i=1}^{n} X_i\right] \\
&= \sum_{i=1}^{n} E[x_i]
\end{aligned}
$$

- What is $E[X_i]$?

02-13: Hiring Problem

- What is $E[X_i]$?
  - $E[X_i]$ = Probability that the ith candidate is hired
  - When is the ith candidate hired?

02-14: Hiring Problem

- What is $E[X_i]$?
  - $E[X_i]$ = Probability that the ith candidate is hired
  - ith candidate hired when s/he is better than the $i - 1$ candidates that came before
  - Assuming that all permutations of candidates are equally likely, what is the probability that the ith candidate is the best of the first i candidates?

02-15: Hiring Problem

- What is $E[X_i]$?
  - $E[X_i]$ = Probability that the ith candidate is hired
  - ith candidate hired when s/he is better than the $i-1$ candidates that came before
  - Assuming that all permutations of candidates are equally likely, what is the probability that the ith candidate is the best of the first i candidates?
    - $\frac{1}{i}$

Hiring Problem

Probability that the ith candidate is best of first i is $\frac{1}{i}$

- Sanity Check: (Doing a few concrete examples as a sanity check is often a good idea)
  - $i = 1$, probability that the first candidate is the best so far = 1/1 = 1
  - $i = 2$: (1,2), (2,1) In one of the two permutations, 2nd candidate is the best so far
  - $i = 3$: (1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1) In two of the 6 permutations, the 3rd candidate is the best so far
- Note that a few concrete examples do not *prove* anything, but a counter-example can show that you have made a mistake

02-17: Hiring Problem

- Now that we know that $E[X_i] = \frac{1}{i}$, we can find the expected number of hires:

$$
\begin{aligned}
E[X] &= E\left[\sum_{i=1}^{n} X_i\right] \\
&= \sum_{i=1}^{n} E[x_i] \\
&= \sum_{i=1}^{n} 1/i \\
&= \ln n + O(1)
\end{aligned}
$$

02-18: Randomized Algorithms

- In average-case analysis, we often assume that all inputs are equally likely
- In actuality, some inputs might be much more likely
  - If we're really unlucky, the most likely inputs can be the most costly (as in some implementations of quicksort)
- What can we do?

02-19: Randomized Algorithms

- In average-case analysis, we often assume that all inputs are equally likely
- In actuality, some inputs might be much more likely
  - If we're really unlucky, the most likely inputs can be the most costly (as in some implementations of quicksort)
- What can we do?
  - Force all inputs to be equally likely, by randomizing the input

02-20: Randomized Algorithms

- In the hire-assistant problem, we can first randomly permute the lists of candidates, and then run the algorithm
- Then, for any input, we'd be guaranteed that the expected number of hires would be $\ln n + O(1)$
- How can we randomly permute a list, so that every permutation is equally as likely?
  - That is, how can we shuffle a list, so that every permutation is equally likely? Assume that we have a good random number generator.

02-21: Randomized Algorithms

- To create a random permutation (method 1):
  - Assign each element in the list a random priority
  - Sort based on the priority

```
n <- length(A)
for i <-1 to n do
     Priority[i] = Random(1,n*n*n)
sort A (using Priority as keys)
```

- Why n^3?
- Time?

02-22: Randomized Algorithms

- To create a random permutation (method 2):

```
n <- length(A)
for i <- 1 to n do
   swap(A[i], A[Random(i,n)])
```

02-23: On-line Hiring Problem

- Interview candidates one at a time
- After each person is interviewed:
  - Tell them at once they are not wanted
  - Hire them (and stop the interview process)
- How can we maximize the probability that we get the best person (assume that they come in random order – we can always randomize the input to insure this)

02-24: On-line Hiring Problem

Algorithm:

- Interview first k candidates, reject them all
- Continue to interview candidates, hiring the first one that is better than anyone seen so far

Problems? Can we do better?

