Thursday, June 4, 2009

Note Title 6/4/2009

Functions — Exercises

#3

$$f(\theta) = 4\cos(2\theta) + 1$$

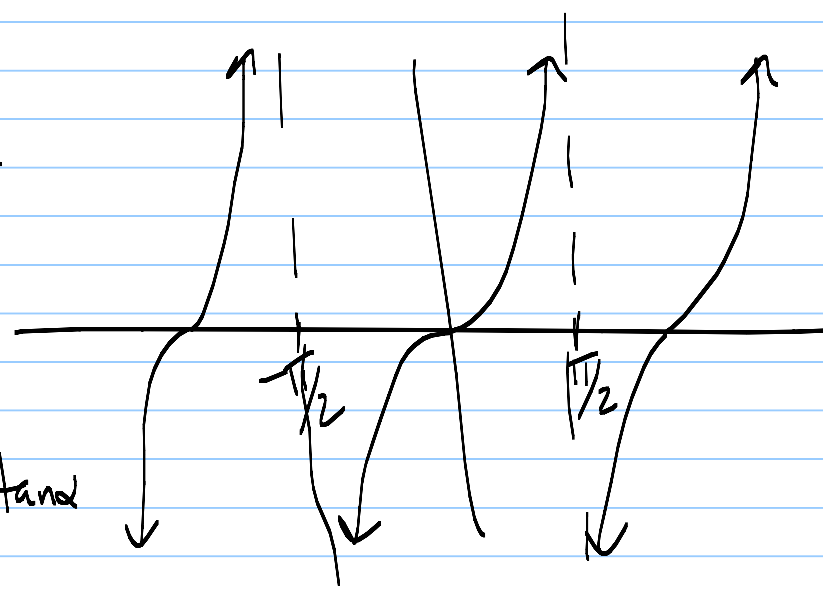
$$f(-\theta) = 4\cos(-2\theta) + 1$$

$$= 4\cos(2\theta) + 1 = f(\theta)$$, so f is EVEN.

$$g(\alpha) = \tan\alpha$$

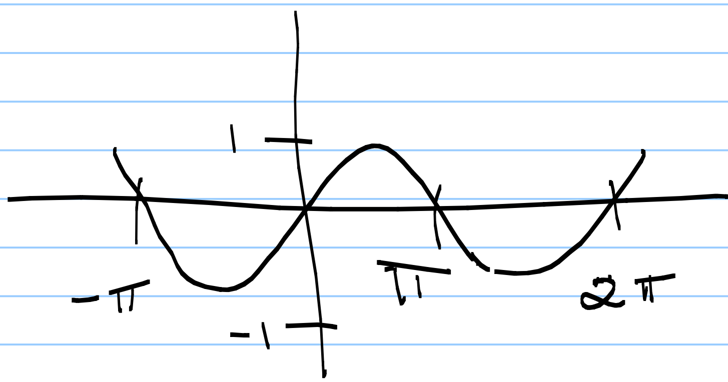
$$g(-\alpha) = \frac{\sin(-\alpha)}{\cos(-\alpha)}$$

