Graduate Algorithms

CS673-2016F-17

Shortest Path Algorithms

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Given a directed weighted graph $G$ (all weights non-negative) and two vertices $x$ and $y$, find the least-cost path from $x$ to $y$ in $G$.

- Undirected graph is a special case of a directed graph, with symmetric edges.
- Least-cost path may not be the path containing the fewest edges.
  - “shortest path” == “least cost path”
  - “path containing fewest edges” = “path containing fewest edges”
17-1: **Shortest Path Example**

- Shortest path ≠ path containing fewest edges

![Graph with labeled edges]

- Shortest Path from A to E?
Shortest path ≠ path containing fewest edges

Shortest Path from A to E:
- A, B, C, D, E
To find the shortest path from vertex $x$ to vertex $y$, we need (worst case) to find the shortest path from $x$ to *all* other vertices in the graph.

Why?
To find the shortest path from vertex $x$ to vertex $y$, we need (worst case) to find the shortest path from $x$ to all other vertices in the graph.

- To find the shortest path from $x$ to $y$, we need to find the shortest path from $x$ to all nodes on the path from $x$ to $y$.
- Worst case, all nodes will be on the path.
17-5: Single Source Shortest Path

- If all edges have unit weight ...
17-6: Single Source Shortest Path

- If all edges have unit weight,
- We can use Breadth First Search to compute the shortest path
- BFS Spanning Tree contains shortest path to each node in the graph
  - Need to do some more work to create & save BFS spanning tree
- When edges have differing weights, this obviously will not work
17-7: Single Source Shortest Path

- General Idea for finding Single Source Shortest Path

  - Start with the distance estimate to each node (except the source) as $\infty$
  - Repeatedly relax distance estimate until you can relax no more
  - To relax an edge $(u, v)$
    - $\text{dist}(v) > \text{dist}(u) + \text{cost}((u, v))$
    - Set $\text{dist}(v) \leftarrow \text{dist}(u) + \text{cost}((u, v))$
**17-8: Single Source Shortest Path**

- Dijkstra’s algorithm
  - Relax edges from source
- *Remarkably* similar to Prim’s MST algorithm
  - Pretty neat – algorithms are doing different things, but code is almost identical
Divide the vertices into two sets:
- Vertices whose shortest path from the initial vertex is known
- Vertices whose shortest path from the initial vertex is not known

Initially, only the initial vertex is known

Move vertices one at a time from the unknown set to the known set, until all vertices are known
Start with the vertex A
 Known vertices are circled in red

We can now extend the known set by 1 vertex
Why is it safe to add D, with cost 1?
• Why is it safe to add D, with cost 1?
  • Could we do better with a more roundabout path?
Why is it safe to add D, with cost 1?

- Could we do better with a more roundabout path?
- No – to get to any other node will cost at least 1
- No negative edge weights, can’t do better than 1
We can now add another vertex to our known list ...
How do we know that we could not get to B cheaper by going through D?
17-17: Single Source Shortest Path

- How do we know that we could not get to B cheaper by going through D?
  - Costs 1 to get to D
  - Costs at least 2 to get anywhere from D
    - Cost at least \((1+2 = 3)\) to get to B through D
17-18: Single Source Shortest Path

Next node we can add ...

<table>
<thead>
<tr>
<th>Node</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>1</td>
</tr>
<tr>
<td>E</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td></td>
</tr>
<tr>
<td>G</td>
<td></td>
</tr>
</tbody>
</table>
(We also could have added E for this step)

Next vertex to add to Known ...
Cost to add F is 8 (through C)
Cost to add G is 5 (through D)
17-21: Single Source Shortest Path

- Last node...
We now know the length of the shortest path from A to all other vertices in the graph.
Dijkstra’s Algorithm

Keep a table that contains, for each vertex
  
  - Is the distance to that vertex known?
  - What is the best distance we’ve found so far?

