Graduate Algorithms

CS673-2016F-07

Red/Black Trees

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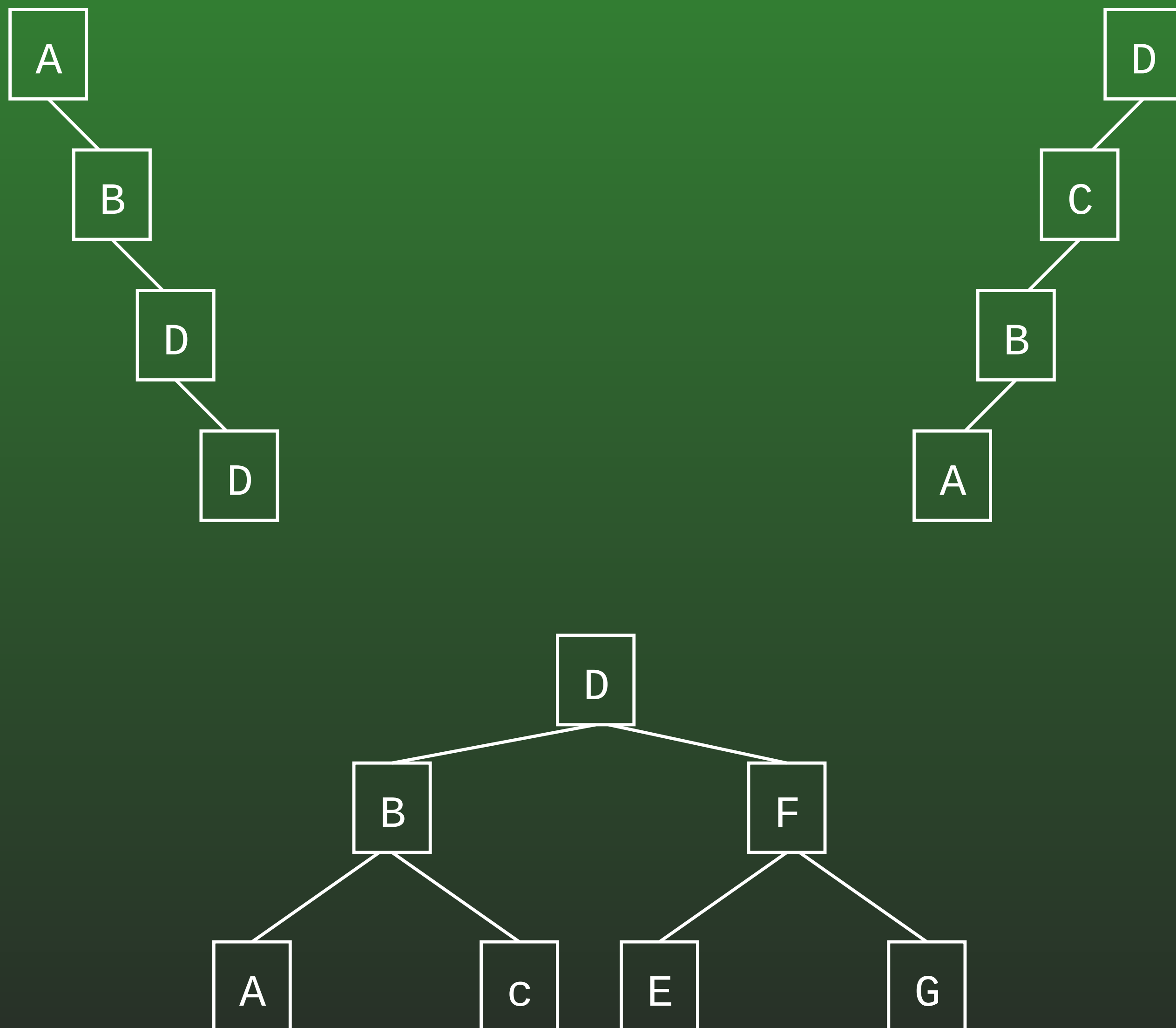
07-0: Binary Search Trees

- Binary Trees
- For each node n, (value stored at node n) $>$ (value stored in left subtree)
- For each node n, (value stored at node n) $<$ (value stored in right subtree)

07-1: Example Binary Search Trees



07-2: Example Binary Search Trees

- Examples:
  - Finding an element
  - Inserting an element
  - Deleting an element

07-3: Running Times

- Best-Case upper limit on the time for insert/delete/find of an element for a BST with n elements?
- Worst-Case upper limit on the time for insert/delete/find for a BST with n elements?
- Expected upper limit on the time for insert/delete/find for a BST with n elements?
    - What would we mean by “expected” in this instance?

07-4: Running Times

- Best-Case upper limit on the time for insert/delete/find of an element for a BST with n elements?
  - $O(\lg n)$, if the tree is balanced
- Worst-Case upper limit on the time for insert/delete/find for a BST with n elements?
  - $O(n)$, if the tree is a list
- Expected upper limit on the time for insert/delete/find for a BST with n elements?
  - $O(\lg n)$, if elements are inserted in random order

07-5: Balanced BSTs

- We can guarantee $O(\lg n)$ running time for insert/find/delete if we can guarantee the tree is balanced
- Several methods for guaranteeing a balanced tree
  - AVL trees & Red-Black trees are the most common
  - We'll look at Red-Black Trees

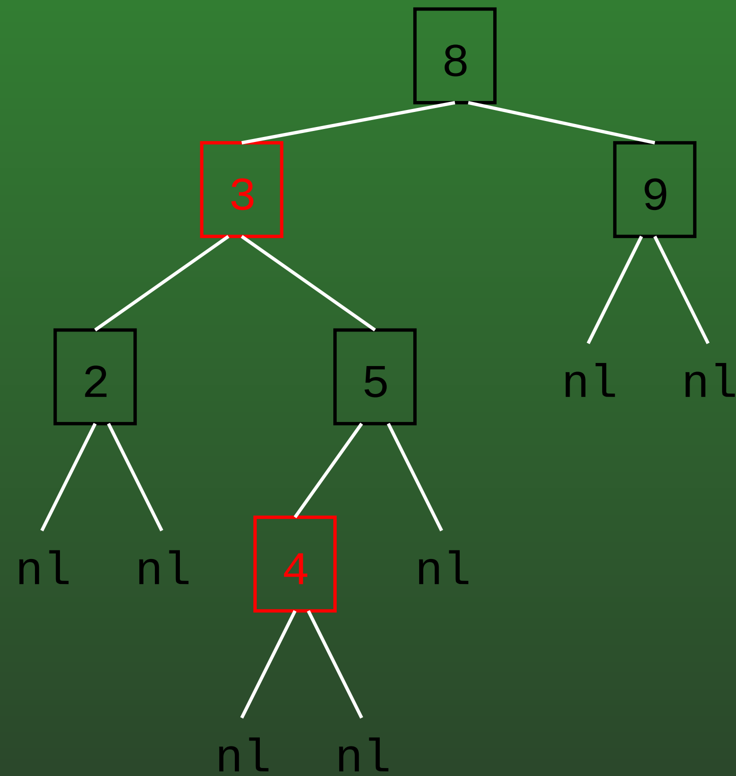
07-6: Red-Black Trees

- Red-Black Trees as Binary Search trees, with “Null Leaves”
  - Examples of BSTs with “Null Leaves”
  - (Null leaves are mostly a notational convenience)

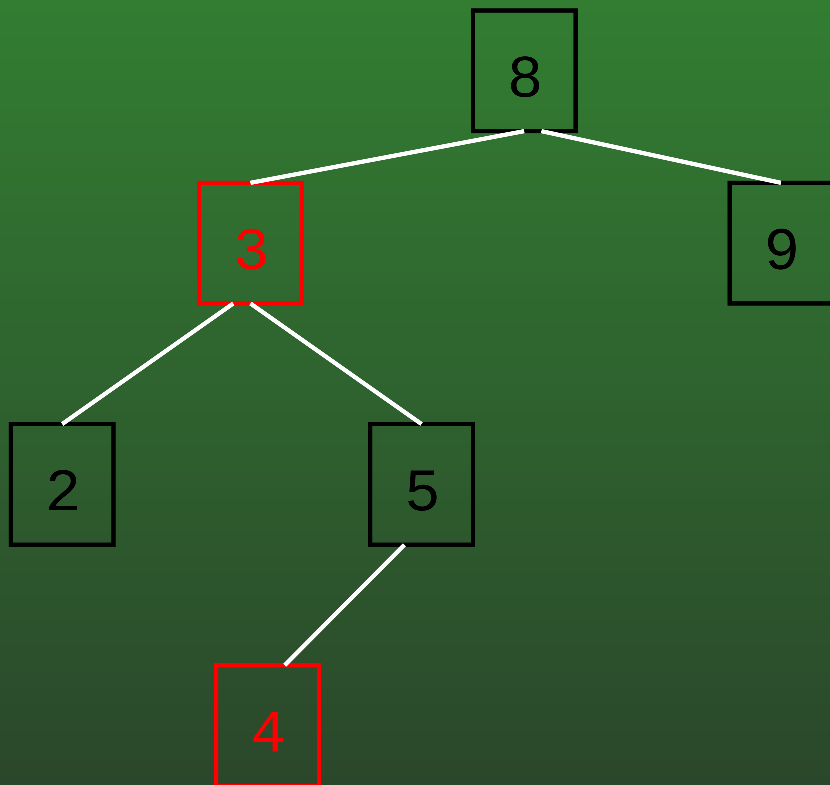
07-7: Red-Black Trees

- Red-Black Trees are Binary Search trees, with "Null Leaves", and the following properties:
  - Every Node is either Red or Black
  - (Root is Black) $<$Not strictly required$>$
  - Each null "leaf" is Black
  - If a node is red, both children are black
  - For each node, all paths from the node to descendant leaves contain the same number of black nodes

07-8: Red-Black Trees



07-9: Red-Black Trees



- Example Red-Black tree ("Null Leaves" left out for clarity)