$$= \frac{-\sin(\alpha)}{\cos(\alpha)} = -\tan\alpha$$



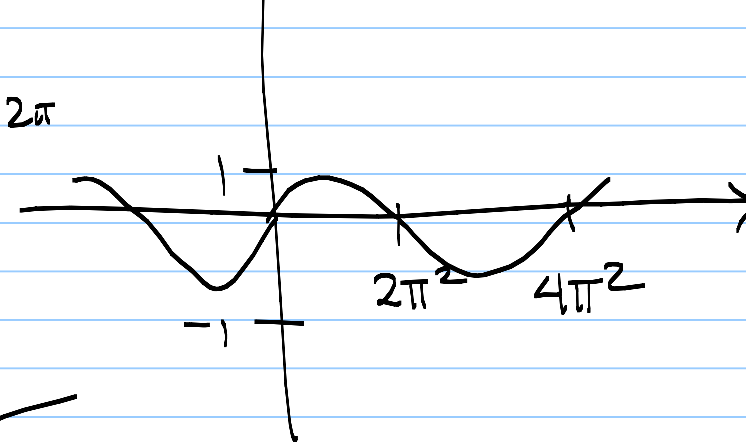
#4

$\sin(t)$



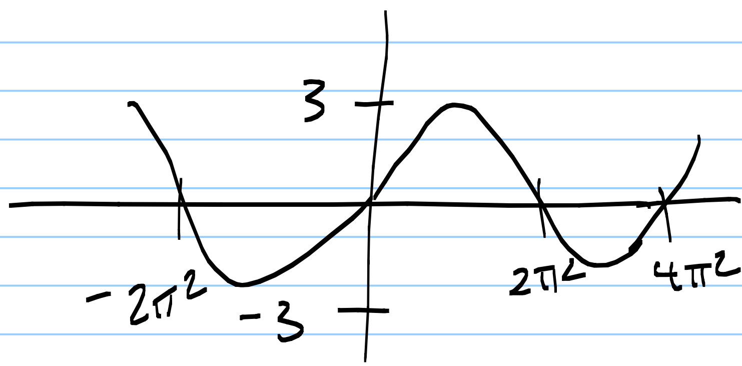
strech
hori. $\tilde{z}$ by 2π

$\sin\left(\frac{t}{2\pi}\right)$

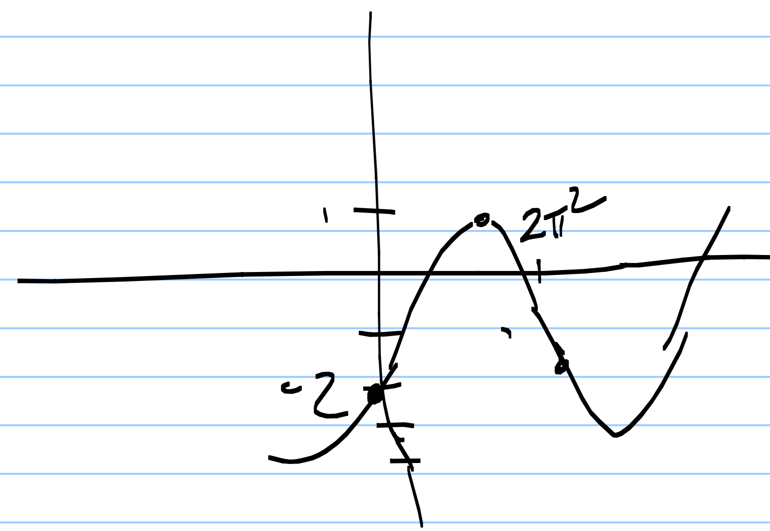


strech vert
by 3

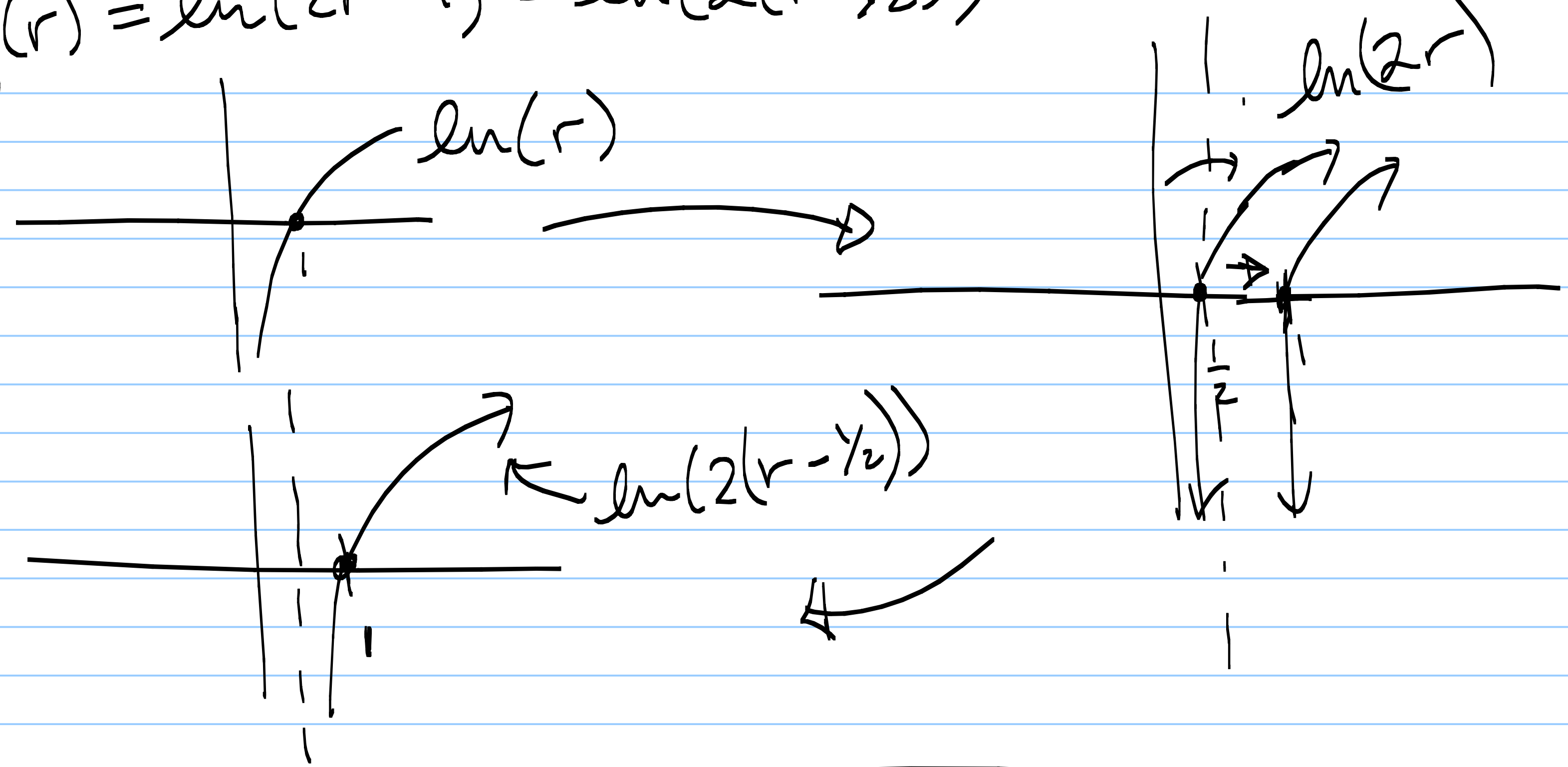
$3\sin\left(\frac{t}{2\pi}\right)$



SHIFT
DOWN
BY
2

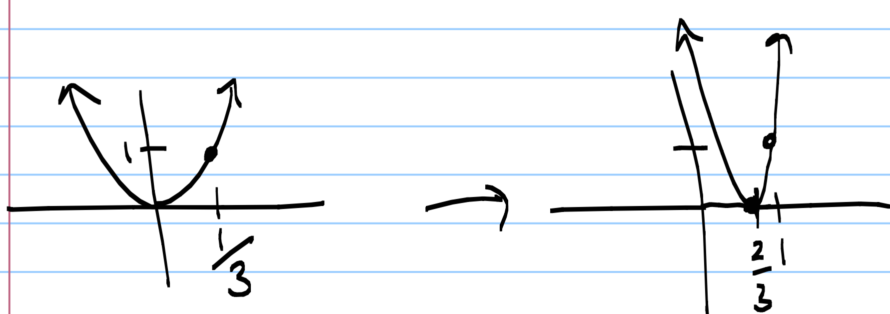


$$g(r) = \ln(2r-1) = \ln(2(r - 1/2))$$



2r shifted by 1 is shifting r by 1/2

$$f(x) = (3x-2)^2 = \left(3\left(x - \tfrac{2}{3}\right)\right)^2$$



$$g(x) = x^2$$

$$\tilde{g}(x) = (3x)^2 = g(3x)$$

$$f(x) = \tilde{g}\left(x - \tfrac{2}{3}\right) = \left(3\left(x - \tfrac{2}{3}\right)\right)^2$$

$$y = f(x-k) + h$$

$$y - h = f(x-k)$$

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5. $y = 1000 + 250(x-1)$; $x \geq 1$, x IN Tb

ceiling?

