Graduate Algorithms

CS673-2016F-06

Selection Probelm

David Galles

Department of Computer Science
University of San Francisco

06-0: Finding Max & Min

- What is the smallest exact number of comparisons required to find the maximum element of a list with n elements?
- What is the smallest exact number of comparisons required to find the minimum element of a list of n elements?
- What is the smallest number of comparisons required to find the maximum and minimum elements of a list?

06-1: Finding Max & Min

- What is the smallest number of comparisons required to find the maximum element of a list?
  $(n-1)$
- What is the smallest number of comparisons required to find the minimum element of a list?
  $(n-1)$
- What is the smallest number of comparisons required to find the maximum and minimum elements of a list?
  - Compare pairs: then compare the largest to the current largest, and smallest to the current smallest $\lceil \frac{n}{2} \rceil + (n-2)$

06-2: Selection Problem

- What if we want to find the kth smallest element?
  - Median: $k = \lceil \frac{n}{2} \rceil$th smallest element, for odd n
- What is the obvious method?
- Can we do better?

06-3: Selection Problem

- What if we want to find the kth smallest element?
  - Median: $k = \lceil \frac{n}{2} \rceil$th smallest element, for odd n
- What is the obvious method?
  - Sort the list, select the element at index k
- Can we do better?

06-4: Selection Problem

```
Quicksort(A, low, high)
    if (low < high) then
        pivotindex ← Partition(A, low, high)
        Quicksort(A, low, pivotindex 1)
        Quicksort(A, pivotindex + 1, high)

Partition(A, low, high)
    pivot = A[high]
    i ← low - 1
    for j ← low to high - 1 do
        if (A[j] ≤ pivot) then
            i ← i + 1
            swap A[i] ↔ A[j]
    swap A[i+1] ↔ A[high]
    return i+1
```

06-5: Selection Problem

```
Select(A,low,high,k)
    if (low = high)
        return A[low]
    pivot = Partition(A, low, high)
    adj_pivot = piv - low + 1
    if (k = adj_pivot) then
        return A[pivot]
    if (k < adj_pivot) then
        return Select(A,low,pivot-1, k)
    else
        return Select(A,pivot+1, high, k-adj_pivot)
```

Running time (Best and worst case)?

06-6: Selection Problem

- Best case time:

$$T(n) = T(n/2) + c * n \in \Theta(n)$$

- Worst case time:

$$T(n) = T(n-1) + c * n \in \Theta(n^2)$$

- Average case time turns out to be $\Theta(n)$, but we'd like to get the worst-case time down.

06-7: Selection Problem

- Improving worst-case time for selection
  - We need to guarantee a “good” pivot to get $\Theta(n)$ time for selection
  - How much time can we spend to find a good pivot, and still get $\Theta(n)$ time for selection?

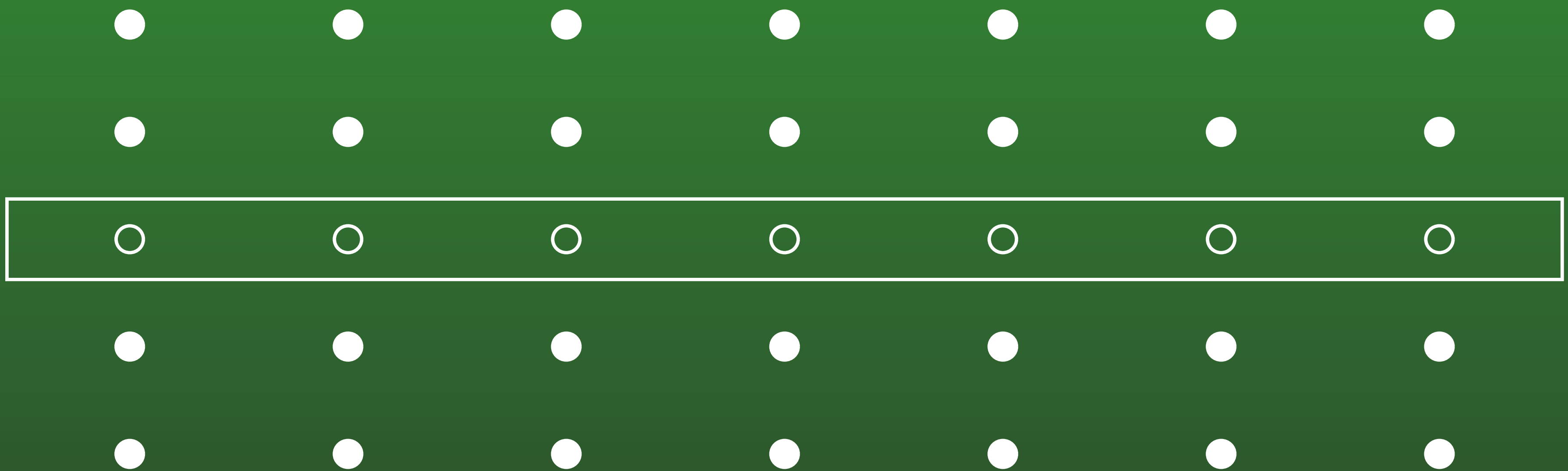
06-8: Selection Problem

- Improving worst-case time for selection
  - We need to guarantee a “good” pivot to get $\Theta(n)$ time for selection
  - How much time can we spend to find a good pivot, and still get $\Theta(n)$ time for selection?
  - $O(n)$!

