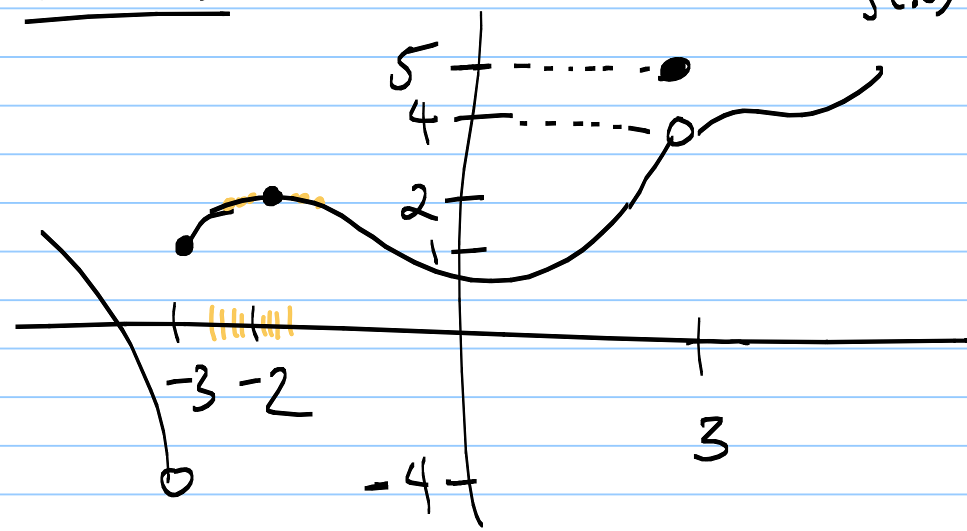
Note Title 6/4/2009

Thursday, June 4, 2009

LIMITS



$$\lim_{x \to -2} f(x) = 2$$

As x gets close to −2, y gets close to 2.

$$\lim_{x \to 3} f(x) = 4$$

$$f(3) = 5$$

$$\lim_{x \to -3} f(x) \text{ D.N.E.}$$

$$\lim_{x \to -3^{+}} f(x) = 1$$

APPROACH FROM RIGHT

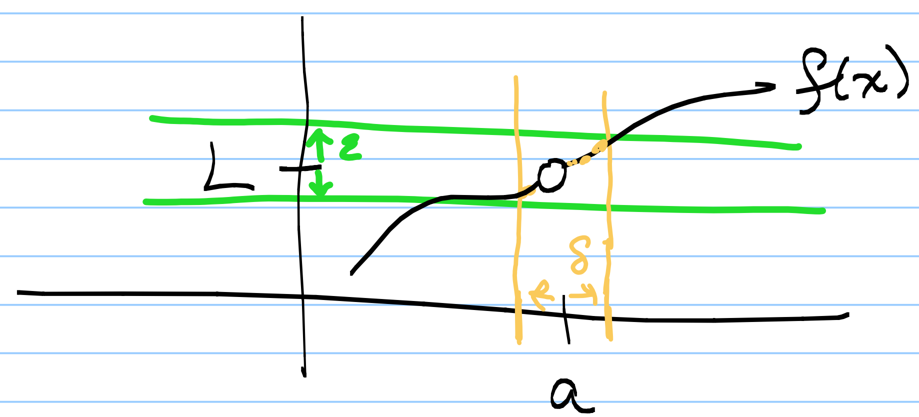
APPROACH FROM LEFT $\lim_{x \to -3^-} f(x) = -4$

