MONDAY, JUNE 1 AM

Note Title 6/1/2009

Book "HIGH SCHOOL MATHEMATICS FROM AN ADVANCED PERSPECTIVE"

NUMBER SYSTEMS

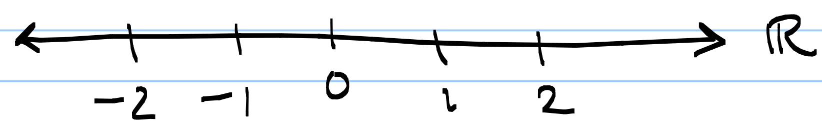
$\mathbb{N}$ - NATURAL NUMBERS - COUNTING NUMBERS
$0, 1, 2, 3, \ldots$

$\mathbb{Z}$ - INTEGERS - WHOLE NUMBERS
$\ldots -2, -1, 0, 1, 2, 3, \ldots$

$\mathbb{Q}$ - RATIONAL NUMBERS - FRACTIONS
↳ RATIOS OF INTEGERS $\frac{2}{1}, \frac{0}{1}, \frac{-5}{3}$

$\mathbb{P}$ - IRRATIONAL NUMBERS
$\pi, e, \sqrt{2}, \sqrt{3}, \sqrt[3]{9}$

$\mathbb{R}$ - RATIONALS + IRRATIONALS



(CANTOR: $2^{\aleph_0} = \aleph_1$)

$\mathbb{C}$ - COMPLEX NUMBERS

$i = \sqrt{-1}$, $2i, 3i, -\pi i$ $\Big\}$ $b \cdot i$ IMAGINARY NUMBERS

imaginary unit

$3+\sqrt{-1} = 3+i \rightarrow$ COMPLEX NUMBER

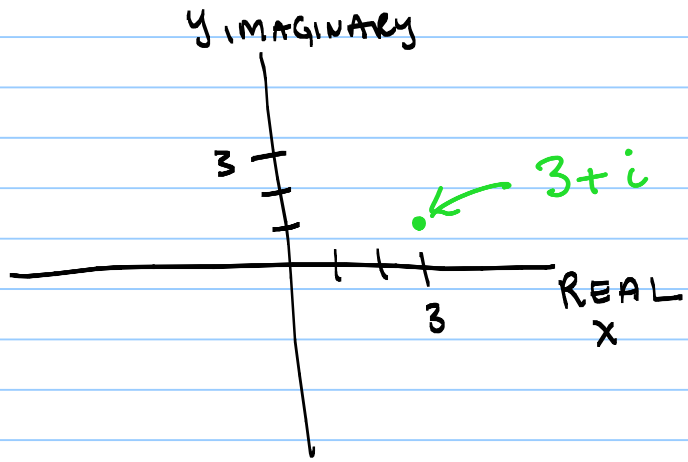
CLOSURE OF REAL NUMBERS AND IMAGINARY NUMBERS UNDER $+$.

$z = x + yi$

real part

imaginary part.

COMPLEX PLANE



$i = \sqrt{-1} = i^5$

$i^2 = -1 = i^6$

$i^3 = -i = i^7$

$i^4 = 1 = i^8$

OPERATIONS ON COMPLEX NUMBERS

$+ :\quad (3+2i) + (7-3i) = 10 - i$

$- :\quad (3+2i) - (7-3i) = -4 + 5i$

$\cdot :\quad (3+2i)(7-3i) = 21 - 9i + 14i - 6i^2 \quad +6$

$= 27 + 5i$

$\div :\quad \dfrac{3+2i}{7-3i} = \dfrac{3+2i}{(7-3i)} \cdot \dfrac{7+3i}{(7+3i)}$

$= \dfrac{21+9i+14i-6}{49-(9i^2)}$

$= \dfrac{15+23i}{58} = \dfrac{15}{58} + \dfrac{23}{58}i$

$x+yi$ is conjugate to $x-yi$

COMPLEX CONJUGATE

$a^2 - b^2 = (a-b)(a+b)$

SETS OF NUMBERS

- DESCRIPTION WITH WORDS "THE EVEN NUMBERS."
- LIST (FINITE, INFINITE) $\{0, 2, 4, 6, \ldots\} = A$

  $\{6\} = B$

- SET BUILDER NOTATION

$$\{x \in \mathbb{N} \mid \underbrace{x \text{ is even.}}_{\text{condition.}}\}$$

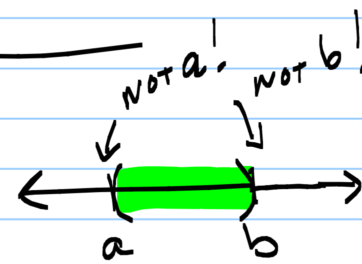
"IS A MEMBER OF" ($\in$)

UNIVERSE ($\mathbb{N}$)

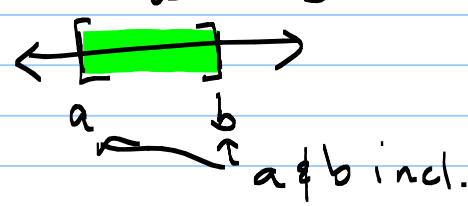
$$\mathbb{Q} = \left\{\frac{a}{b} \,\middle|\, a, b \in \mathbb{Z}\right\}$$