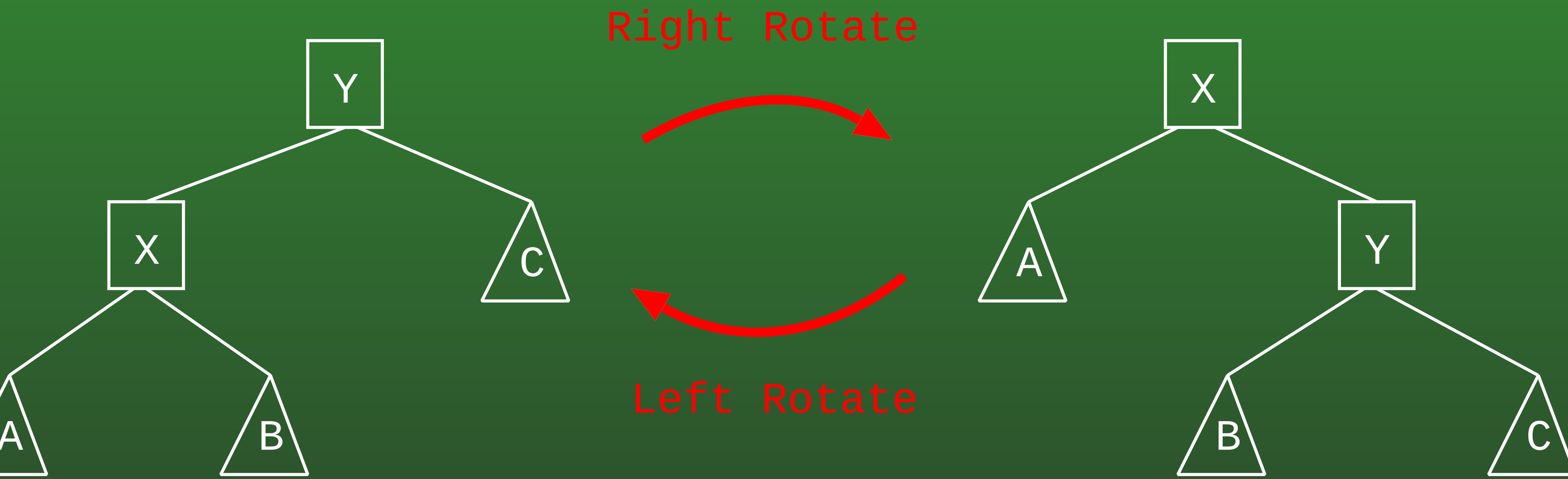
07-10: **Red-Black Trees**

- In a Red-Black tree, what is the greatest possible difference in the length of the path from the root to two different leaves?
- What is the largest height of a Red-Black tree that contains n elements?

07-11: Red-Black Trees

- Let $bh(X)$ be the "Black Height" of a node – the number of black nodes on a path from that node to a leaf (not including the node itself)
- The subtree rooted at any node X has at least $2^{bh(X)} - 1$ internal (non-leaf) nodes
  - Proof by induction (on board)

07-12: Tree Rotations



07-13: Tree Insertions

- Always insert red nodes
- Which property would be violated by inserting a red node?

07-14: Tree Insertions

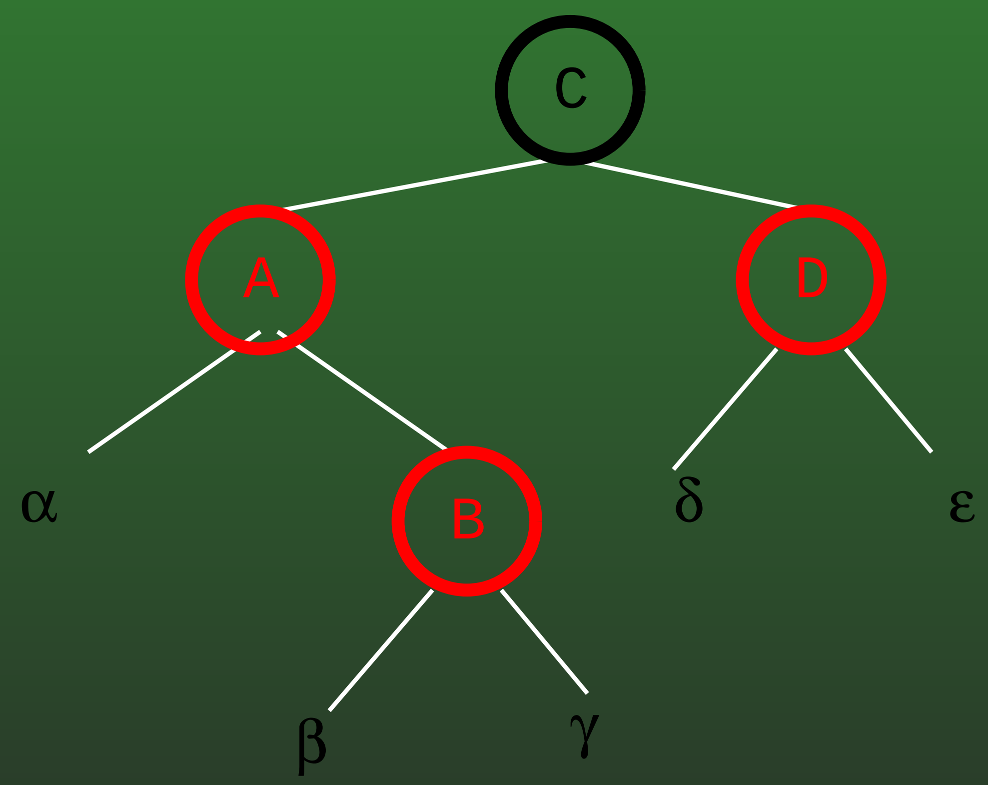
- Always insert red nodes
- Which property would be violated by inserting a red node?
  - Could have a red node with a red child
- Fix using tree rotations

07-15: Tree Insertions

- To fix a red node with red child:
  - Case 1: Uncle is red
  - Case 2: Uncle is black, Inserted node is right child of parent, and parent is a left child of Grandparent (or node is left child, parent is right child)
  - Case 3: Uncle is black, Node is left child of parent, parent is left child of Grandparent (or node is righ child, parent is right child)