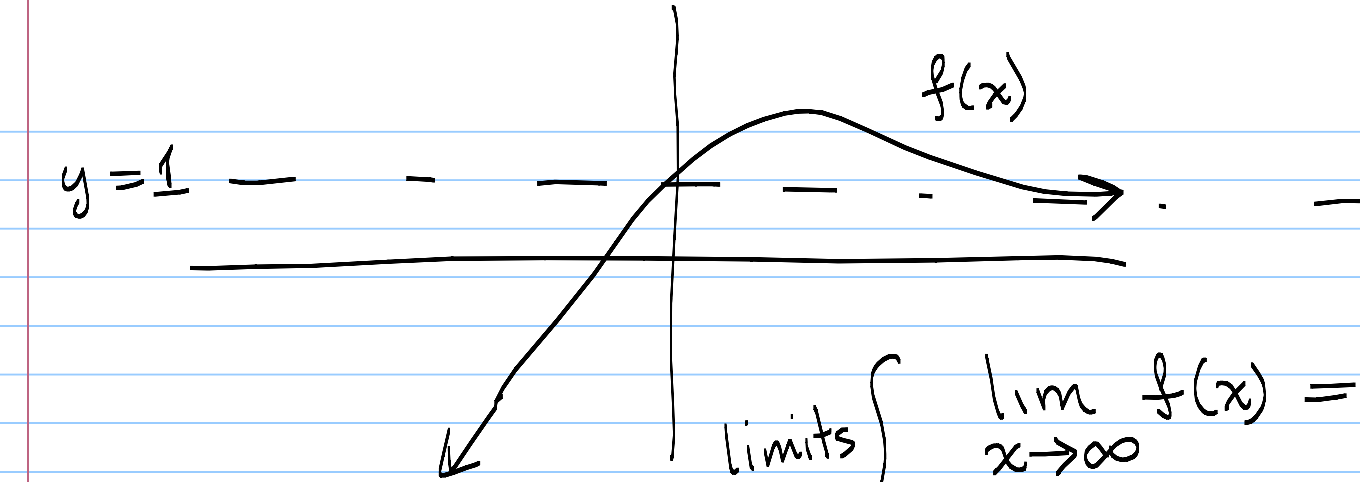
DEFINITION OF $\lim_{x \to a} f(x) = L$ IF AND ONLY IF

FOR ANY $\varepsilon > 0$, THERE IS A $\delta > 0$, SO THAT

AS LONG AS $0 < |x - a| < \delta$,

WE HAVE THAT $|f(x) - L| < \varepsilon$.



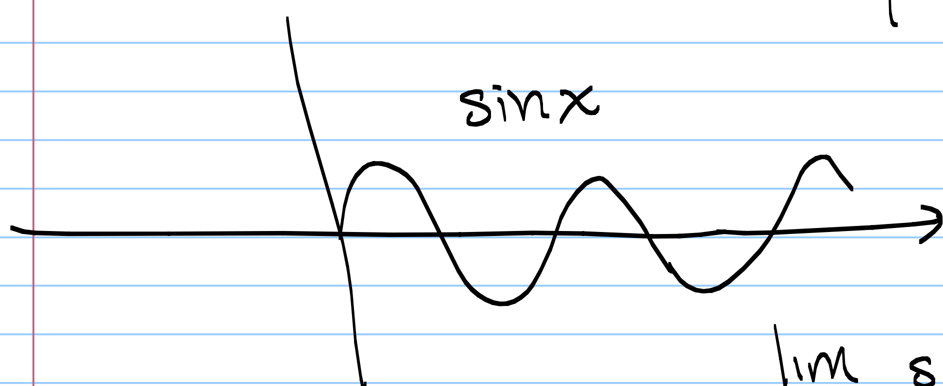


limits at infinity

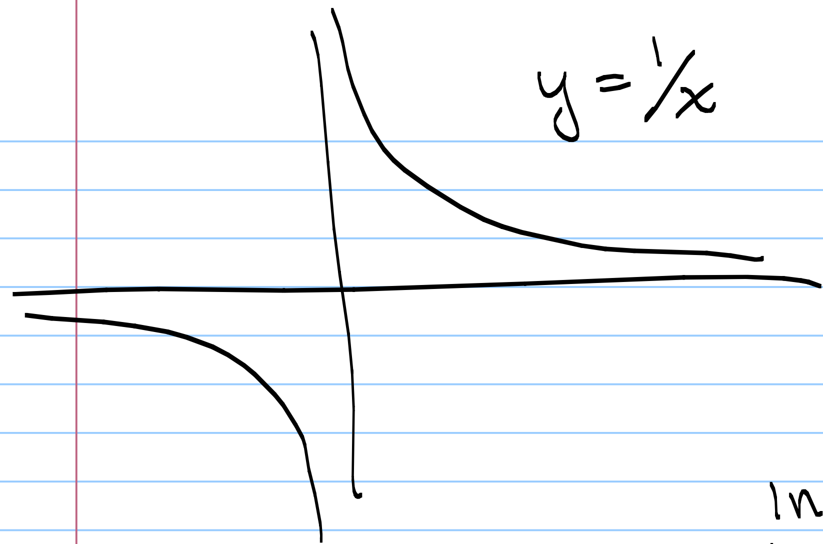
$$\lim_{x\to\infty} f(x) = 1$$

$$\lim_{x\to -\infty} f(x) = -\infty$$

LIMIT DOESN'T EXIST, BUT IN A SPECIAL WAY.