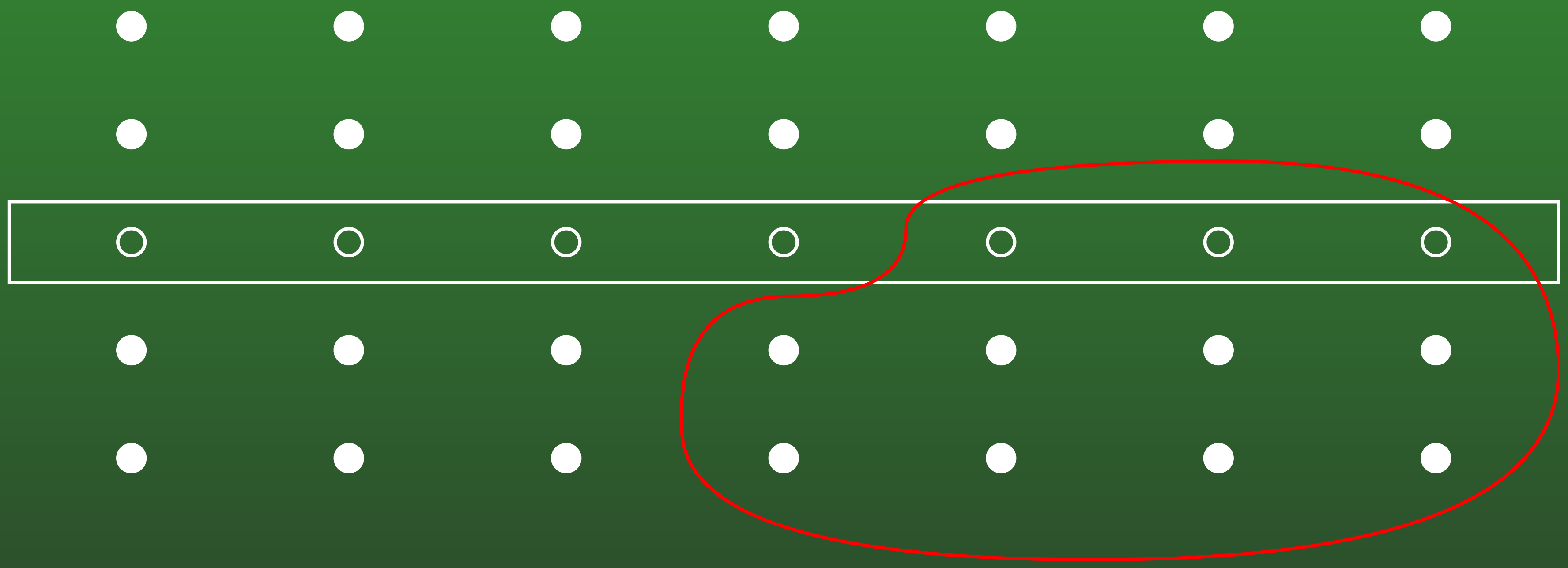
06-9: Selection Problem

- Finding a “Good” pivot (one that is near the median) in linear time:
  - Split the list into $\frac{n}{5}$ list of length 5
  - Do an insertion sort on each of the $\frac{n}{5}$ lists to find the median of each of these lists
  - Call select recursively to find the median of the $\frac{n}{5}$ medians

