Graduate Algorithms

CS673-2016F-03

Heaps

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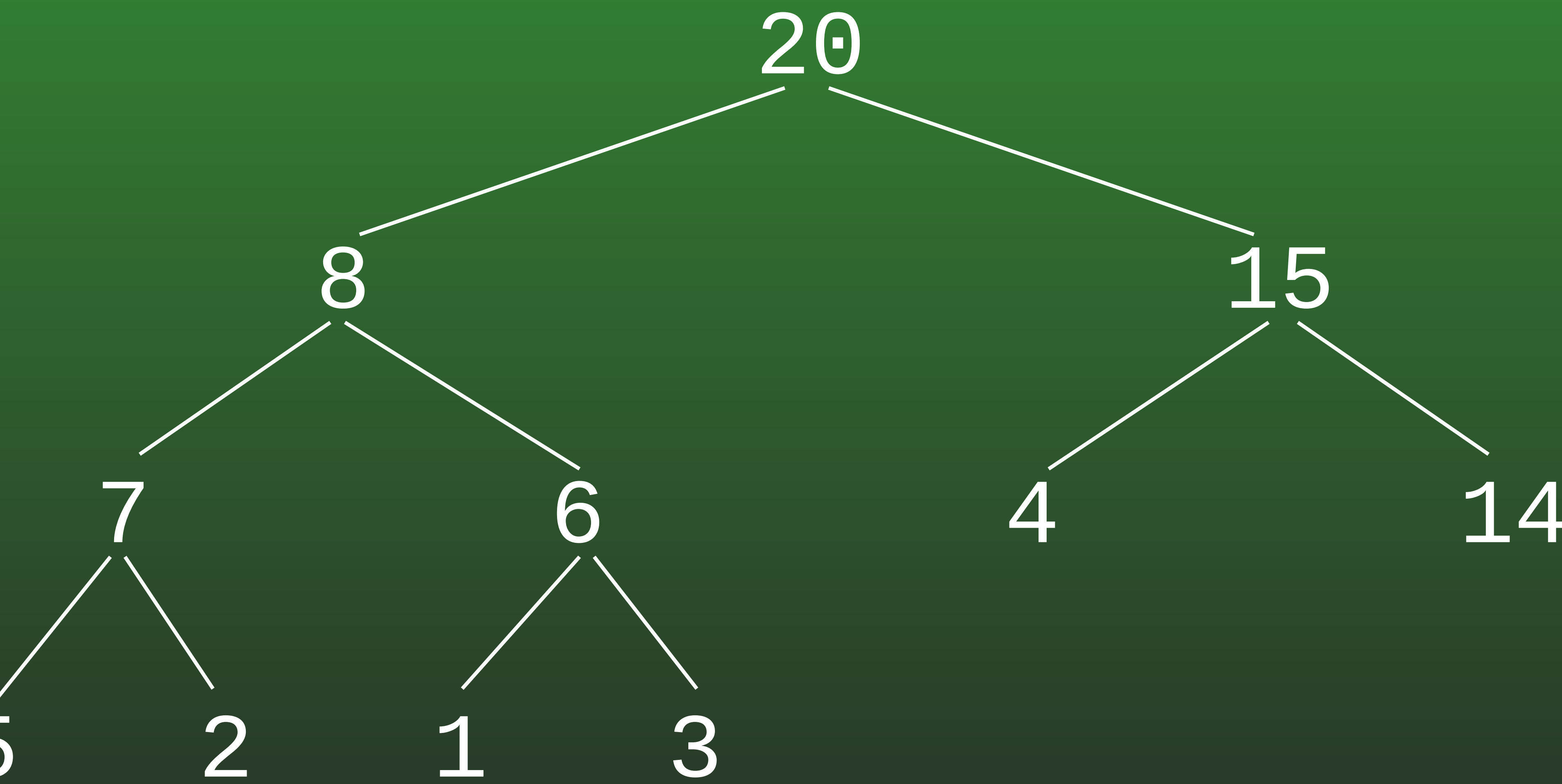
03-0: Heap Definition