$$y = 1000 + 250\left(\left\lceil \frac{x}{1000} \right\rceil \cdot 1000 - 1\right)$$

use $flr(x) = \{ \ \ldots$

PIECEWISE

$$f(x) = \begin{cases} 1000 ; & x \leq 1000 \\ \lceil x-1000 \rceil \cdot .25 + 1000 ; & x > 1000 \end{cases}$$

HERE, x IS IN Gb. $. \; x \geq 0$

$\lceil \cdot \rceil \leftarrow$ CEILING FUNCTION - ROUND UP

$\lfloor \cdot \rfloor \leftarrow$ FLOOR FUNCTION - GREATEST INTEGER FUNCTION

$$81^{x-2} = 9 \cdot 3^{4x}$$

$$(3^4)^{x-2} = 3^2 \cdot 3^{4x}$$

$$3^{4x-8} = 3^{2+4x}$$

$4x - 8 = 2 + 4x$

$-8 = 2$ ← FALSE

NO SOLUTION.

#8

$8^x = 5^{2x^2-1}$

$\ln(8^x) = \ln(5^{2x^2-1})$

Log Rule

$\ln(a^b) = b \ln a$

$x \ln 8 = (2x^2 - 1)\ln 5$

$x \ln 8 = 2x^2 \ln 5 - \ln 5$

$0 = \underbrace{(2\ln 5)}_{a} x^2 \underbrace{-(\ln 8)}_{b} x \underbrace{-\ln 5}_{c}$

$$x = \frac{\ln 8 \pm \sqrt{(\ln 8)^2 + 8(\ln 5)^2}}{4 \ln 5}$$

ANSWER

$(f+g)(x) = f(x) + g(x)$

$(f+g)(4) = f(4) + g(4) = 9 + 6 = 15$

$f(g(4)) = f(6) = 2$

$g(f(4)) = 13$

10

$g(x) = 3x^2 - 7$

$f(x) = 5\sin x$

$f(g(x)) = 5\sin(3x^2 - 7)$

$g(x) = \sin(3x^2 - 7)$

$f(x) = 5x$

$g(x) = x^2$

$f(x) = 5\sin(3x - 7)$

MULT. CHOICE

#10

① $25 + 3t = y_1$

② $t = 2,\ y_2 = 18$

$t = 3,\ y_2 = 22$

SLOPE = 4

$y_2 - 18 = 4(t - 2)$

$y_2 = 4t + 10$

t is yrs since 2000

$T = t - 2$

↖ yrs since 2002

$25 + 3(T + 2) = 4(T + 2) + 10$

$25 + 3T + 6 = 4T + 8 + 10$

$25 + 3T = 4T + 12$

$31 + 3T = 18 + 4T$

13

$$P(t) = 250 \cdot (3.04)^{t/1.98}$$

STRUCT RESP.

1. 100 TREES , AVG YIELD 70 ORG/TREE.

101 , AVG YIELD 68 ORG/TREE.