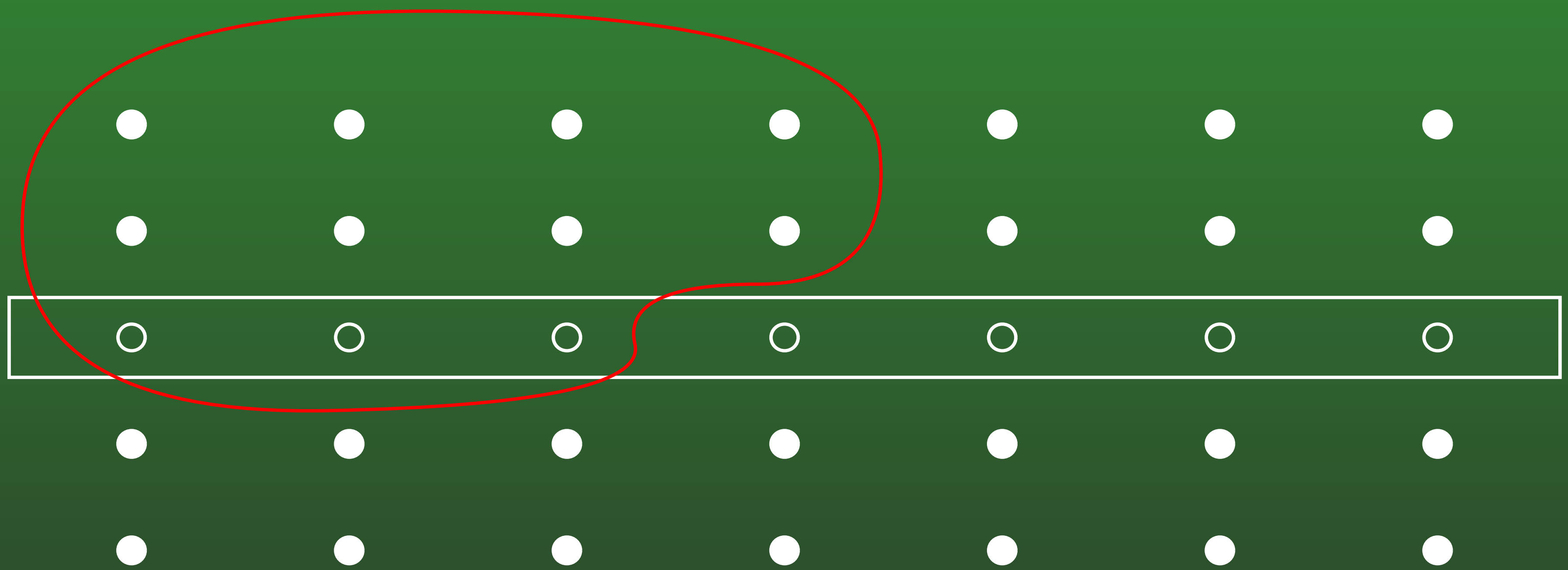
06-10: Selection Problem



06-11: Selection Problem



06-12: Selection Problem



06-13: Selection Problem

- How good is the pivot chosen by this method? How many elements are guaranteed to be less than the pivot?
  - Each row has 5 elements
  - Half of the rows will have a median less than the pivot
  - Each of these rows will have 3 elements less than the pivot

$$3 * \left\lceil \left\lceil \frac{n}{5} \right\rceil \frac{1}{2} \right\rceil$$

06-14: Selection Problem

$3 * \left\lceil \left\lceil \frac{n}{5} \right\rceil \frac{1}{2} \right\rceil$

- Not all of those rows have exactly 3 elements less than the pivot:
  - The total number of elements might not be divisible by 5 (so one row would have < 5 elements
  - The row containing the pivot itself only has 2 elements less than the pivot (not 3)
- So, we will omit those two rows, leaving: $3 * \left(\left\lceil \left\lceil \frac{n}{5} \right\rceil \frac{1}{2} \right\rceil - 2 \right) \geq \frac{3n}{10} - 6$

06-15: Selection Problem

- Worst case time for selection for a problem of size n:
  - $\Theta(n)$ time to do partition, n/5 insertion sorts
  - Time to find the median of medians (looking at $n/5$ elements)
  - Time to make the recursive call (to a problem of no more than size $7n/10 + 6$

06-16: Selection Problem

$$T(n) \leq \begin{cases} C_1 & \text{if } n < 140 \\ T(\lceil \frac{n}{5} \rceil) + T(\frac{7n}{10} + 6) + C_2 * n & \text{otherwise} \end{cases}$$

06-17: Selection Problem

$$
\begin{aligned}
T(n) &\leq T\left(\lceil\frac{n}{5}\rceil\right) + T\left(\frac{7n}{10} + 6\right) + C_2 * n \\
&\leq \frac{C * n}{5} + C + \frac{7 * C * n}{10} + 6 * C + C_2 * n \\
&= \frac{9 * C * n}{10} + 7 * C + C_2 * n \\
&= C * n + \left(7 * C + C_2 * n - \frac{C * n}{10}\right)
\end{aligned}
$$

06-18: Selection Problem

$$
\begin{aligned}
7 * C + C_2 * n - \frac{C * n}{10} &\leq 0 \\
C * \left(7 - \frac{n}{10}\right) &\leq -C_2 * n \\
C * \left(\frac{n}{10} - 7\right) &\geq C_2 * n \\
C &\geq C_2 * n/(n/10 - 7) \\
C &\geq 10 * C_2 * (n/(n - 70))
\end{aligned}
$$

Note that we must insist that $n > 70$. If $n \geq 140$, then this is true if $C > 20 * C_2$

06-19: Selection Problem

- Selection takes time $O(n)$
  - in fact, $\Theta(n)$, since each recursion steps takes time $\Omega(n)$
- So, we can use Selection to make Quicksort take time $\Theta(n \lg n)$ worst case
  - Would that be a good idea?