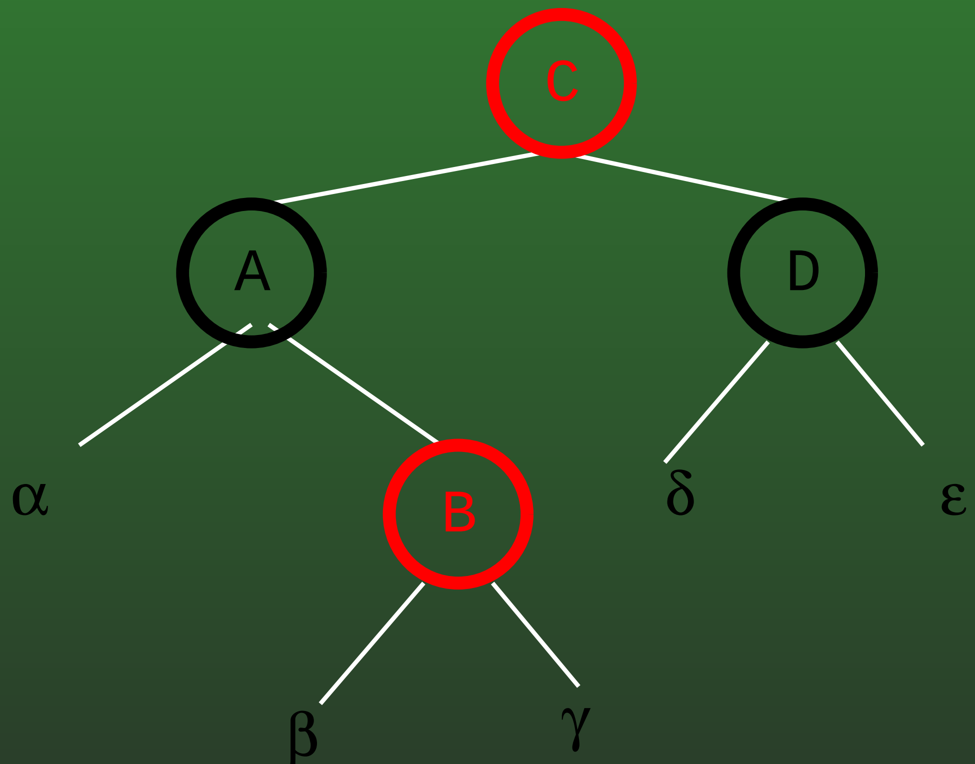
07-16: Case 1

- Red Uncle



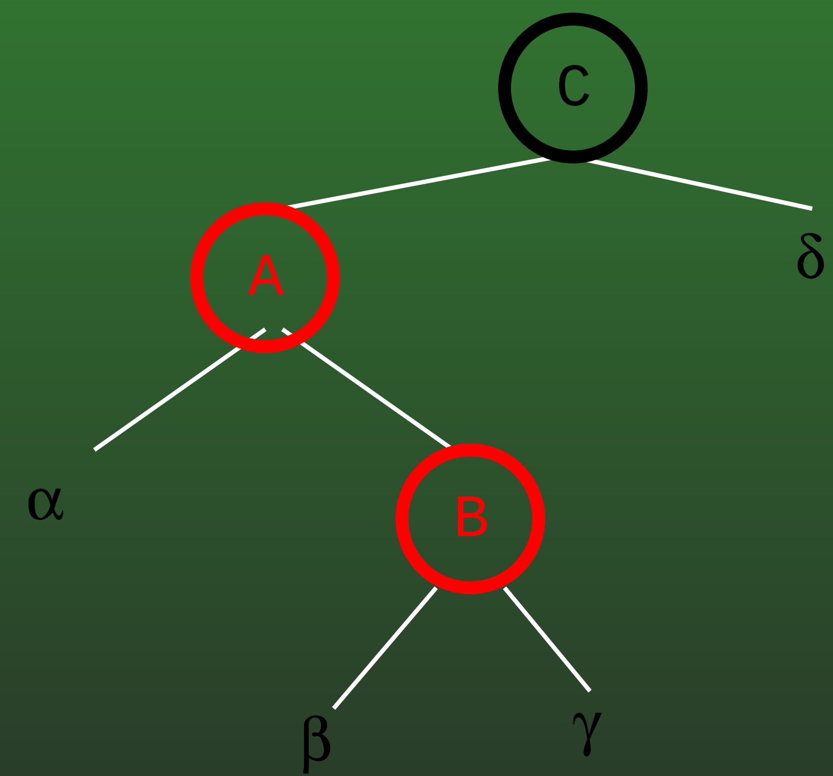
Case 1

- Red Uncle



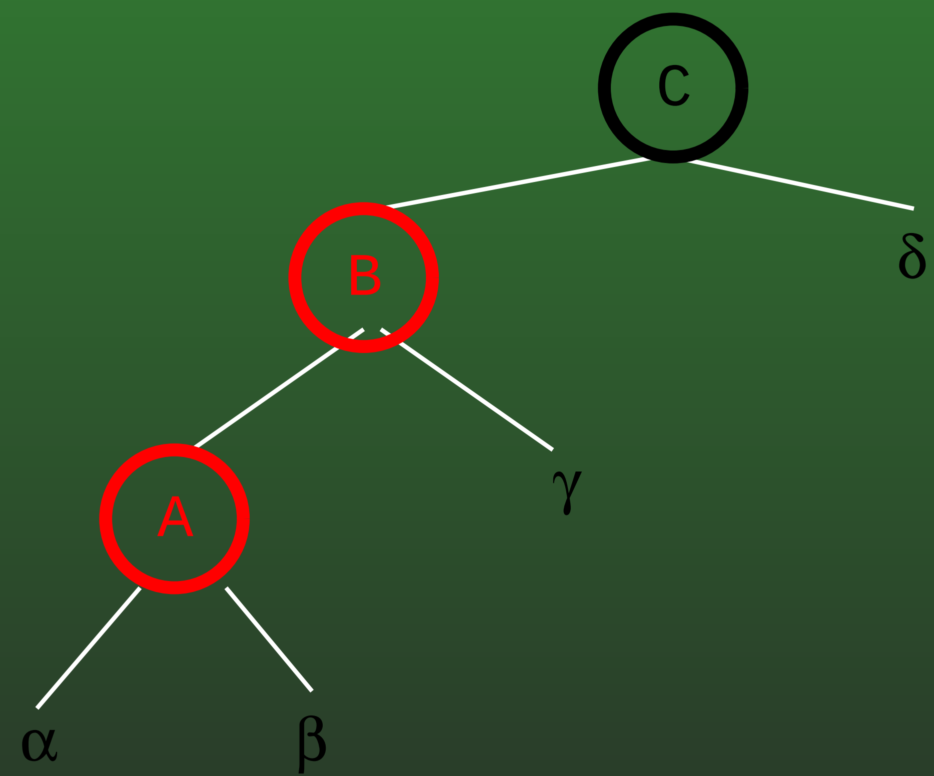
07-18: Case 2

- Black Uncle / parent child different handedness



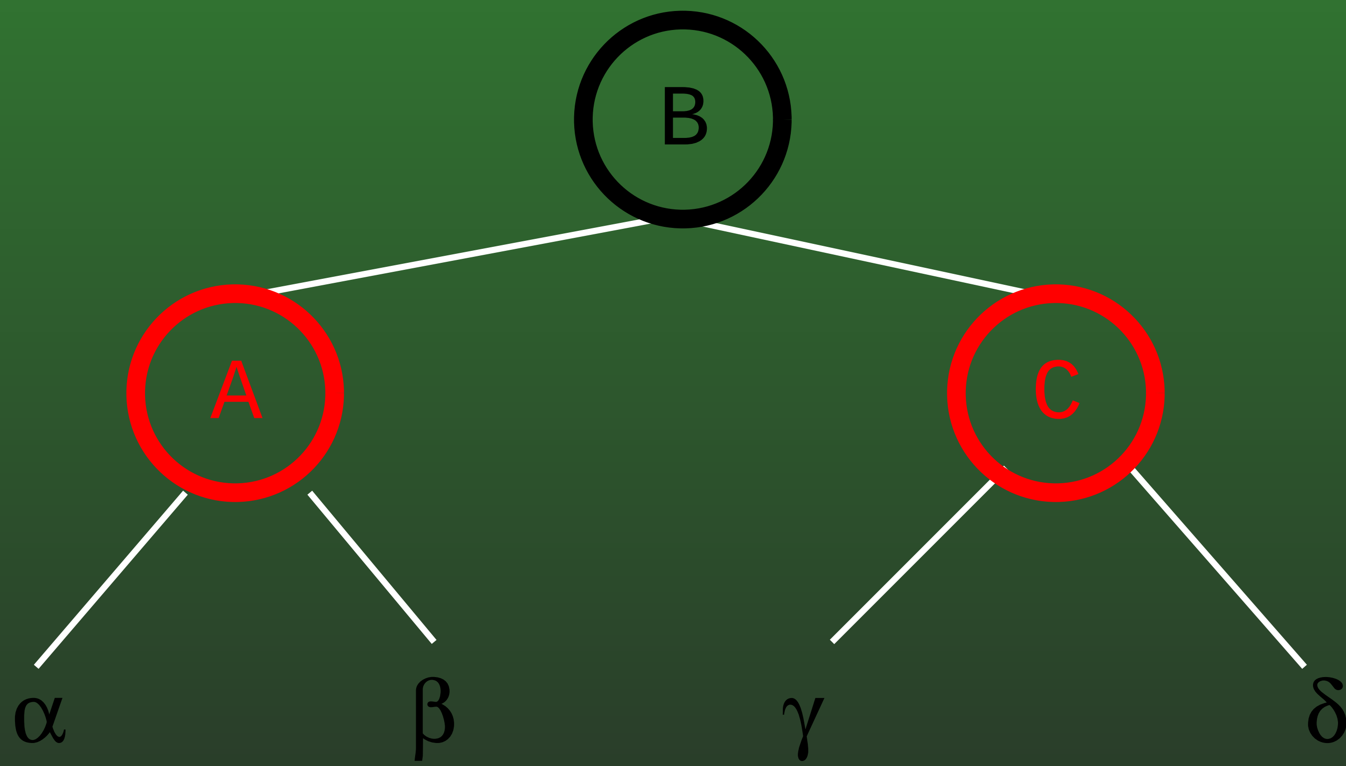
07-19: Case 3

- Black Uncle / parent child same handedness



07-20: Case 3

- Black Uncle / parent child same handedness



07-21: Deleting nodes

- Deleting nodes
  - Delete nodes just like in standard BST
  - Which properties could be violated by deleting a red node?
    - Each node red or black
    - Black Root
    - Each red node has 2 black children
    - Black path length to leaves same for each node

07-22: Deleting nodes

- Deleting nodes
  - Delete nodes just like in standard BST
  - Which properties could be violated by deleting a red node?
    - None!

07-23: Deleting Nodes

- Deleting nodes
  - Delete nodes just like in standard BST
  - Which properties could be violated by deleting a black node?
    - Each node red or black
    - Black Root
    - Each red node has 2 black children
    - Black path length to leaves same for each node

07-24: Deleting Nodes

- Deleting black node
  - If the child of the deleted node is red ... (show example on board)

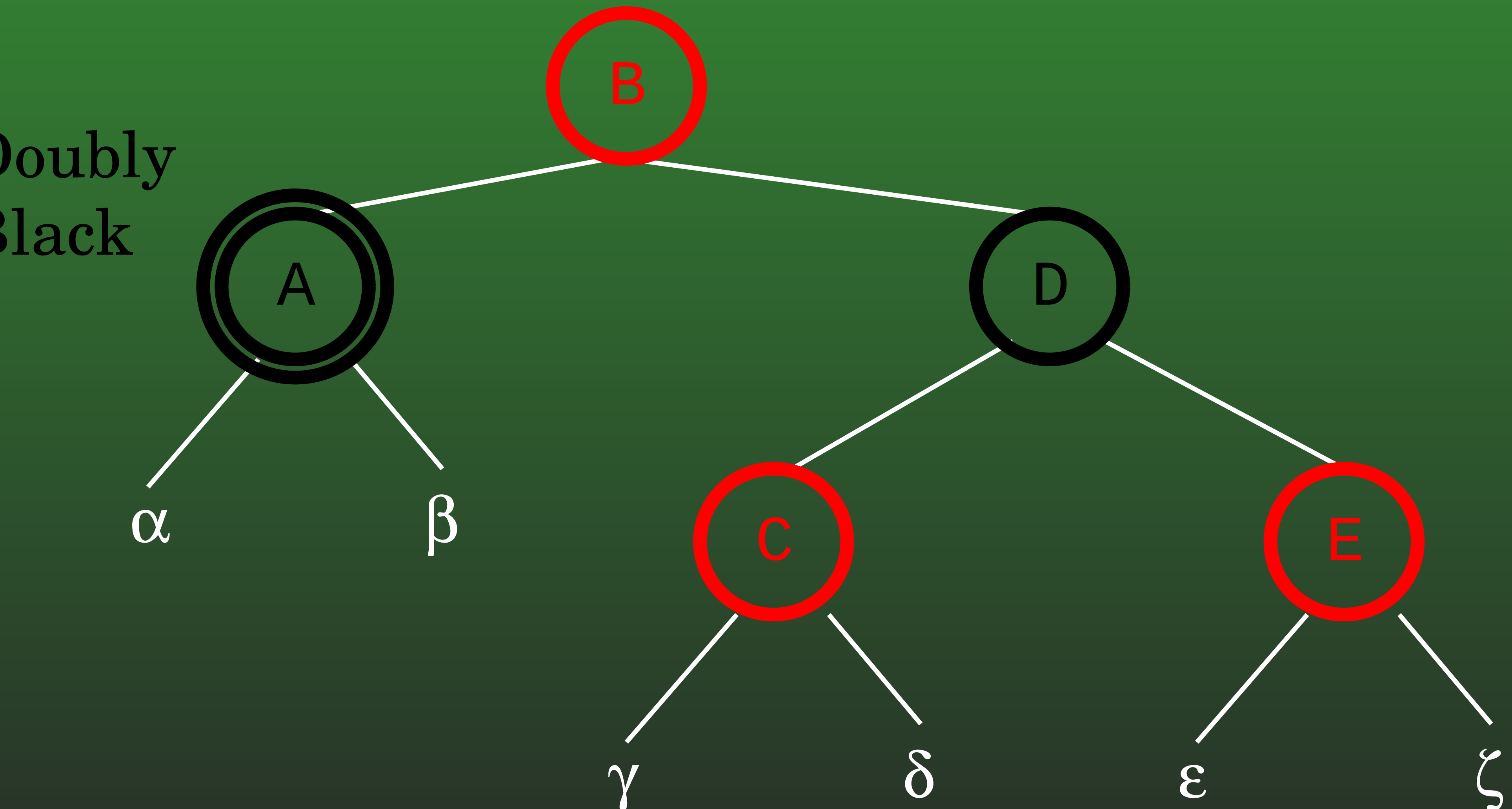
07-25: Deleting Nodes

- Deleting black node
  - If the child of the deleted node is black
    - Make the child “doubly black”
    - Push “extra blackness” up the tree until it can be removed by a rotation