INTERVALS

open interval $(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$



closed interval $[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$



half-open $(a,b] = \{x \in \mathbb{R} \mid a < x \leq b\}$

OPERATION: HAS INPUT AND OUTPUT: $+, \cdot, -, \div$

RELATION: HAS INPUT & OUTPUTS TRUE OR FALSE

$2 < 3 \rightarrow$ TRUE

$2 < -3 \rightarrow$ FALSE

PROPERTIES OPERATIONS CAN HAVE RELATIVE TO A CHOSEN NUMBER SYSTEM

COMMUTATIVITY $a+b=b+a$ ← $+$ is commutative

$a-b \neq b-a$ ← $-$ is not.

ASSOCIATIVITY $(a+b)+c = a+(b+c)$

CLOSURE $\mathbb{N}$ ARE CLOSED UNDER $+, \cdot$

$a+b$, $a \cdot b$ WILL BE IN $\mathbb{N}$

$\mathbb{N}$ ARE NOT CLOSED UNDER $-, \div$

$5-7=-2 \notin \mathbb{N}$

$\mathbb{Z}$ $+, \cdot, -$

$\mathbb{Q}$ $+, \cdot, -, \div$ ← NO SOL'N $x^2-2=0$

$\mathbb{R}$ $+, \cdot, -, \div$ ← HAS A SOL'N

$\mathbb{C}$ $+, \cdot, -, \div$ $a+bi$

DISTRIBUTIVITY OF $\cdot$ OVER $+$

$a(b+c) = ab+ac$

<u>Ex</u> $(a,b) \equiv_{def} a^b$

$(a,(b,c)) = a^{(b^c)}$ $((a,b),c) = (a^b)^c$

COMMUTATIVE? No. $(2,3) \neq (3,2)$, $2^3 \neq 3^2$ } Counter-example

ASSOCIATIVE? No.

$(2,(3,2)) \stackrel{?}{=} ((2,3),2)$

$2^9 \stackrel{?}{=} 8^2$

$2^9 \stackrel{?}{=} (2^3)^2 = 2^6$ No!

$a^{b^c} = a^{(b^c)}$

$\neq$

$(a^b)^c = a^{bc}$

PROPERTIES OF RELATIONS ($<$)

TRANSITIVE: IF $a < b$ & $b < c$ THEN $a < c$ — IF aRb AND bRc THEN aRc.

REFLEXIVE: $x < x$ NO! — aRa

$x \leq x$ YES!

SYMMETRIC: $x = y$ THEN $y = x$ — IF aRb THEN bRa

ANTISYMMETRIC: IF $x < y$ THEN $y \not< x$ — IF aRb THEN $\neg(bRa)$

$\neg$ NOT.

• EXAMPLE: $x \sim y$ IFF $y - x$ IS AN INTEGER (IN $\mathbb{R}$)

SYMMETRIC? YES - $|y-x| = |x-y|$

TRANSITIVE? $x \sim y$ & $y \sim z$, IS $x \sim z$?

IF $\underbrace{y-x \text{ IS A WHOLE NUMBER} \ \&\ z-y \text{ IS A WHOLE NUMBER}}$ THEN $x \sim z$.

SCRATCH

$1.25 \sim -3.75$

$-3.75 \sim -0.75 \to 1.25 \sim -0.75$

SINCE $z - x = (z - \underbrace{y) + (y}_{\text{ADD } 0.} - x)$

AND $z-y$ AND $y-x$ ARE INTEGERS

AND THE INTEGERS ARE CLOSED UNDER $+$,

$z-x$ MUST ALSO BE AN INTEGER.

PROOF.

REFLEXIVE? $x \sim x$? Yes!

IF A RELATION IS TRANS., REFL., <u>AND</u> SYMMETRIC

THEN IT IS AN <u>EQUIVALENCE RELATION</u>.

ALGEBRA

EXPONENTS / RADICALS

$$a^b c^b = (ac)^b$$

$$a^b a^c = a^{b+c} \rightsquigarrow \underbrace{a \cdot a \cdots a}_{b} \cdot \underbrace{a \cdots a}_{c} = a^{b+c}$$

$$(a^b)^c = a^{bc}$$

$$\frac{a^b}{a^c} = a^{b-c}$$

$$a^{-b} = \frac{1}{a^b}$$

$$\sqrt[p]{a} = a^{1/p}$$

$$\sqrt{a} = a^{1/2}$$

?

PRINCIPAL SQUARE ROOTS (LE)

$$\sqrt{4} = 2$$

$$-\sqrt{4} = -2$$

$$x^2 = 4$$

$$x = \pm 2 \quad \checkmark$$

ALGEBRAIC EXPRESSIONS

POLYNOMIALS IN 1 VARIABLE

coeff

$$3x^2 - 2x + 1$$

CONSTANT TERM

Coeff.