$\lim_{x\to\infty} \sin x$ D.N.E.



$$\lim_{x\to 0} \frac{1}{x} \quad \text{D.N.E.}$$

infinite limits

$$\begin{cases} \lim_{x\to 0^+} \frac{1}{x} = \infty \\ \lim_{x\to 0^-} \frac{1}{x} = -\infty \end{cases}$$

COMPUTING LIMITS

- $\lim_{x\to a} f(x) \underset{\times}{\pm} g(x) = \lim_{x\to a} f(x) \underset{\times}{\pm} \lim_{x\to a} g(x)$

ASSUMING BOTH EXIST!

- $\lim_{x\to a} k\cdot f(x) = k\cdot \lim_{x\to a} f(x)$

- $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$, PROVIDED $\lim\limits_{x \to a} g(x) \neq 0$.

- $\lim\limits_{x \to a} k = k$; $\lim\limits_{x \to a} x = a$. $\lim\limits_{x \to a} f(g(x)) = f\left(\lim\limits_{x \to a} g(x)\right)$.

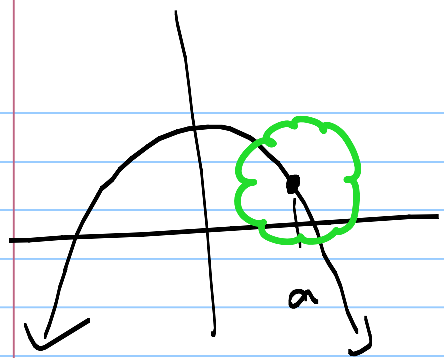
ex $\lim\limits_{x \to 3} 2x^7 - 3x + 1 = 2 \cdot \overbrace{\lim\limits_{x \to 3} x^7}^{3^7} - 3 \lim\limits_{x \to 3} x + \lim\limits_{x \to 3} 1$

$= 2 \cdot 3^7 - 3 \cdot 3 + 1$

CONTINUITY

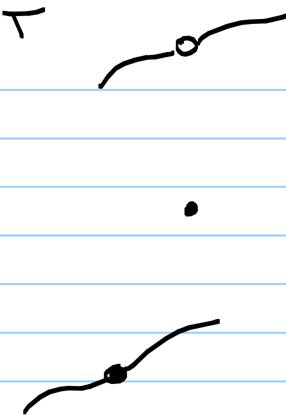
INTUITION:

A FUNCTION, $f(x)$, IS CONTINUOUS AT $x = a$ IF WE CAN DRAW THAT PART OF THE PICTURE WITHOUT PICKING UP THE PEN.



FORMALLY

1. $\lim_{x \to a} f(x)$ MUST EXIST
2. $f(a)$ MUST EXIST
3. $f(a) = \lim_{x \to a} f(x)$



MOST FUNCTIONS THAT WE KNOW ABOUT ARE <u>CONTINUOUS ON THEIR DOMAINS</u>.

↳ CONTINUOUS AT EVERY POINT IN THE DOMAIN.

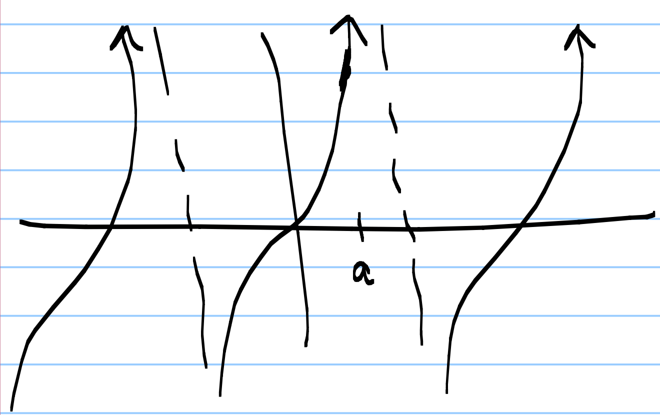
POLYNOMIALS	EXPONENTIAL FUNCTIONS
ROOT FUNCTIONS	LOGARITHMIC FUNCTIONS
RATIONAL FUNCTIONS	sine, cosine, tangent

ALGEBRAIC FUNCTIONS → BUILD UP FROM ROOT / POWER BY $+, -, \cdot, \div$, RAISING TO A integer POWER

OR ROOTING

$$f(x) = \sqrt{\frac{x^2 - \sqrt{x} + \frac{1}{3}}{2x^{1/4} - 6x^2}}$$

A NICE THING ABOUT FUNCTIONS THAT ARE CONTINUOUS ON THEIR DOMAIN : $\lim\limits_{x \to a} f(x) = f(a)$ FOR a in the domain of $f(x)$.