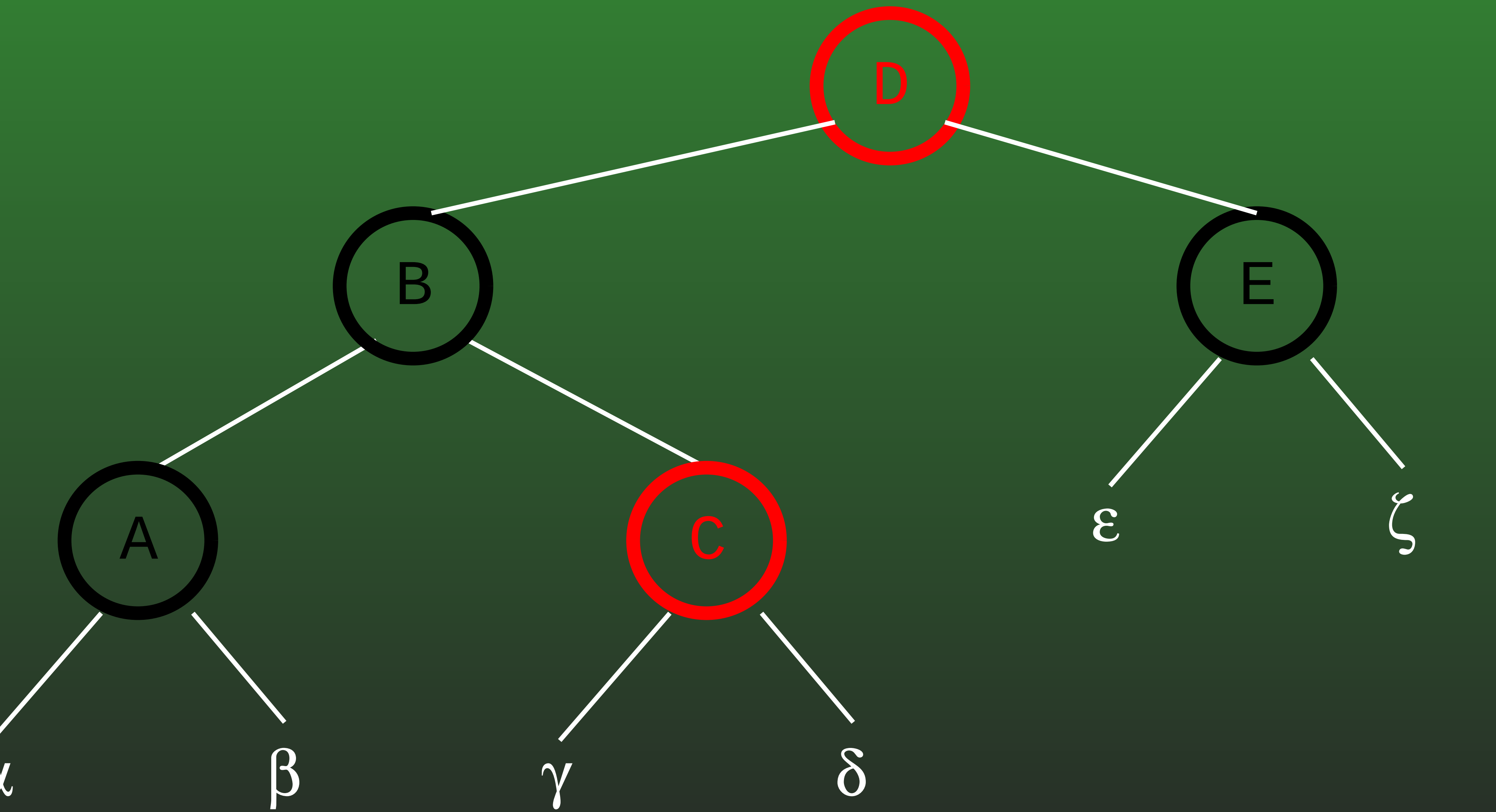
07-26: Deleting Nodes

- X is “doubly black” node, X is a left child
  - Case 4:
    - X’s sibling W is black, and W’s right child is red
    - Can remove “double-blackness” of X with a single rotation