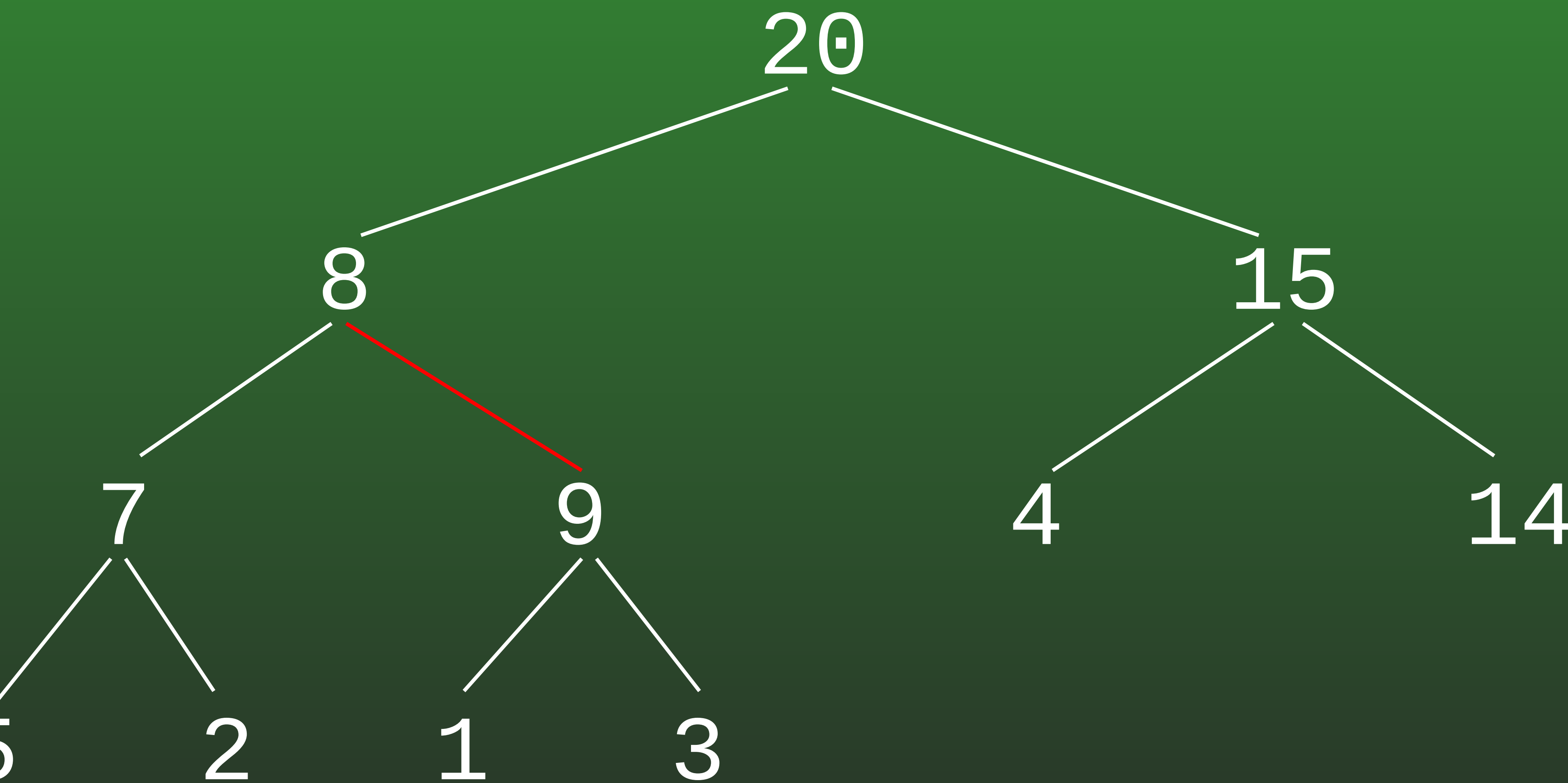
- Complete Binary Tree
- Heap Property
  - Max Heap:
    - For every subtree in a tree, each value in the subtree is <= value stored at the root of the subtree
  - Min Heap:
    - For every subtree in a tree, each value in the subtree is >= value stored at the root of the subtree