On-line Hiring Problem

- Interview first k candidates, reject them all
- Continue to interview candidates, hiring the first one that is better than anyone seen so far

Analysis:

- The bigger k is, the larger the chance that we see the best person in the first k (and don't hire the best person).
- The smaller k is, the larger the chance that we stop too soon.
- How should we pick k?

02-26: On-line Hiring Problem

- S = we pick the best applicant
- S_i = the best applicant is i, and we pick i.

$$Pr\{S\} = \sum_{i=k+1}^{n} Pr\{S_i\}$$

- Why $k+1$ instead of 1?
- When is the best person picked?

On-line Hiring Problem

$$Pr\{s\} = \sum_{i=k+1}^{n} Pr\{S_i\}$$

- Why $k + 1$ instead of 1?
  - $Pr\{S_I\} = 0$ if $i < k$, since we never pick the first k people
- When is the best person picked?
  - If the best person is interviewed, s/he will be picked. The best person is interviewed when candidates $k + 1..best - 1$ are all worse than the best in $1..k$

02-28: On-line Hiring Problem

- B_i == ith candidate is the best
- O_i == none of applicants in $k + 1..i - 1$ are picked

S_i (in terms of B_i and O_i) = ?

On-line Hiring Problem

- B_i = ith candidate is the best
- O_i = none of applicants in $k+1..i-1$ are picked

$S_i = B_i \wedge O_i$

$$
\begin{aligned}
Pr\{S_i\} &= Pr\{B_i \wedge O_i\} \\
&= Pr\{B_i\} * Pr\{O_i | B_i\} \\
&= Pr\{B_i\} * Pr\{O_i\} \\
&= (1/n) * k/(i-1)
\end{aligned}
$$

02-30: On-line Hiring Problem

$$\begin{aligned}
Pr\{S_i\} &= (1/n) * k/(i-1) \\
Pr\{S\} &= \sum_{i=k+1}^{n} \frac{k}{n(i-1)} \\
&= \frac{k}{n} \sum_{i=k+1}^{n} \frac{1}{i-1} \\
&= \frac{k}{n} \sum_{i=k}^{n-1} \frac{1}{i}
\end{aligned}$$

How do we find a value of a variable to maximize a function?

02-31: On-line Hiring Problem

Hard to take a derivative of a summation. However:

$$\int_{m}^{n+1} f(x)dx \leq \sum_{i=m}^{n} f(i) \leq \int_{m-1}^{n} f(x)dx$$

(if $f(x)$ is monotonically decreasing)
Looking at just the lower bound:

$$\frac{k}{n}\int_{k}^{n} \frac{1}{x}dx \;\;\leq\;\; \frac{k}{n}\sum_{i=k}^{n-1}\frac{1}{i}$$

$$\frac{k}{n}(\ln n - \ln k) \;\;\leq\;\; Pr\{S\}$$

02-32: On-line Hiring Problem

Maximizing the lower bound:

- To maximize $k/n(\ln n - \ln k)$: Take first derivative with respect to k, set to 0.
- (recall the product rule for derivatives: $D[f(k)g(k)] = D[f(k)]g(k) + f(k)D[g(k)]$)

02-33: On-line Hiring Problem

- To maximize $k/n(\ln n - \ln k)$: Take first derivative with respect to k, set to 0.
- (recall the product rule for derivatives: $D[f(k)g(k)] = D[f(k)]g(k) + f(k)D[g(x)]$)

$$\begin{aligned}
\frac{1}{n}(\ln n - \ln k - 1) &= 0 \\
\ln k &= \ln n - 1 \\
\ln k &= \ln n - \ln e \\
\ln k &= \ln \frac{n}{e} \\
k &= \frac{n}{e}
\end{aligned}$$

02-34: On-line Hiring Problem

- Interview just under 1/3 of the applicants (hiring none of them)
- Hire the first person better than anyone seen so far
- Probability of getting the best person $\geq$ $(n/e)/n(\ln n - \ln(n/e)) = 1/e(\ln e) = 1/e \approx 0.37$