Note Title 8/14/2014

THM A SET, A, IS COMPUTABLE IFF A AND $\overline{A}$ ARE BOTH c.e.

- A comp: PROGRAM, ON INPUT n, OUTPUTS 1 IF $n \in A$, 0 IF $n \notin A$.
- A c.e.: PROGRAM: ON INPUT n, OUTPUTS 1 IF $n \in \overline{A}$, $\uparrow$ IF $n \notin \overline{A}$.
  $\overline{A}$
  OR
  ON INPUT n, OUTPUTS n^{th} ELEMENT OF $\overline{A}$.

PROOF: $\rightarrow$ ASSUME A IS COMP.

INPUT n.
IF $\chi_A(n) = 1^{0}$, HALT
IF $\chi_A(n) = 0^{1}$, $\uparrow$

$\leftarrow$ A, $\overline{A}$ ARE c.e.

LET f_1 AND f_2 BE COMPUTABLE FUNCTIONS WITH RANGE A, $\overline{A}$, RESPECTIVELY.

INPUT n
SET $i = 0$.
1. IS $f_1(i) = n$? YES, OUTPUT 1
IS $f_2(i) = n$? YES, OUTPUT 0.
$i = i+1$, GOTO 1.

THM THE HALTING SET $\mathbb{K} = \{ e \mid \varphi_e(e)\downarrow \}$ IS NOT COMPUTABLE.

PROOF BY WAY OF CONTRADICTION, ASSUME $\mathbb{K}$ IS COMPUTABLE.

THEN, THERE IS A COMPUTABLE FUNCTION $K(n) = \begin{cases} 1 & n \in \mathbb{K} \\ 0 & n \notin \mathbb{K} \end{cases}$.

LET e BE THE GÖDEL INDEX FOR THE FOLLOWING PROGRAM:

INPUT n
1. IF $K(n) = 0$, OUTPUT 1.
ELSE, GOTO 1.

NOW, IF K IS COMPUTABLE, PROGRAM e CAN BE COMPUTED BY A TM. CONSIDER THE EXECUTION OF THE PROGRAM ON INPUT e...

$$\varphi_e(e)\downarrow \text{ iff } \varphi_e(e)\uparrow.$$

THIS IS A CONTRADICTION; WE CONCLUDE $\mathbb{K}$ IS NOT COMPUTABLE.

FIND COMPUTABLE FUNCTION. $f(n,s)$. S.T.

$$\lim_{s\to\infty} f(n,s) = n^{th} \text{ smallest element of } \mathbb{K}.$$

INPUTS n, s.

RUN $\varphi_{i,s}(i)$.

OUTPUT THE n^{th} i S.T. IS NOT -1.

IN THE LIMIT, IF $\varphi_i(i)$ IS GOING TO HALT, EVENTUALLY, s IS BIG ENOUGH SO THAT $\varphi_{i,s}(i) \neq -1$.

Example. Let $B_n = \{n\}$.

Example. Let $A_i = \{n \in \mathbb{N} \mid n \neq i\}$.

INPUT i

COMPUTE THE CODE OF THE PROGRAM:

" INPUT x

1. IF $x \neq i$, OUTPUT 1.

ELSE, GOTO 1. "

COMPUTING INDICES FOR SETS IN A FAMILY.

INPUT n

COMPUTE THE CODE OF THE PROGRAM:

" INPUT x

1. IF $x = n$, OUTPUT 1.

ELSE, GOTO 1. "

OUTPUT THE CODE.