03-1: Heap Examples



Valid Heap

03-2: Heap Examples



Invalid Heap

03-3: Heap Insert

- There is only one place we can insert an element into a heap, so that the heap remains a complete binary tree
- Inserting an element at the “end” of the heap might break the heap property

03-4: Heap Insert

- There is only one place we can insert an element into a heap, so that the heap remains a complete binary tree
- Inserting an element at the "end" of the heap might break the heap property
  - Swap the inserted value up the tree

03-5: Heap Remove Largest

- Removing the Root of the heap is hard
- Removing the element at the “end” of the heap is easy

03-6: Heap Remove Largest

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  - Move last element into root
    - May break the heap property

03-7: Heap Remove Largest

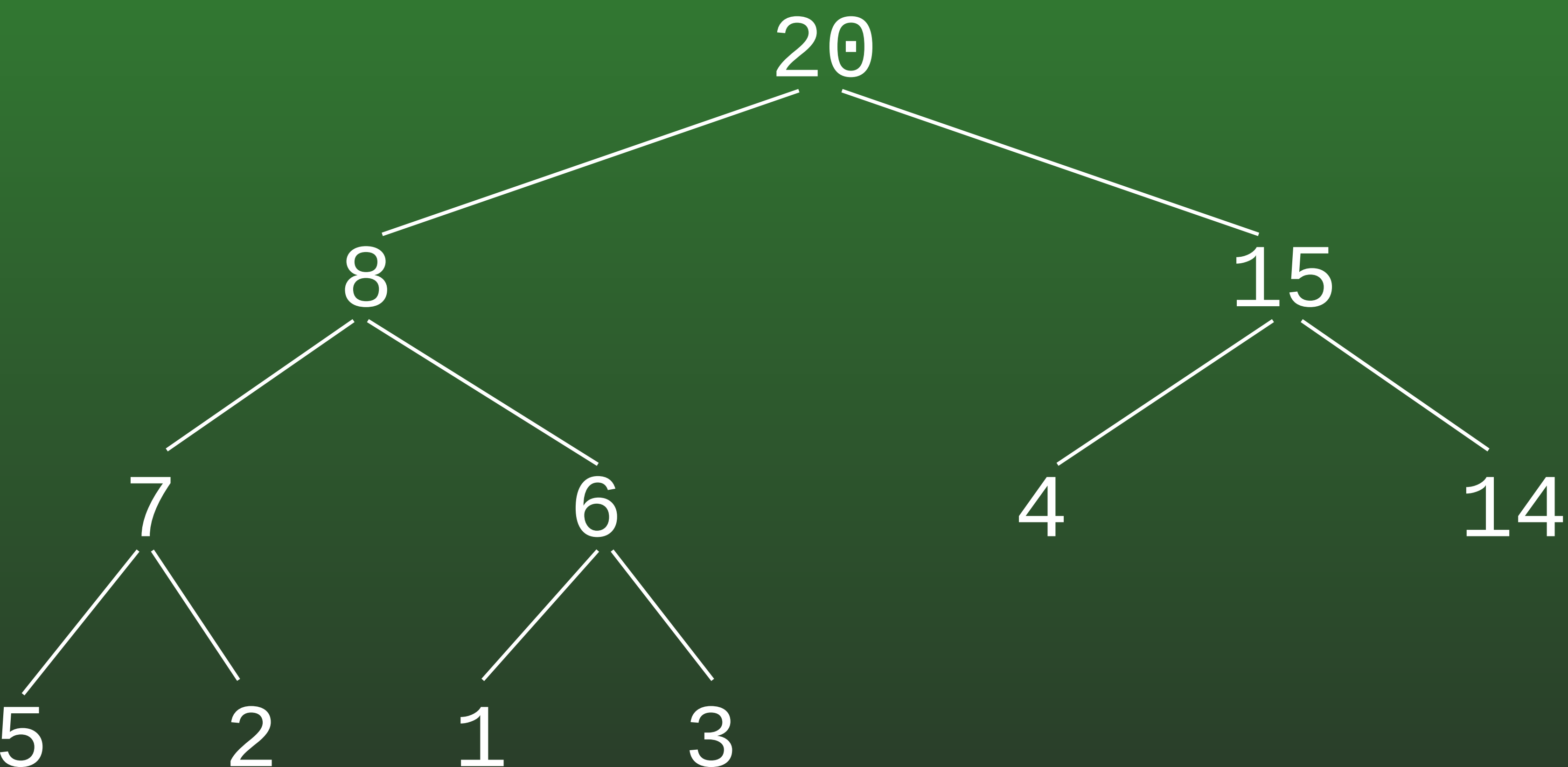
- Removing the Root of the heap is hard
- Removing the element at the “end” of the heap is easy
  - Move last element into root
    - Shift the root down, until heap property is satisfied