Repeat:
  
  - Pick the smallest unknown distance
  - mark it as known
  - update the distance of all unknown neighbors of that node

Until all vertices are known
Dijkstra’s Algorithm Example

Node | Known | Distance
A    | false | 0
B    | false | ∞
C    | false | ∞
D    | false | ∞
E    | false | ∞
F    | false | ∞
Dijkstra’s Algorithm Example

Node | Known | Distance
--- | --- | ---
A | true | 0
B | false | 7
C | false | 5
D | false | ∞
E | false | ∞
F | false | 1
### Dijkstra’s Algorithm Example

![Diagram of a graph with node distances and known status]

<table>
<thead>
<tr>
<th>Node</th>
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</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>true</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>false</td>
<td>7</td>
</tr>
<tr>
<td>C</td>
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<td>5</td>
</tr>
<tr>
<td>D</td>
<td>false</td>
<td>8</td>
</tr>
<tr>
<td>E</td>
<td>false</td>
<td>3</td>
</tr>
<tr>
<td>F</td>
<td>true</td>
<td>1</td>
</tr>
</tbody>
</table>
17-27: Dijkstra’s Algorithm Example

Node | Known | Distance
-----|-------|--------
A    | true  | 0      
B    | false | 7      
C    | false | 4      
D    | false | 8      
E    | true  | 3      
F    | true  | 1      

Diagram showing weighted graph and Dijkstra's algorithm example.
Dijkstra’s Algorithm Example

Node | Known | Distance
--- | --- | ---
A | true | 0
B | false | 5
C | true | 4
D | false | 6
E | true | 3
F | true | 1
17-29: Dijkstra’s Algorithm Example

### Table

<table>
<thead>
<tr>
<th>Node</th>
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<th>Distance</th>
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</thead>
<tbody>
<tr>
<td>A</td>
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<tr>
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<tr>
<td>F</td>
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</tr>
</tbody>
</table>
17-30: Dijkstra’s Algorithm Example

### Node Table

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</tr>
<tr>
<td>F</td>
<td>true</td>
<td>1</td>
</tr>
</tbody>
</table>

Diagram:

- A to B: 7
- A to C: 1
- A to D: 1
- A to E: 2
- A to F: 5
- B to D: 1
- B to E: 3
- C to D: 2
- C to E: 1
- C to F: 2
- D to B: 3
- D to E: 7
- E to C: 1
- E to F: 2
- F to D: 7

Known distances:
- A: 0
- B: 5
- C: 4
- D: 6
- E: 3
- F: 1
After Dijkstra’s algorithm is complete:

- We know the *length* of the shortest path
- We do not know *what* the shortest path is

How can we modify Dijkstra’s algorithm to compute the path?
After Dijkstra’s algorithm is complete:

- We know the *length* of the shortest path
- We do not know *what* the shortest path is

How can we modify Dijkstra’s algorithm to compute the path?

- Store not only the distance, but the immediate parent that led to this distance
Dijkstra’s Algorithm Example

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Dijkstra’s Algorithm Example

Node | Known | Dist | Path
--- | --- | --- | ---
A | true | 0 | A
B | false | 5 | A
C | false | 3 | A
D | false | ∞ | 
E | false | ∞ | 
F | false | ∞ | 
G | false | ∞ | 

Diagram:
- A to B: 5
- A to C: 3
- A to D: 1
- B to C: 2
- B to E: 4
- C to D: 1
- C to F: 5
- D to E: 5
- D to G: 3
- E to G: 2
- F to G: 1
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Dijkstra’s Algorithm Example

Node | Known | Dist | Path
--- | --- | --- | ---
A | true | 0 | A
B | false | 5 | A
C | true | 3 | A
D | true | 4 | C
E | false | 9 | D
F | false | 9 | D
G | false | 7 | D
## Dijkstra’s Algorithm Example

### Graph

![Graph](image)

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</tr>
<tr>
<td>G</td>
<td>true</td>
<td>7</td>
<td>D</td>
</tr>
</tbody>
</table>
Dijkstra’s Algorithm