TERMS

LEADING TERM

$3x^2$

LEADING COEFFICIENT

3

Degree is 2.

$$\frac{-5x^7 - 3x}{x} = -5x^6 - 3 \quad , x \neq 0.$$

PASCAL'S TRIANGLE $\qquad (x+y)^n$

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
```

$n=0$: 1

$n=1$: $x+y$

$n=2$: $x^2+2xy+y^2$

$n=3$: $x^3+3x^2y+3xy^2+y^3$

$n=4$: $\cdots$

QUADRATIC FORMULA

IF $\quad ax^2+bx+c=0$

THEN $\quad x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

EQUATION

$\sim\sim = \sim\sim$

$2x = 4$ $\leftarrow$ SOLUTION : $x=2$

$x^2 = 4$ $\leftarrow$ SOLUTION SET : $\{2, -2\}$

HANDY PROPERTY OF 0 :

IF $x \cdot y = 0$ THEN EITHER x OR $y = 0$.

$x^2 - 3 = 7x + 4$

$x^2 - 7x - 7 = 0$

$x^4 - x^3 = 0$

IF $x^3(x-1) = 0$

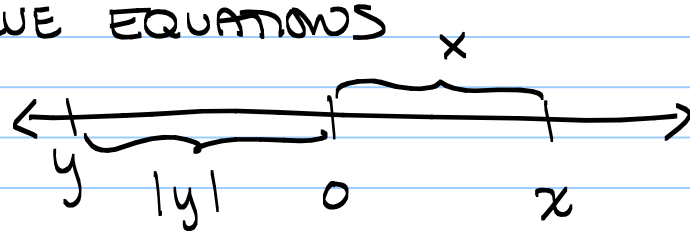
THEN $x^3 = 0$ OR $(x-1) = 0$

$x = 0$ OR $x = \underline{1}$.

ABSOLUTE VALUE EQUATIONS

$|x|$

$\uparrow$ DISTANCE BETWEEN 0 AND x ON THE NUMBER LINE



EX $2|x-4| + 7 = 3$

$|x-4| = -2$

$\{\} = \emptyset$ EMPTY SET

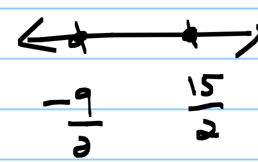
ISOLATE $|\cdot|$

EX $|2x-3| = 12$

$2x-3 = 12 \rightsquigarrow x = \frac{15}{2}$

OR $2x-3 = -12 \rightsquigarrow x = \frac{-9}{2}$

$\left\{ \frac{15}{2}, \frac{-9}{2} \right\}$



ROOT EQUATION

$(\sqrt{x})^2 = (4)^2$

$x = 16$

$x = 2 \leftarrow$ 1 SOL'N

$\downarrow$ SQUARING INTRODUCED AN EXTRANEOUS SOL'N

$x^2 = 4 \leftarrow$ 2 SOL'NS

INEQUALITIES

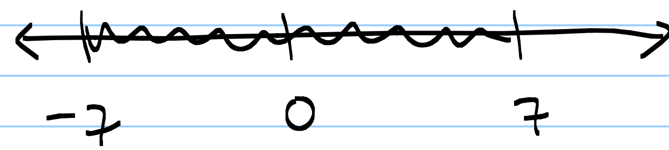
~ $<$ ~

$\leq$

$>$

$\geq$

$|2x-3| < 7$

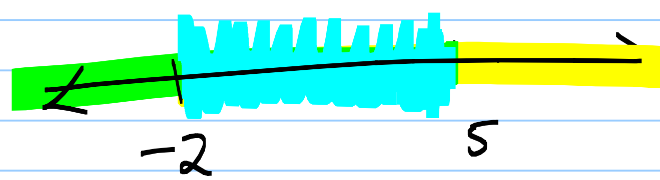


$2x-3 > -7$ <u>AND</u> $2x-3 < 7$

$2x > -4$

$2x < 10$

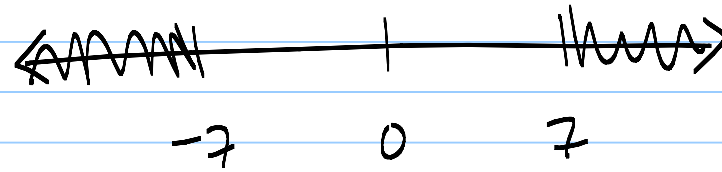
$x > -2$ <u>AND</u> $x < 5$



$(-2, 5)$

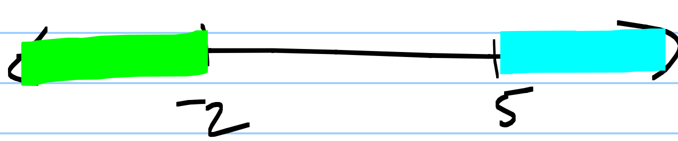
$\{x \in \mathbb{R} \mid -2 < x < 5\}$

$|2x-3| > 7$



$2x-3 < -7$ <u>OR</u> $2x-3 > 7$

$x < -2$ $x > 5$



$(-\infty, -2) \cup (5, \infty)$ UNION