$100 + x$, AVG YIELD $70 - 2x$

(a) 7000 ORANGES

(b) 103 TREES , 64/TREE $= 103 \cdot 64 = 6592$

DIFF $= 408$

(c) x is # of add'l trees.

$$\text{TOTAL YIELD} = \left(\text{TOTAL NUM of TREES}\right) \cdot \left(\text{AVG/YIELD}\right)$$

$$= (100 + x)(70 - 2x)$$

$$= 7000 - 130x - 2x^2$$

$$y = \overset{c}{7000} - \overset{b}{130}x \overset{a}{-2}x^2$$

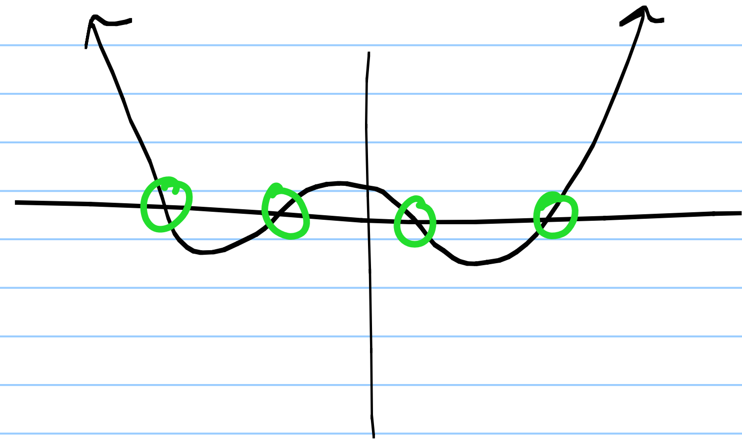
$$x_v = \frac{-b}{2a} = -\frac{130}{4} = -32.5$$

so $100 - 32.5 = 67.5$

2

$$f(x) = \frac{x^4}{2} - 3x^3 + 15x + c$$

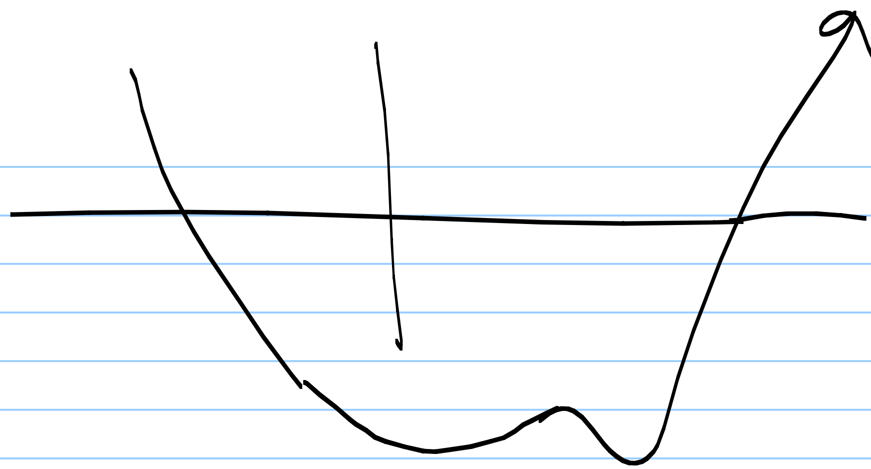
(a)

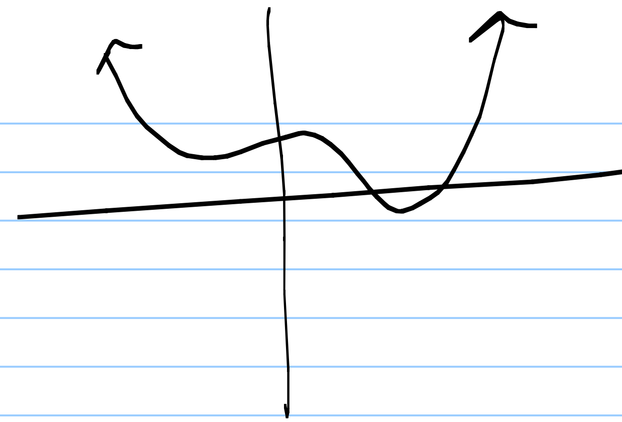


$c = 0.$

(b) $c = 100$

(c) $c = -100$





PROOFS BY CONTRAPOSITIVE.

ANTECEDENT

#3 IF n IS A POSITIVE INTEGER WITH $n \bmod 3 = 2$

THEN n IS NOT A PERFECT SQUARE. CONSEQUENT

PROOF. SUPPOSE n IS A PERFECT SQUARE.