- Given the “path” field, we can construct the shortest path
  - Work backward from the end of the path
  - Follow the “path” pointers until the start node is reached
    - We can use a sentinel value in the “path” field of the initial node, so we know when to stop
void Dijkstra(Edge G[], int s, tableEntry T[]) {
    int i, v;
    Edge e;
    for(i=0; i<G.length; i++) {
        T[i].distance = Integer.MAX_VALUE;
        T[i].path = -1;
        T[i].known = false;
    }
    T[s].distance = 0;
    for (i=0; i < G.length; i++) {
        v = minUnknownVertex(T);
        T[v].known = true;
        for (e = G[v]; e != null; e = e.next) {
            if (T[e.neighbor].distance >
                T[v].distance + e.cost) {
                T[e.neighbor].distance = T[v].distance + e.cost;
                T[e.neighbor].path = v;
            }
        }
    }
}
If minUnknownVertex(T) is calculated by doing a linear search through the table:

- Each minUnknownVertex call takes time $\Theta(|V|)$
- Called $|V|$ times – total time for all calls to minUnkownVertex: $\Theta(|V|^2)$
- If statement is executed $|E|$ times, each time takes time $O(1)$
- Total time: $O(|V|^2 + |E|) = O(|V|^2)$.
17-44: Dijkstra Running Time

- If minUnknownVertex(T) is calculated by inserting all vertices into a min-heap (using distances as key) updating the heap as the distances are changed
  - Each minUnknownVertex call takes time $\Theta(lg |V|)$
  - Called $|V|$ times – total time for all calls to minUnknownVertex: $\Theta(|V| lg |V|)$
- If statement is executed $|E|$ times – each time takes time $O(lg |V|)$, since we need to update (decrement) keys in heap
- Total time: $O(|V| lg |V| + |E| lg |V|) \in O(|E| lg |V|)$
If minUnknownVertex(T) is calculated by inserting all vertices into a Fibonacci heap (using distances as key) updating the heap as the distances are changed

- Each minUnknownVertex call takes amortized time $\Theta(\lg |V|)$
  - Called $|V|$ times – total amortized time for all calls to minUnknownVertex: $\Theta(|V| \lg |V|)$
- If statement is executed $|E|$ times – each time takes amortized time $O(1)$, since decrementing keys takes time $O(1)$.
- Total time: $O(|V| \lg |V| + |E|)$
17-46: **Negative Edges**

- Does Dijkstra’s algorithm work when edge costs can be negative?
  - Give a counterexample!

- What happens if there is a negative-weight cycle in the graph?
Bellman-Ford

• Bellman-Ford allows us to calculate shortest paths in graphs with negative edge weights, as long as there are no negative-weight cycles

• As a bonus, we will also be able to detect negative-weight cycles
For each node $v$, maintain:
- A “distance estimate” from source to $v$, $d[v]$
- Parent of $v$, $\pi[v]$, that gives this distance estimate

Start with $d[v] = \infty$, $\pi[v] = \text{nil}$ for all nodes

Set $d[\text{source}] = 0$

Update estimates by “relaxing” edges
• Relaxing an edge \((u, v)\)
  • See if we can get a better distance estimate for \(v\) by going through \(u\)

\[
\text{Relax}(u,v,w) \\
\text{if } d[v] > d[u] + w(u, v) \\
\quad d[v] \leftarrow d[u] + w(u, v) \\
\quad \pi[v] \leftarrow u
\]
Bellman-Ford

- Relax all edges edges in the graph (in any order)
- Repeat until relax steps cause no change
  - After first relaxing, all optimal paths from source of length 1 are computed
  - After second relaxing, all optimal paths from source of length 2 are computed
  - After \(|V| - 1\) relaxing, all optimal paths of length \(|V| - 1\) are computed
  - If some path of length \(|V|\) is cheaper than a path of length \(|V| - 1\) that means ...
Bellman-Ford