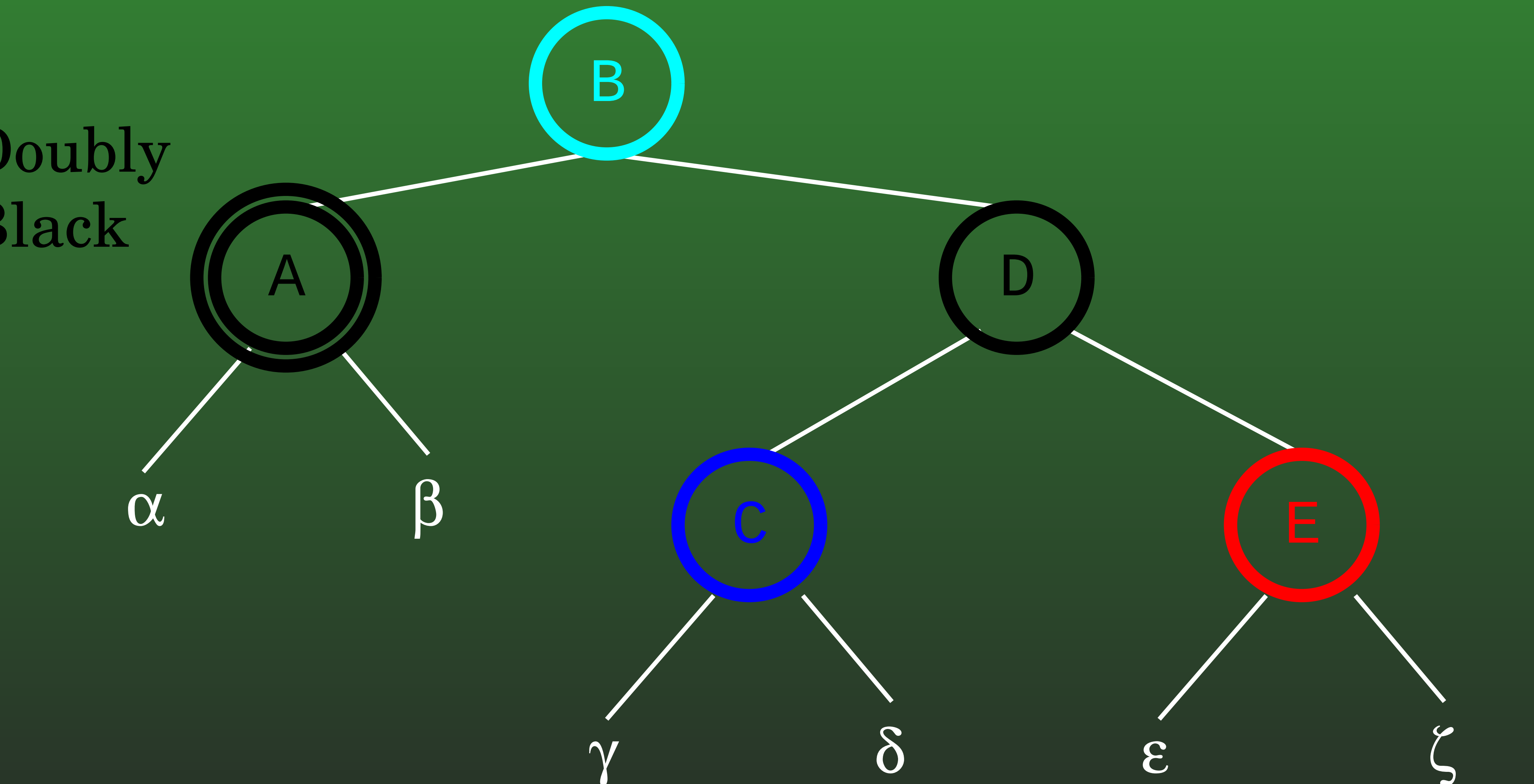
07-27: Deleting Nodes



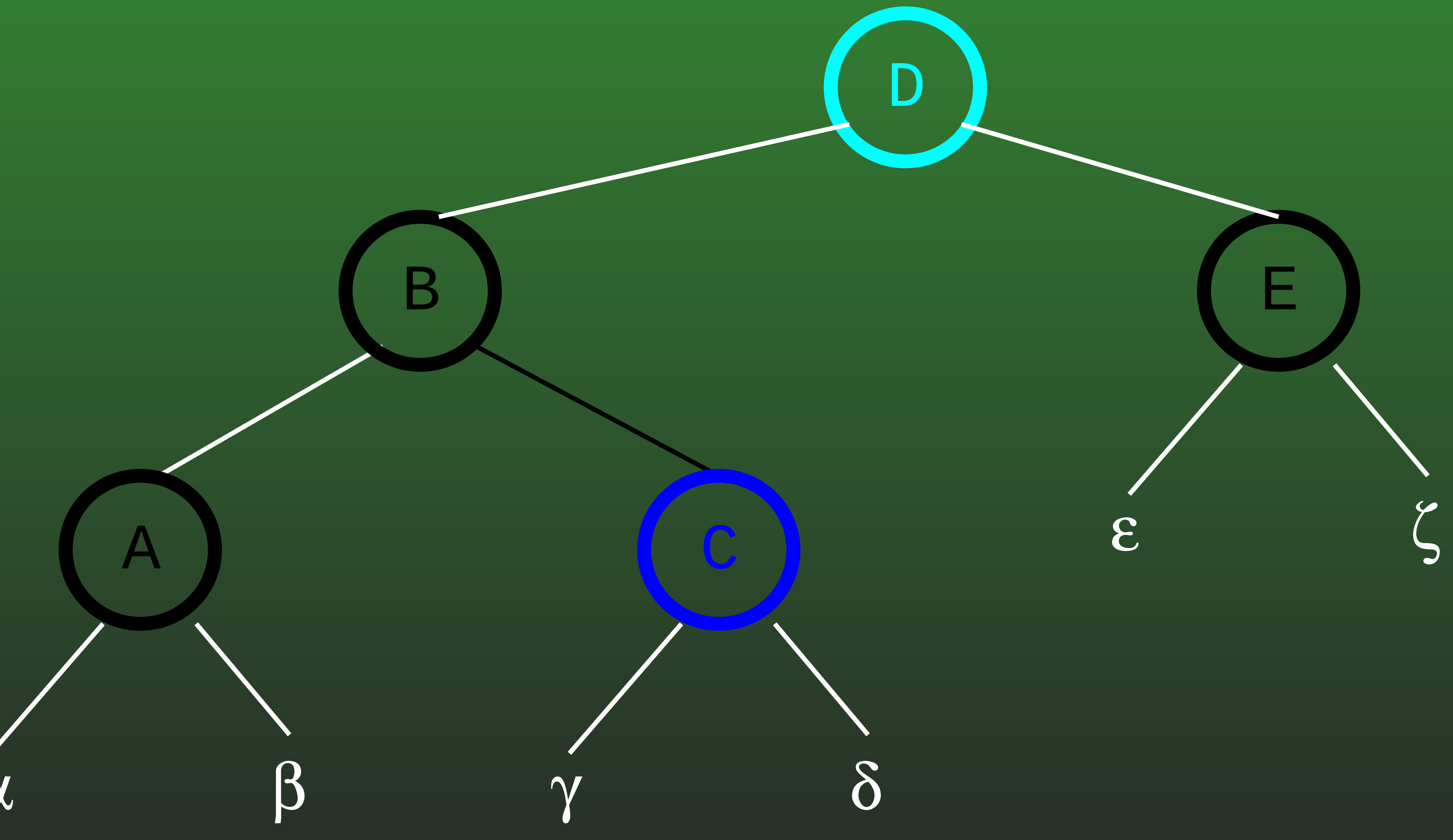
07-28: Deleting Nodes



Deleting Nodes



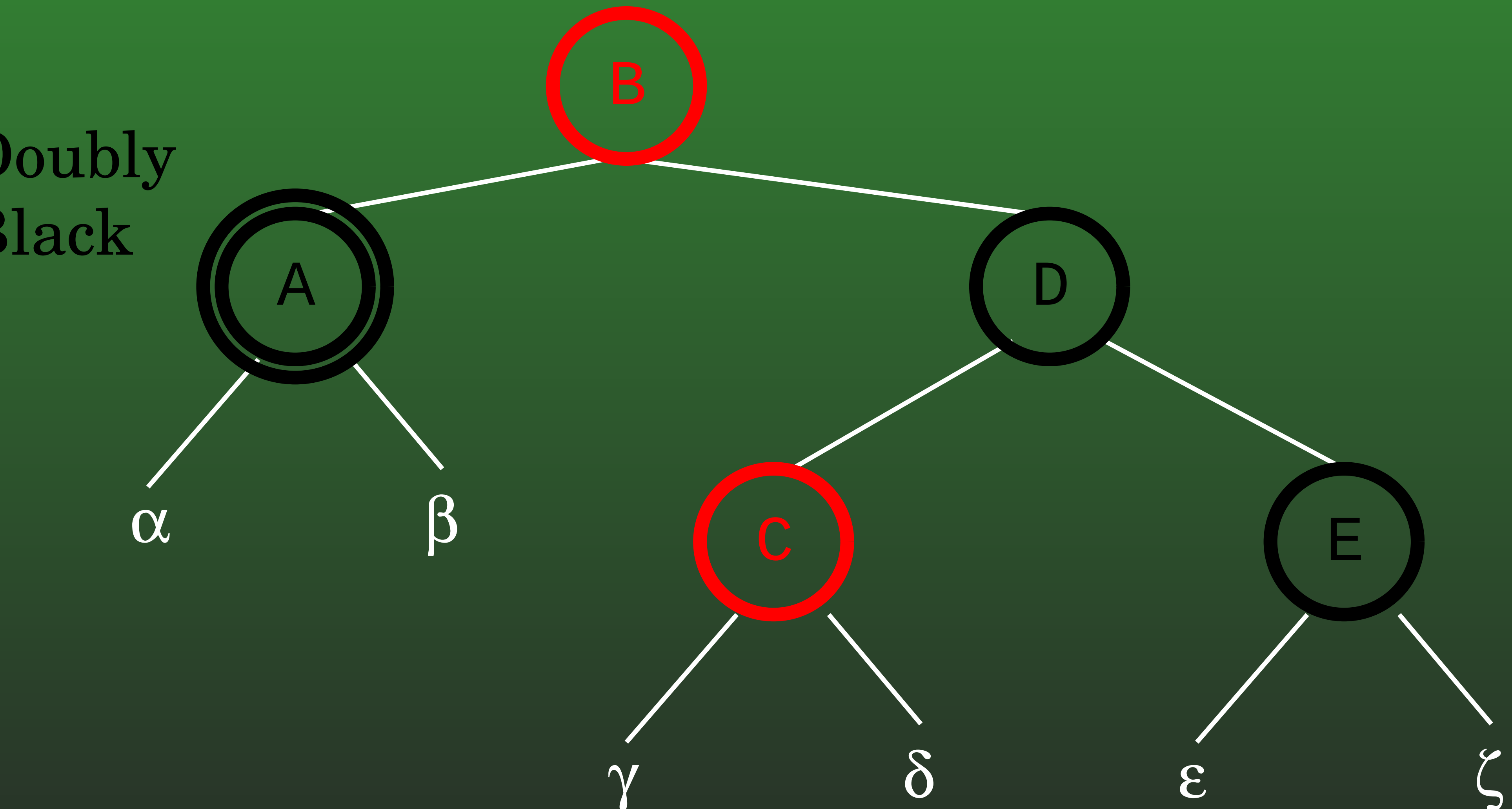
Deleting Nodes



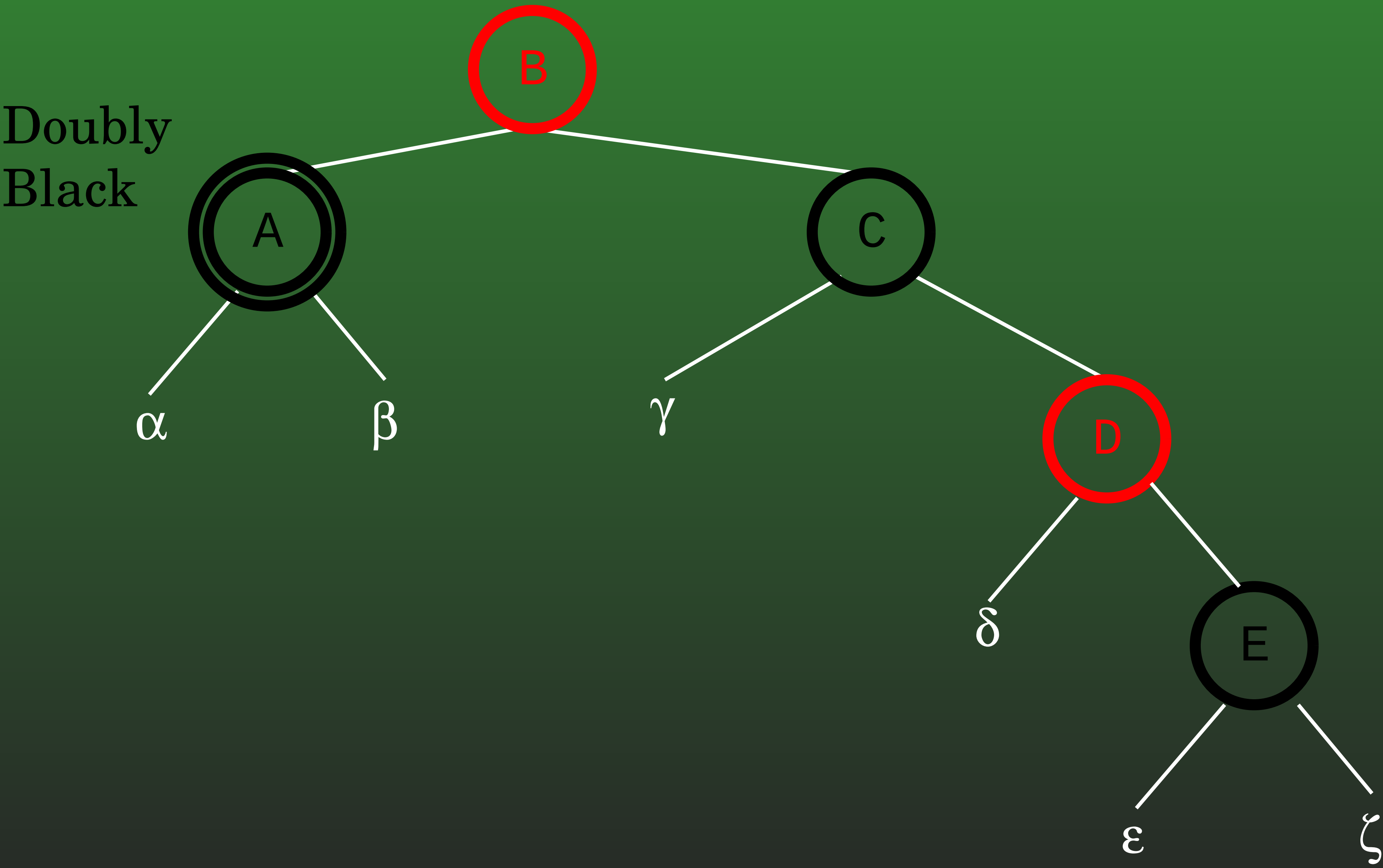
07-31: Deleting Nodes

- X is “doubly black” node, X is a left child
  - Case 3:
    - X’s sibling W is black, and W’s left child is red, and right child is black
    - Single rotation to get to previous case