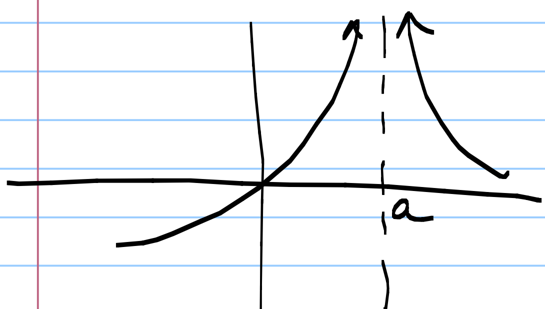
$\leftarrow$ $\tan x$ is continuous on its domain

$$f(x) = \frac{3x^2 - 2x + 1}{2x^4 - 3x}$$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{3x^2 - 2x + 1}{2x^4 - 3x} = \frac{3 - 2 + 1}{2 - 3} = \frac{2}{-1} = -2$$

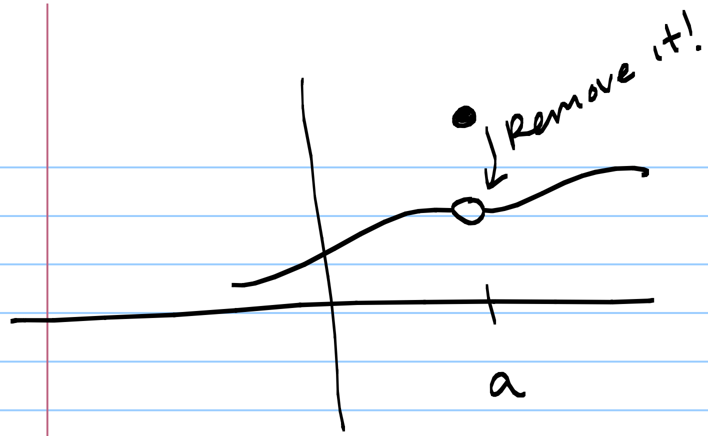
$\lim_{x \to 0} f(x)$ hmm... can't plug in!

DISCONTINUITIES



$$\lim_{x \to a^{\pm}} f(x) = \pm\infty$$

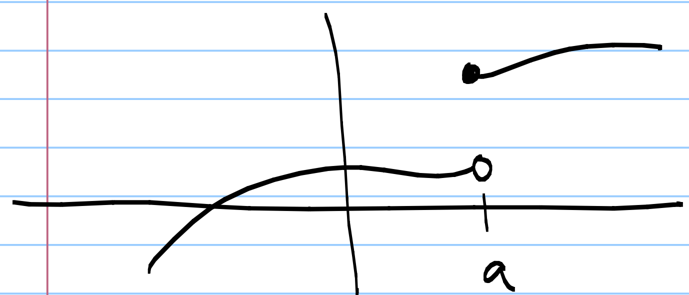
VERTICAL ASYMPTOTE, INFINITE DISCONTINUITY.



REMOVABLE DISCONTINUITY

EITHER 2 FAILS.

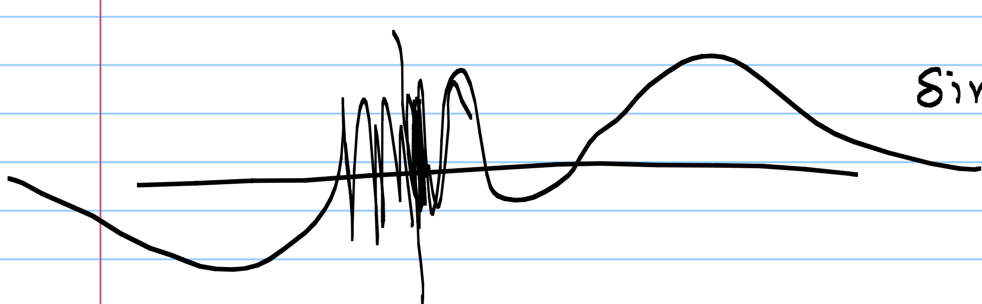
OR 3 FAILS.