- Relax all edges in the graph (in any order)
- Repeat until relax steps cause no change
  - After first relaxing, all optimal paths from source of length 1 are computed
  - After second relaxing, all optimal paths from source of length 2 are computed
  - After $|V| - 1$ relaxing, all optimal paths of length $|V| - 1$ are computed
  - If some path of length $|V|$ is cheaper than a path of length $|V| - 1$ that means ...
    - Negative weight cycle
Begin Bellman-Ford:

BellamanFord(\(G, s\))

Initialize \(d[], \pi[]\)

for \(i \leftarrow 1\) to \(|V| - 1\) do

for each edge \((u, v) \in G\) do

if \(d[v] > d[u] + w(u, v)\)

\(d[v] \leftarrow d[u] + w(u, v)\)

\(\pi[v] \leftarrow u\)

for each edge \((u, v) \in G\) do

if \(d[v] > d[u] + w(u, v)\)

return false

return true
17-53: Bellman-Ford

• Running time:
  • Each iteration requires us to relax all $|E|$ edges
  • Each single relaxation takes time $O(1)$
  • $|V| - 1$ iterations ($|V|$ if we are checking for negative weight cycles)
  • Total running time $O(|V| \times |E|)$
Finding Single Source Shorest path in a Directed, Acyclic graph

Very easy! How can we do this quickly?
Finding Single Source Shortest path in a Directed, Acyclic graph

Very easy!

How can we do this quickly?
  - Do a topological sort
  - Relax edges in topological order
  - We’re done!
All-Source Shortest Path

- What if we want to find the shortest path from all vertices to all other vertices?
- How can we do it?
All-Source Shortest Path

• What if we want to find the shortest path from all vertices to all other vertices?

• How can we do it?
  • Run Dijkstra’s Algorithm $V$ times
  • How long will this take?
What if we want to find the shortest path from all vertices to all other vertices?

How can we do it?

- Run Dijkstra’s Algorithm $V$ times
- How long will this take?
  - $\Theta(V^2 \log V + V \cdot E)$ (using Fibonacci heaps)
  - Doesn’t work if there are negative edges!

Running Bellman-Ford $V$ times (which does work with negative edges) takes time $O(V^2 E)$ – which is $\Theta(V^4)$ for dense graphs.
Let $L^{(m)}[i, j]$ (in text, $l^{(m)}_{i,j}$) be cost of the shortest path from $i$ to $j$ that contains at most $m$ edges.

If $m = 0$, there is a shortest path from $i$ to $j$ with no edges iff $i = j$.

$$L^{(0)}[i, j] = \begin{cases} 0 & \text{if } i = j \\ \infty & \text{otherwise} \end{cases}$$

How can we calculate $L^m[i, j]$ recursively?
Multi-Source Shortest Path

- Let \( L^{(m)}[i, j] \) (in text, \( l^{(m)}_{i,j} \)) be cost of the shortest path from \( i \) to \( j \) that contains at most \( m \) edges

\[
L^{(0)}[i, j] = \begin{cases} 
0 & \text{if } i = j \\
\infty & \text{otherwise}
\end{cases}
\]

- How can we calculate \( L^m[i, j] \) recursively?

\[
L^{(m)}[i, j] = \min \left( L^{(m-1)}[i, j], \min_{1 \leq k \leq n} (L^{(m-1)}[i, k] + w_{kj}) \right)
\]

\[
= \min_{1 \leq k \leq n} (L^{(m-1)}[i, k] + w_{kj})
\]
Multi-Source Shortest Path

- Create $L^{(m+1)}$ from $L^{(m)}$:

```
Extend-Shortest-Paths($L, W$)

n ← rows[$L$]
$L'$ ← new $n \times n$ matrix

for $i ← 1$ to $n$ do
  for $j ← 1$ to $n$ do
    for $k ← 1$ to $n$ do
      $L'[i, j] ← \infty$
    for $k ← 1$ to $n$ do
      $L'[i, j] ← \min(L'[i, j], L[i, k] + W[k, j])$

return $L'$
```
17-62: Multi-Source Shortest Path

- Need to calculate $L^{(n-1)}$
- Why $L^{(n-1)}$, and not $L^{(n)}$ or $L^{(n+1)}$?