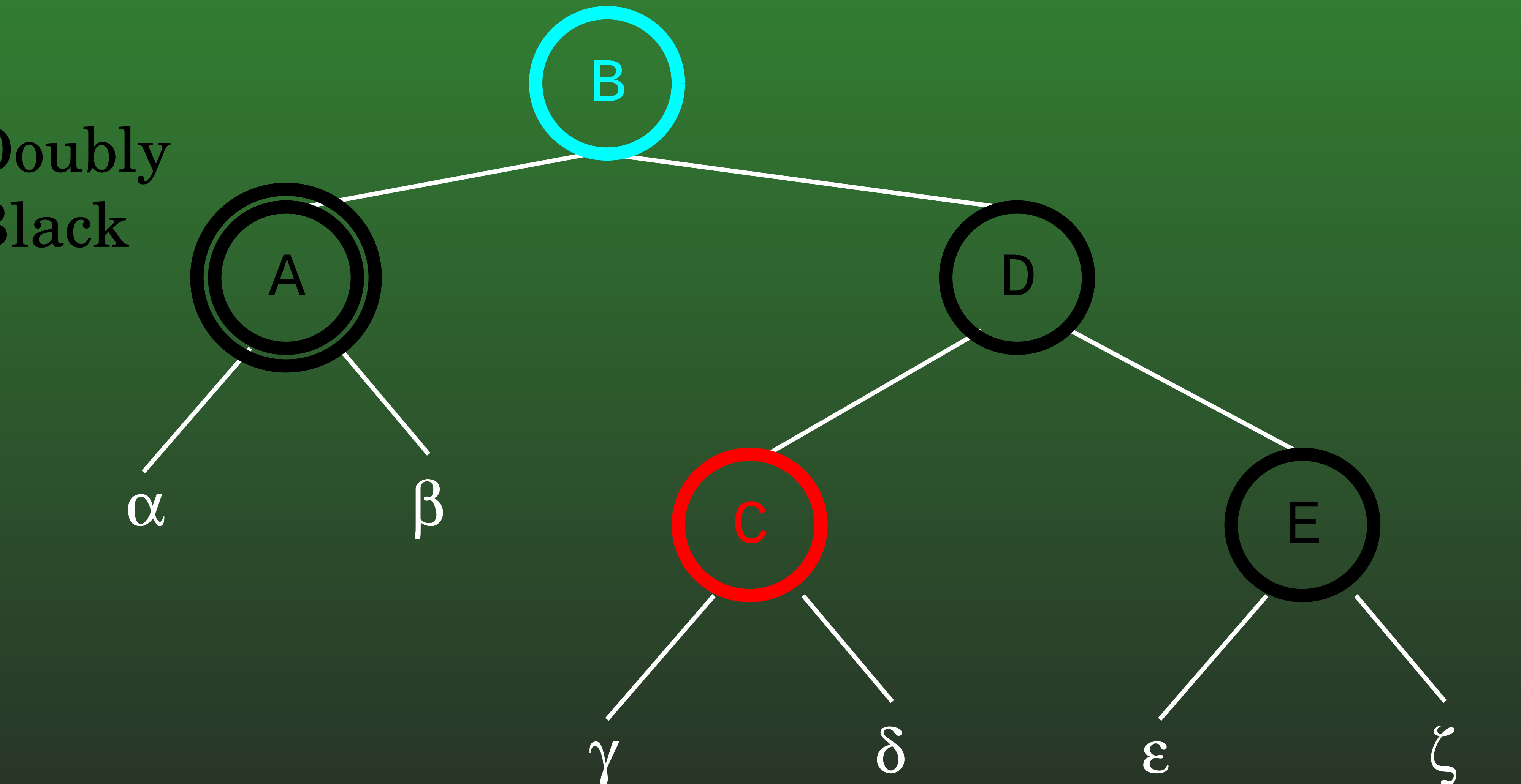
07-32: Deleting Nodes



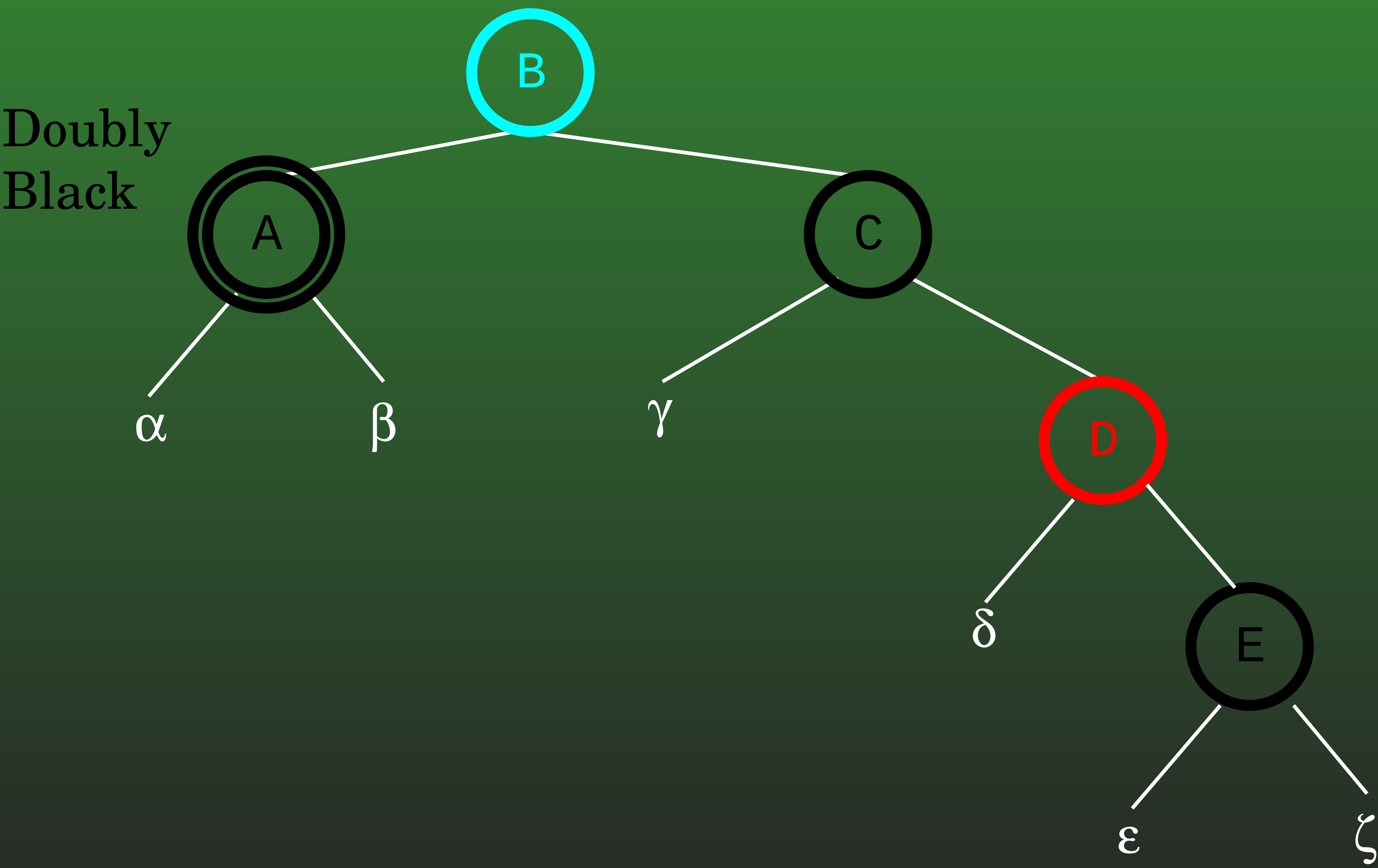
07-33: Deleting Nodes



07-34: Deleting Nodes



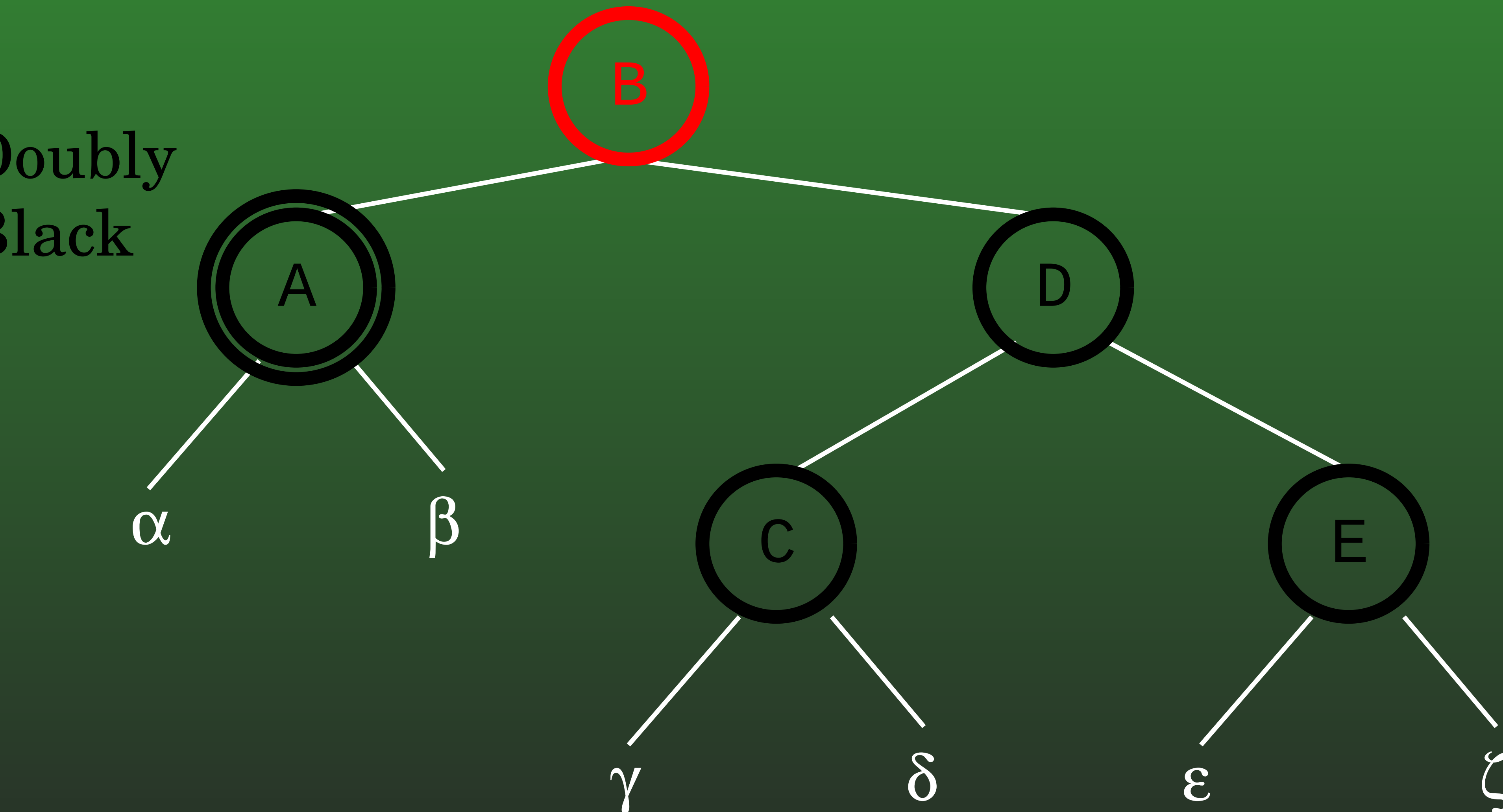
07-35: Deleting Nodes



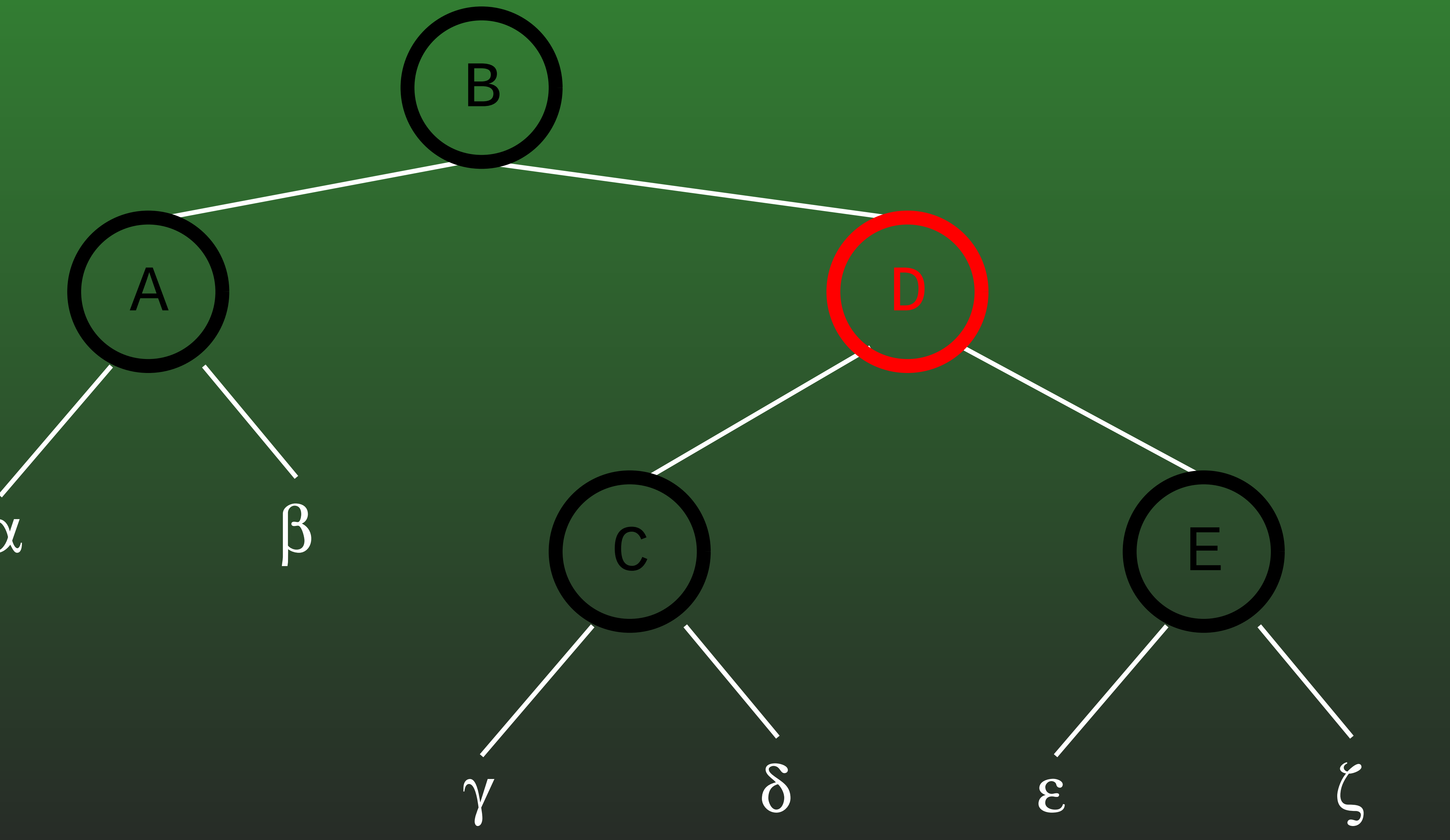
Deleting Nodes

- X is “doubly black” node, X is a left child
  - Case 2:
    - X’s sibling W is black, and both of W’s children are black
    - Push “Blackness” of X and W to parent