SO WE CAN WRITE $n = k^2$.

CASES.

1. $k \bmod 3 = 0$. THEN $k = 3m$. So

$$k^2 = (3m)^2 = 9m^2$$, WHICH IS DIVISIBLE BY 3, SO $k^2 \bmod 3 = 0$.

2. $k \bmod 3 = 1$. THEN $k = 3m+1$. So

$$k^2 = (3m+1)^2 = 9m^2 + 6m + 1 = 3(3m^2+2m) + 1$$

So, $k^2 \bmod 3 = 1$, since .

3. $k \bmod 3 = 2$. THEN $k = 3m+2$. So

$$k^2 = (3m+2)^2 = 9m^2 + 12m + 4$$

$$= \underbrace{9m^2 + 12m + 3}_{\text{DIVISIBLE BY 3}} + 1$$, THUS $k^2 \bmod 3 = 1$.

Since a perfect SQUARE ALWAYS HAS REMAINDER 0 OR 1 AFTER DIVIDING BY 3, IF $n \bmod 3 = 2$, n can't BE A PERFECT SQUARE.

#2

<u>THM</u> IF X AND Y HAVE ODD PRODUCT, THEN THEY MUST BOTH BE ODD.

<u>PROOF</u>. ASSUME AT LEAST ONE IS EVEN. WITHOUT LOSS OF GENERALITY, ASSUME X IS EVEN, $x = 2k$.

THEN $x \cdot y = 2k \cdot y = 2(ky)$.

SINCE $k \cdot y$ IS AN INTEGER, $2(ky)$ IS AN EVEN INTEGER.

IF x IS EVEN, SO IS THE PRODUCT, THUS IF THE PRODUCT IS ODD, NEITHER FACTOR CAN BE EVEN.

MATHEMATICAL INDUCTION

IDEA WANT TO SHOW THAT SOME PROPERTY, P, HOLDS OF ALL NATURAL NUMBERS.

INDUCTION IS A 2-STEP TECHNIQUE FOR DOING THIS:

1. BASE CASE: PROVE THAT PROPERTY, P, HOLDS OF THE FIRST NUMBER YOU CARE ABOUT.

INDUCTION HYPOTHESIS

2. INDUCTION STEP: ASSUME THAT n HAS THE PROPERTY, AND SHOW THAT IT HOLDS OF $n+1$.

THM FOR ANY POSITIVE INTEGER n,

$$1+2+\dots+n = \frac{n(n+1)}{2}.$$

<u>PROOF</u> BY INDUCTION.

BASE CASE: FOR $n=1$, WE HAVE

$$1 = \frac{1(1+1)}{2},$$ WHICH IS TRUE.

INDUCTIVE STEP: ASSUME THAT $1+2+\dots+n = \frac{n(n+1)}{2}$.

WANT TO SHOW THAT $1+2+\dots+n+(n+1) = \frac{(n+1)((n+1)+1)}{2} = \frac{(n+1)(n+2)}{2}$

$$\underbrace{1+2+\dots+n}_{\substack{\text{BY I.H.} \\ = \frac{n(n+1)}{2}}} + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2}$$

$$= \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+2)(n+1)}{2},$$

WHICH IS WHAT WE WANTED TO SHOW!

THM THE SEQUENCE GIVEN BY

$$a_0 = 1/4$$

$$a_{n+1} = 2a_n(1-a_n)$$

CAN BE OBTAINED EXPLICITY BY

$$a_n = \frac{\left(1-\left(\frac{1}{2}\right)^{2^n}\right)}{2}.$$

PROOF BY INDUCTION.