All-Pairs-Shortest-Paths($W$)

$n \leftarrow \text{rows}[W]$
$L^{(1)} \leftarrow W$

for $m \leftarrow 2$ to $n - 1$ do

$L^{(m)} \leftarrow \text{Extend-Shortest-Path}(L^{(m-1)}, W)$

return $L^{(n-1)}$
We really don’t care about any of the $L$ matrices except $L^{(n-1)}$

We can save some time by not calculating all of the intermediate matrices $L^{(1)} \ldots L^{(n-2)}$

Note that Extend-Shortest-Path looks a lot like matrix multiplication
Square-Matrix-Multiply($A$, $B$)

$n \leftarrow \text{rows}[A]$

$C \leftarrow \text{new } n \times n \text{ matrix}$

for $i \leftarrow 1$ to $n$ do

  for $j \leftarrow 1$ to $n$ do

    $C[i, j] \leftarrow 0$

    for $k \leftarrow 1$ to $n$ do

      $C[i, j] \leftarrow C[i, j] + A[i, k] \times B[k, j])$

return $L'$

- Replace min with $+$, $+$ with $\times$
Using our “Extend-Multiplication”
- Replace + with min, * with +

\[
\begin{align*}
L^{(1)} &= L^{(0)} \ast W = W \\
L^{(1)} &= L^{(1)} \ast W = W^2 \\
L^{(2)} &= L^{(2)} \ast W = W^3 \\
L^{(3)} &= L^{(3)} \ast W = W^4 \\
&\vdots \\
L^{(n-1)} &= L^{(n-2)} \ast W = W^{n-1}
\end{align*}
\]
Multi-Source Shortest Path

\[ \begin{align*}
L^{(1)} &= W \\
L^{(2)} &= W^2 = W \cdot W \\
L^{(4)} &= W^4 = W^2 \cdot W^2 \\
L^{(8)} &= W^8 = W^4 \cdot W^4 \\
&\vdots \\
L^{2^{\lfloor \log(n-1) \rfloor}} &= L^{2^{\lfloor \log(n-1) \rfloor}} = L^{2^{\lfloor \log(n-1) \rfloor} - 1} \cdot L^{2^{\lfloor \log(n-1) \rfloor} - 1}
\end{align*} \]

- Since \( L^{(n-1)} = L^{(n)} = L^{(n+1)} = \ldots \), it doesn’t matter if \( n \) is an exact power of 2 – we just need to get to at least \( L^{(n-1)} \), not hit it exactly.
All-Pairs-Shortest-Paths($W$)

$n \leftarrow \text{rows}[W]$
$L^{(1)} \leftarrow W$
$m \leftarrow 1$

while $m < n - 1$ do

$L^{(2m)} \leftarrow \text{Extend-Shortest-Path}(L^{(m)}, L^{(m)})$

$m \rightarrow m \times 2$

return $L^{(m)}$
Each call to Extend-Shortest-Path takes time:

# of calls to Extend-Shortest-Path:

Total time:
17-69: Multi-Source Shortest Path

- Each call to Extend-Shortest-Path takes time $\Theta(|V|^3)$
- # of calls to Extend-Shortest-Path: $\Theta(\lg |V|)$
- Total time: $\Theta(|V|^3 \lg |V|)$
17-70: *Floyd’s Algorithm*

- Alternate solution to all pairs shortest path
- Yields $\Theta(V^3)$ running time for all graphs
Floyd’s Algorithm

- Vertices numbered from 1..n
- $k$-path from vertex $v$ to vertex $u$ is a path whose intermediate vertices (other than $v$ and $u$) contain only vertices numbered $k$ or less
- 0-path is a direct link
- Shortest 0-path from 1 to 5: 5
- Shortest 1-path from 1 to 5: 5
- Shortest 2-path from 1 to 5: 4
- Shortest 3-path from 1 to 5: 4
- Shortest 4-path from 1 to 5: 3
• Shortest 0-path from 1 to 3: 7
• Shortest 1-path from 1 to 3: 7
• Shortest 2-path from 1 to 3: 6
• Shortest 3-path from 1 to 3: 6
• Shortest 4-path from 1 to 3: 6
• Shortest 5-path from 1 to 3: 4
17-74: Floyd’s Algorithm