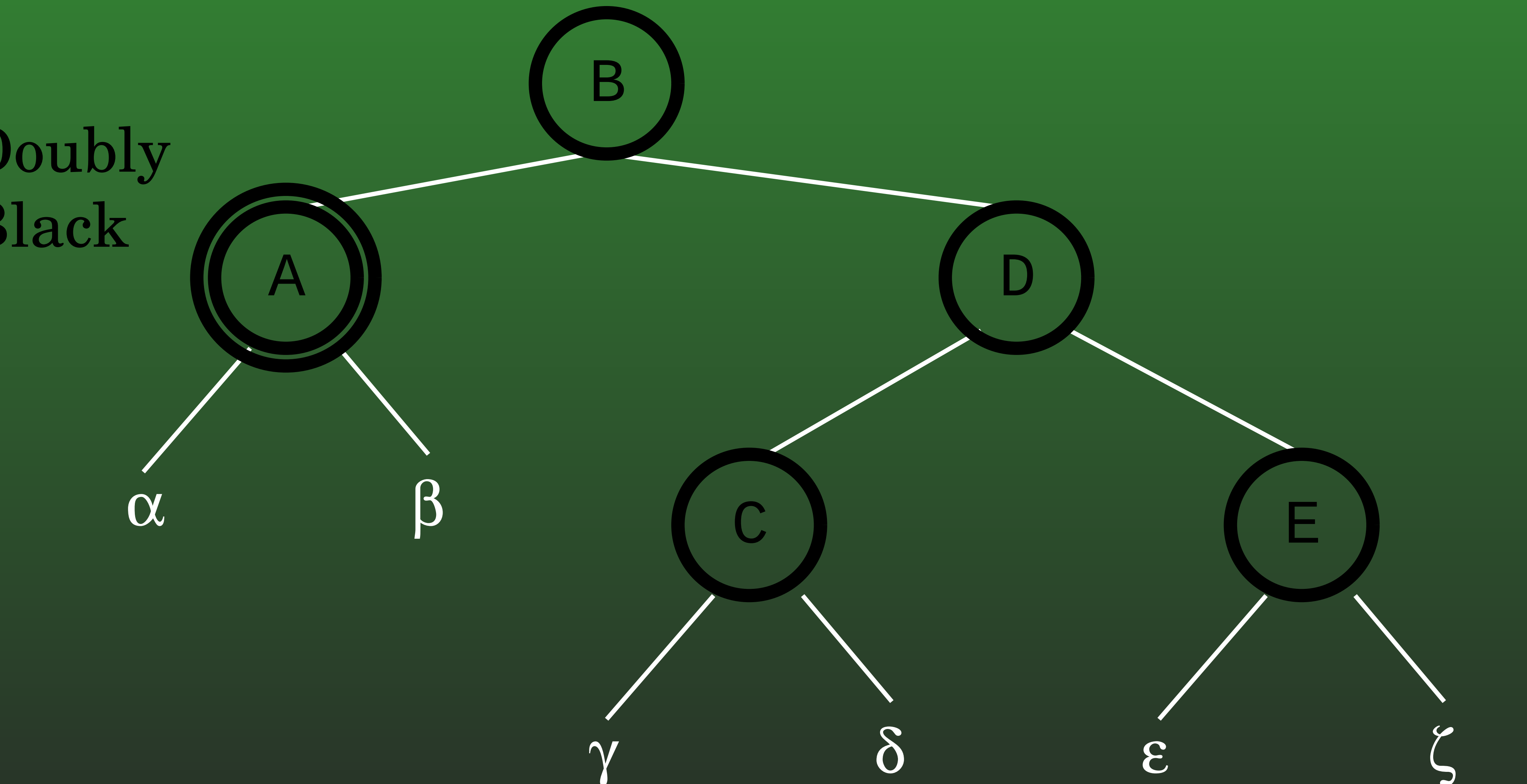
Deleting Nodes



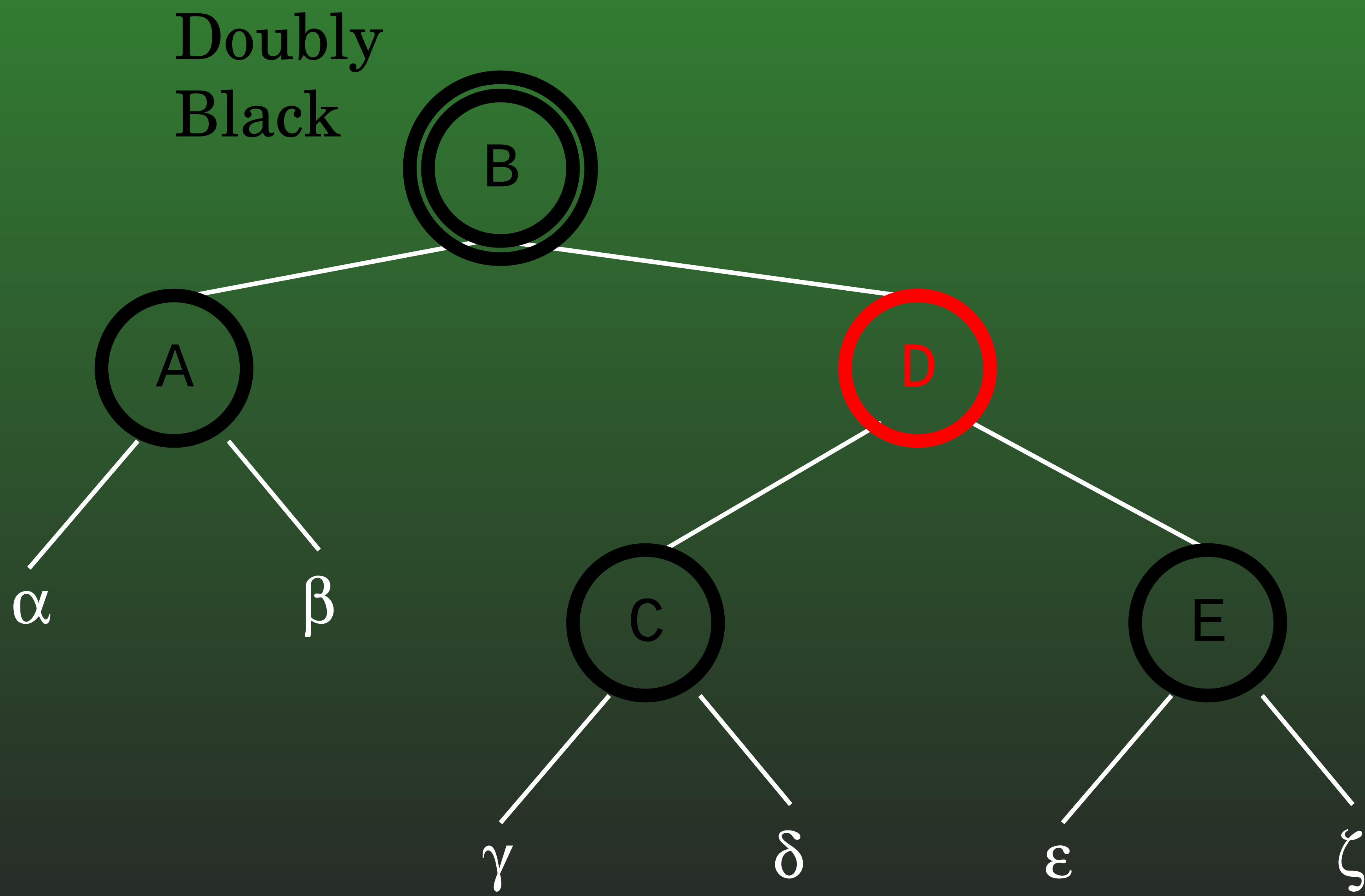
07-38: Deleting Nodes



07-39: Deleting Nodes



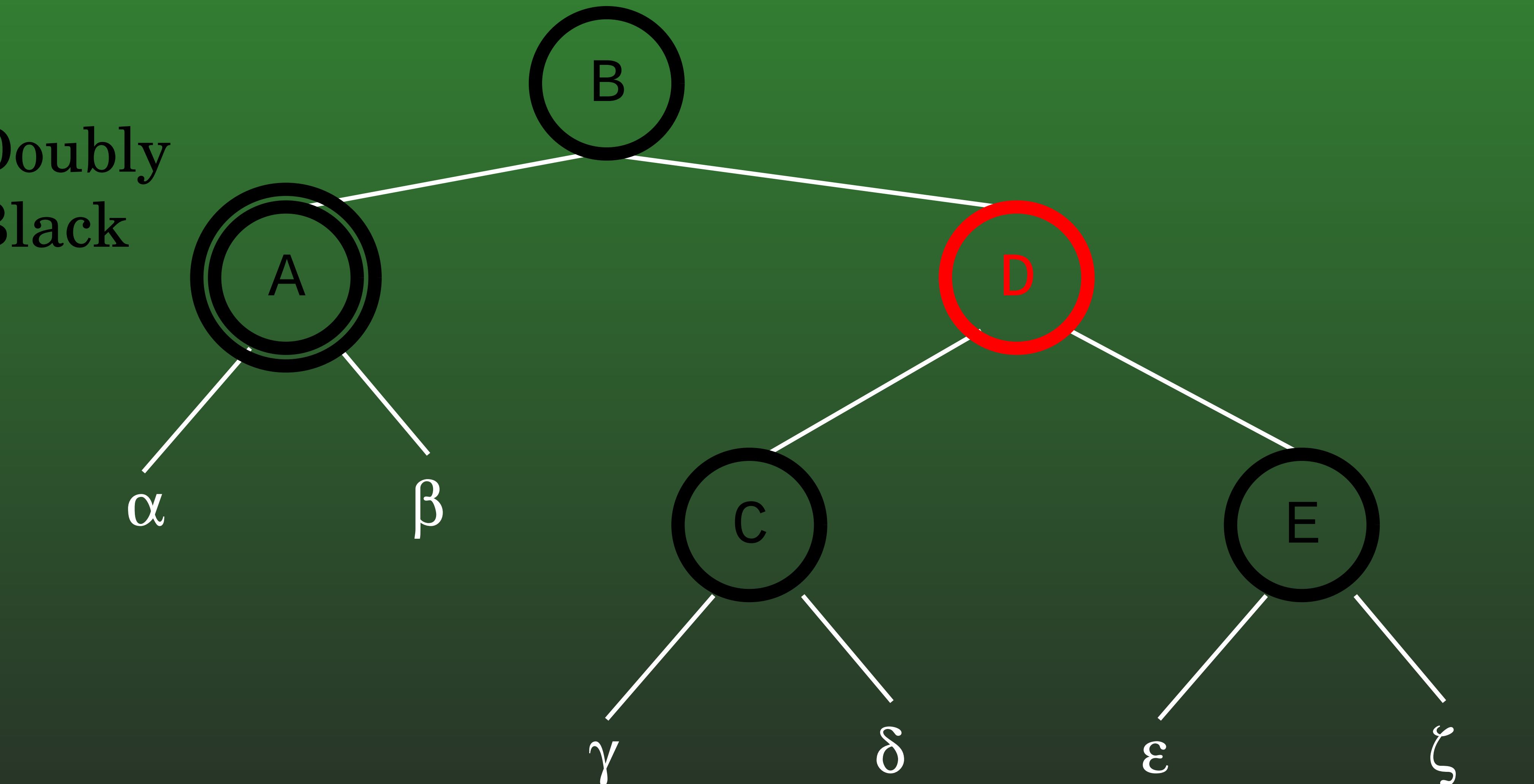
07-40: Deleting Nodes



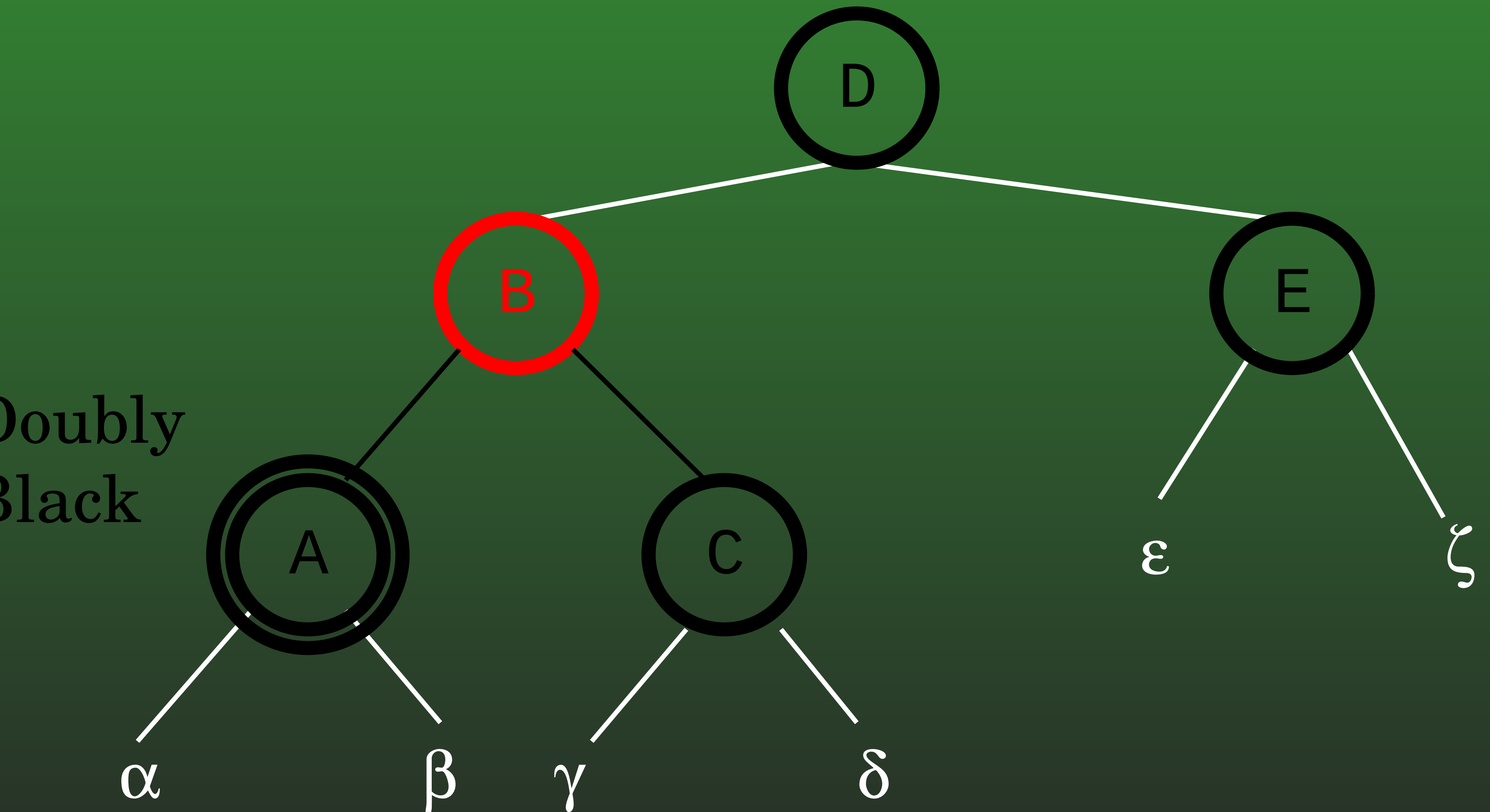
07-41: Deleting Nodes

- X is “doubly black” node, X is a left child
  - Case 2:
    - X’s sibling W is Red
    - Do a rotation, to make W black. Then one of the other cases will apply.

07-42: Deleting Nodes



07-43: Deleting Nodes



07-44: Deleting Nodes

- Need to include symmetric cases
  - In all of the previous examples, swap left/right
  - (Go over at least one example)