03-8: Representing Heaps

- Represent heaps using pointers
  - Need to add parent pointers for insert to work correctly
  - Space needed to store pointers – 3 per node – could be greater than the space need to store the data in the heap!
  - Memory allocation and deallocation is slow
- There is a better way!

03-9: Representing Heaps

A Complete Binary Tree can be stored in an array:



20	8	15	7	6	4	14	5	2	1	3			
0	1	2	3	4	5	6	7	8	9	10	11	12	13

03-10: CBTs as Arrays

- The root is stored at index 0
- For the node stored at index i:
  - Left child is stored at index $2 * i + 1$
  - Right child is stored at index $2 * i + 2$
  - Parent is stored at index $\lfloor (i - 1)/2 \rfloor$

03-11: CBTs as Arrays

Finding the parent of a node

```
int parent(int n) {
  return (n - 1) / 2;
}
```

Finding the left child of a node

```
int leftchild(int n) {
  return 2 * n + 1;
}
```

Finding the right child of a node

```
int rightchild(int n) {
  return 2 * n + 1;
}
```

03-12: Building a Heap

Build a heap out of n elements

03-13: Building a Heap

Build a heap out of n elements

- Start with an empty heap
- Do n insertions into the heap

```
MaxHeap H = new MaxHeap();
for(i=0 < i<A.size(); i++)
   H.insert(A[i]);
```

Running time?

03-14: Building a Heap

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Running time? $O(n \lg n)$ – is this bound tight?

03-15: Building a Heap

Total time: $c_1 + \sum_{i=1}^{n} c_2 \lg i$

03-16: Building a Heap

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$$
\begin{aligned}
c_1 + \sum_{i=1}^{n} c_2 \lg i &\geq \sum_{i=n/2}^{n} c_2 \lg i \\
&\geq \sum_{i=n/2}^{n} c_2 \lg(n/2) \\
&= (n/2) c_2 \lg(n/2) \\
&= (n/2) c_2 ((\lg n) - 1) \\
&\in \Omega(n \lg n)
\end{aligned}
$$

Running Time: $\Omega(n \lg n)$

03-17: Building a Heap

Build a heap from the bottom up

- Place elements into a heap array
- Each leaf is a legal heap
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03-18: Building a Heap

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```
for(i=n/2; i>=0; i--)
  siftdown(i);
```

03-19: Building a Heap

How many swaps, worst case? If every `siftdown` has to swap all the way to a leaf:

$n/4$ elements	1 swap
$n/8$ elements	2 swaps
$n/16$ elements	3 swaps
$n/32$ elements	4 swaps
$\ldots$	

Total # of swaps:

$$n/4 + 2n/8 + 3n/16 + 4n/32 + \ldots + (\lg n)n/n$$

03-20: Heapsort

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  - Build a max-heap out of the array we want to sort (Time $\Theta(n)$)
  - While the heap is not empty:
    - Remove the largest element
    - Place this element in the "empty space" just cleared by the deletion

Total time:

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Total time: $\Theta(n \lg n)$