- Shortest $n$-path = Shortest path
- Shortest 0-path:
  - $\infty$ if there is no direct link
  - Cost of the direct link, otherwise
Floyd’s Algorithm

- Shortest \( n \)-path = Shortest path
- Shortest 0-path:
  - \( \infty \) if there is no direct link
  - Cost of the direct link, otherwise
- If we could use the shortest \( k \)-path to find the shortest \( (k + 1) \) path, we would be set
Floyd’s Algorithm

- Shortest $k$-path from $v$ to $u$ either goes through vertex $k$, or it does not
- If not:
  - Shortest $k$-path = shortest $(k - 1)$-path
- If so:
  - Shortest $k$-path = shortest $k - 1$ path from $v$ to $k$, followed by the shortest $k - 1$ path from $k$ to $w$
If we had the shortest $k$-path for all pairs $(v, w)$, we could obtain the shortest $k + 1$-path for all pairs.

For each pair $v, w$, compare:

- length of the $k$-path from $v$ to $w$
- length of the $k$-path from $v$ to $k$ appended to the $k$-path from $k$ to $w$

Set the $k + 1$ path from $v$ to $w$ to be the minimum of the two paths above.
Floyd’s Algorithm

Let $D_k[v, w]$ be the length of the shortest $k$-path from $v$ to $w$.

$D_0[v, w] =$ cost of arc from $v$ to $w$ ($\infty$ if no direct link)

$D_k[v, w] = \text{MIN}(D_{k-1}[v, w], D_{k-1}[v, k] + D_{k-1}[k, w])$

Create $D_0$, use $D_0$ to create $D_1$, use $D_1$ to create $D_2$, and so on – until we have $D_n$
17-79: Floyd’s Algorithm

- Use a doubly-nested loop to create $D_k$ from $D_{k-1}$
- Use the same array to store $D_{k-1}$ and $D_k$ – just overwrite with the new values
- Embed this loop in a loop from 1..k
Floyd’s Algorithm

Floyd(Edge G[], int D[][])
{
    int i, j, k
    
    Initialize D, D[i][j] = cost from i to j
    
    for (k=0; k<G.length; k++)
        for (i=0; i<G.length; i++)
            for (j=0; j<G.length; j++)
                if ((D[i][k] != Integer.MAX_VALUE) &&
                    (D[k][j] != Integer.MAX_VALUE) &&
                    (D[i][j] > (D[i][k] + D[k][j])))
                    D[i][j] = D[i][k] + D[k][j]
}
17-81: Floyd’s Algorithm

- We’ve only calculated the *distance* of the shortest path, not the path itself.
- We can use a similar strategy to the PATH field for Dijkstra to store the path:
  - We will need a 2-D array to store the paths: $P[i][j] = \text{last vertex on shortest path from } i \text{ to } j$
Johnson’s Algorithm

- Yet another all-pairs shortest path algorithm
- Time $O(|V|^2 \log |V| + |V| \times |E|)$
  - If graph is dense ($|E| \in \Theta(|V|^2)$), no better than Floyd
  - If graph is sparse, better than Floyd
- Basic Idea: Run Dijkstra $|V|$ times
  - Need to modify graph to remove negative edges
17-83: Johnson’s Algorithm

• Reweighing Graph
  • Create a new weight function $\hat{w}$, such that:
    • For all pairs of vertices $u, v \in V$, a path from $u$ to $v$ is a shortest path using $w$ if and only if it is also a shortest path using $\hat{w}$.
    • For all edges $(u, v)$, $\hat{w}(u, v)$ is non-negative
17-84: Johnson’s Algorithm

• Reweighing Graph
  • First Try:
  • Smallest weight is $-w$, for some positive $w$
  • Add $w$ to each edge in the graph
  • Is this a valid reweighing?
17-85: **Johnson’s Algorithm**

- **Reweighing Graph**
  - **First Try:**
  - Smallest weight is $-w$, for some positive $w$
  - Add $w$ to each edge in the graph
  - Is this a valid reweighing?