BASE CASE: $n=0$. THE FORMULA GIVES

$$a_0 = \frac{\left(1-\left(\frac{1}{2}\right)^{2^0}\right)}{2} = \frac{1-\frac{1}{2}}{2} = \frac{\frac{1}{2}}{2} = \frac{1}{4}. \checkmark$$

INDUCTIVE STEP: ASSUME THAT

$$a_n = \frac{\left(1 - \left(\frac{1}{2}\right)^{2^n}\right)}{2}.$$

WANT TO SHOW THAT

$$a_{n+1} = \frac{\left(1 - \left(\frac{1}{2}\right)^{2^{n+1}}\right)}{2}.$$

WE KNOW FROM THE RECURSIVE DEFINITION THAT

$$a_{n+1} = 2\, a_n (1 - a_n)$$

BY I.H., WE KNOW WHAT THIS IS.

SUBSTITUTING IN FOR a_n GIVES

$$a_{n+1} = \cancel{2}\left(\frac{1-\left(\frac{1}{2}\right)^{2^n}}{\cancel{2}}\right)\underbrace{\left(1 - \frac{1-\left(\frac{1}{2}\right)^{2^n}}{2}\right)}_{\text{COMBINE}}$$

$$= \left(1-\left(\tfrac{1}{2}\right)^{2^n}\right)\left(\frac{2-\left(1-\left(\frac{1}{2}\right)^{2^n}\right)}{2}\right)$$

$a^2 - b^2$

$(a-b)(a+b)$

$$= \left(\frac{1-\left(\frac{1}{2}\right)^{2^n}}{1}\right)\left(\frac{2-1+\left(\frac{1}{2}\right)^{2^n}}{2}\right) = \frac{\left(1-\left(\frac{1}{2}\right)^{2^n}\right)\left(1+\left(\frac{1}{2}\right)^{2^n}\right)}{2}$$

$$= \frac{\left(1-\left(\left(\frac{1}{2}\right)^{2^n}\right)^2\right)}{2} = \frac{\left(1-\left(\frac{1}{2}\right)^{2^1\cdot 2^n}\right)}{2}$$

$$= \frac{\left(1-\left(\frac{1}{2}\right)^{2^{n+1}}\right)}{2}$$

, WHICH IS WHAT WE WERE TRYING TO SHOW.

#2

DEFINE A SEQUENCE

$$a_{n+1} = 2a_n - a_n^2 .$$

THEN $a_n = 1 - (1 - a_0)^{2^n}$ FOR $n = 0, 1, \ldots$

PROOF BY INDUCTION.

BASE CASE $n = 0$: $a_0 = 1 - (1 - a_0)^{\underbrace{2^0}_{=1}}$

$= 1 - 1 + a_0 = a_0$ ✓

INDUCTIVE STEP : ASSUME $a_n = 1 - (1 - a_0)^{2^n}$.

WANT TO SHOW : $a_{n+1} = 1 - (1 - a_0)^{2^{n+1}}$

I.H.

WE KNOW THAT

$$a_{n+1} = 2a_n - a_n^2 .$$

WE'LL SUBSTITUTE IN (I.H.)

$$a_{n+1} = 2\left[1-(1-a_0)^{2^n}\right] - \left[1-(1-a_0)^{2^n}\right]^2$$

$$= \left[1-(1-a_0)^{2^n}\right]\left[2-\left[1-(1-a_0)^{2^n}\right]\right]$$

$$= \left[1-(1-a_0)^{2^n}\right]\left[1+(1-a_0)^{2^n}\right]$$

$$= 1-\left((1-a_0)^{2^n}\right)^2 = 1-(1-a_0)^{2\cdot 2^n}$$

$$= 1-(1-a_0)^{2^{n+1}}$$

, WHICH IS WHAT WE WANTED.

THM IF $\underset{\sim\sim}{P}$ THEN $\underset{\sim\sim}{Q}$.

THM P IF AND ONLY IF Q.

(JUST THE SAME AS

IF P THEN Q, AND ALSO

IF Q THEN P.

THM $1 + \tan^2\theta = \sec^2\theta$

PROOF $\sin^2\theta + \cos^2\theta = 1$

iff $\tan^2\theta + 1 = \sec^2\theta$.

$1 + \tan^2\theta \leq \sec^2\theta$

$1 + \tan^2\theta \geq \sec^2\theta$