![Graph with weights](image)
Reweighing Graph

Second Try:

Define some function on vertices $h(v)$

$\hat{w}(u, v) = w(u, v) + h(u) - h(v)$

Does this preserve shortest paths?
Johnson’s Algorithm

- Let $p = v_0, v_1, v_2, \ldots, v_k$ be a path in $G$
- Cost of $p$ under $\hat{w}$:

\[
\hat{w}(p) = \sum_{i=1}^{k} \hat{w}(v_{i-1}, v_i)
\]

\[
= \sum_{i=1}^{k} (w(v_{i-1}, v_i) + h(v_{i-1}) - h(v_i))
\]

\[
= \left( \sum_{i=1}^{k} (w(v_{i-1}, v_i)) + h(v_0) - h(v_k) \right)
\]

\[
= w(p) + h(v_0) - h(v_k)
\]
So, if we can come up with a function \( h(V) \) such that \( w(u, v) + h(u) - h(v) \) is positive for all edges \((u, v)\) in the graph, we’re set.

- Use the function \( h \) to reweigh the graph.
- Run Dijkstra’s algorithm \(|V|\) times, starting from each vertex on the new graph, calculating shortest paths.
- Shortest path in new graph = shortest path in old graph.
17-89: Johnson’s Algorithm

- Add a new vertex $s$ to the graph
- Add an edge from $s$ to every other vertex, with cost 0
- Find the shortest path from $s$ to every other vertex in the graph
- $h(v) = \delta(s, v)$, the cost of the shortest path from $s$ to $v$
  - Using this $h(V)$ function, all new weights are guaranteed to be non-negative
Johnson’s Algorithm

- \( h(v) = \delta(s, v) \), the cost of the shortest path from \( s \) to \( v \)

\[
\hat{w}(u, v) = w(u, v) + h(u) - h(v)
\]

\[
= w(u, v) + \delta(s, u) - \delta(s, v)
\]

- Since \( \delta \) is a shortest path,

\[
\delta(s, v) \leq \delta(s, u) + w(u, v)
\]

\[
0 \leq w(u, v) + \delta(s, u) - \delta(s, v)
\]
17-91: Johnson’s Algorithm
17-93: Johnson’s Algorithm
17-94: Johnson’s Algorithm

The diagram illustrates a network with nodes labeled A, B, C, D, and E, and edges with associated values. The values are given as fractions, indicating costs or weights. The diagram shows directed edges between the nodes with specific values associated with each edge. For example, the edge from A to B has a value of $-2/0$, and the edge from C to B has a value of $-3/0$. The specific values and their directions are as follows:

- From A to B: $-2/0$
- From A to C: $5/4$
- From A to D: $3/4$
- From B to C: $4/1$
- From B to D: $2/1$
- From C to D: $2/2$
- From C to E: $3/5$
- From B to E: $-3/0$
- From D to E: $-2/0$
- From D to A: $1/0$
- From E to A: $-1$
- From E to B: $-3$
- From E to C: $0$

The diagram is used to represent the application of Johnson’s Algorithm, which is a method for solving the shortest path problem in weighted graphs.
17-95: Johnson’s Algorithm
17-96: Johnson’s Algorithm

Johnson(G)
Add s to G, with 0 weight edges to all vertices
if Bellman-Ford(G, s) = FALSE
    There is a negative weight cycle, fail
for each vertex v ∈ G
    set h(v) ← δ(s, v) from B-F
for each edge (u, v) ∈ G
    \( \hat{w}(u, v) = w(u, v) + h(u) - h(v) \)
for each vertex u ∈ G
    run Dijkstra(G, \( \hat{w}, u \)) to compute \( \hat{\delta}(u, v) \)
    \( \delta(u, v) = \hat{\delta}(u, v) + h(v) - h(u) \)