JUMP DISCONTINUITY

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$

1 FAILING.



$\sin\left(\frac{1}{x}\right)$

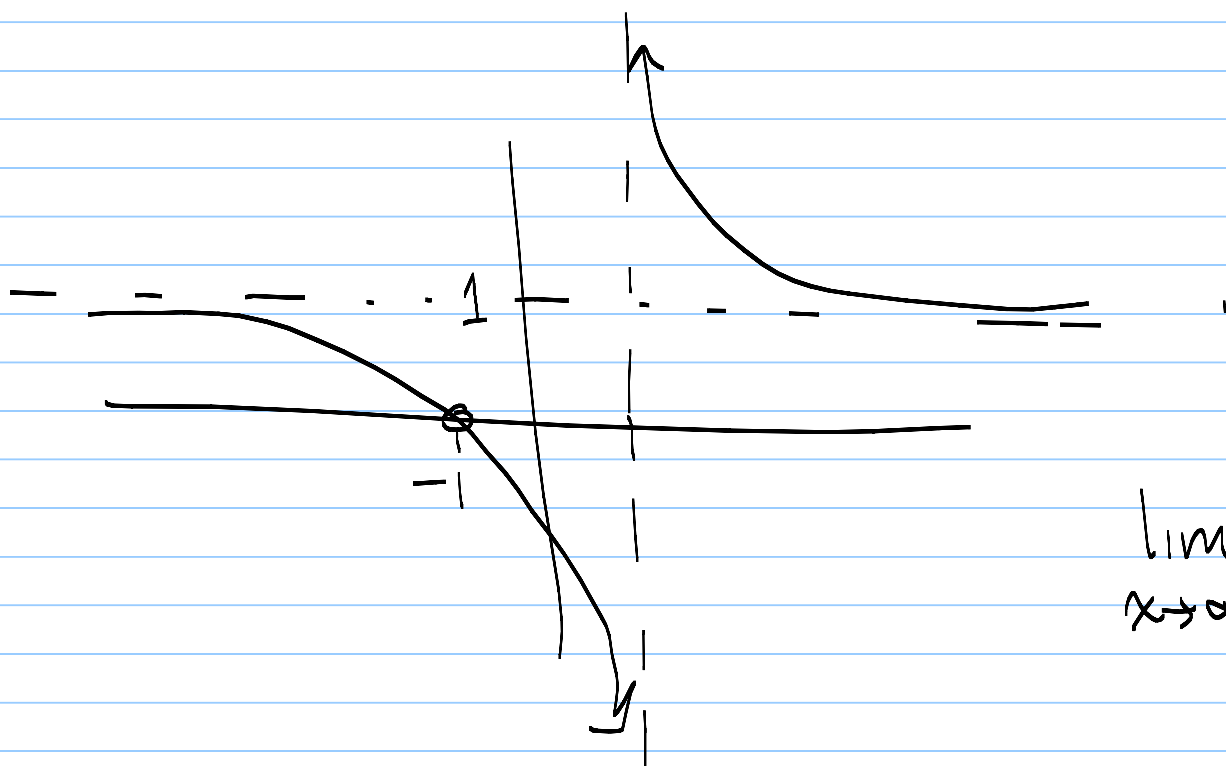
OSCILLATING DISCONTINUITY.

1 FAILS

Ex

$$y = \frac{x^2+2x+1}{x^2-1} = \frac{(x+1)^2}{(x+1)(x-1)} = \frac{x+1}{x-1}, \quad x \neq -1$$

$$\lim_{x\to -1} \frac{(x+1)^2}{(x+1)(x-1)} = \lim_{x\to -1} \frac{x+1}{x-1} = \frac{0}{-2} = 0$$



$$\lim_{x\to\infty} \frac{x+1}{x-1} = 1$$

RATIONAL FUNCTIONS

$$f(x) = \frac{\cancel{(x-3)}(x+2)}{(x+1)(x-2)\cancel{(x-3)}}$$

THERE'S A HOLE IN THE PICTURE AT $x=3$.

THERE ARE VERTICAL ASYMPTOTES AT $x=2,-1$

HORIZONTAL ASYMPTOTES

ie, limits at INFINITY (LOOK FOR HIGHEST POWER OF x IN DEN. & MULT. BY 1.)

- $$\lim_{x\to\infty} \frac{x+1}{3x^2+4} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\overbrace{\frac{1}{x}}^{\to 0} + \overbrace{\frac{1}{x^2}}^{\to 0}}{\underbrace{3}_{3} + \underbrace{\frac{4}{x^2}}_{0}} = \frac{0}{3} = 0$$

- $$\lim_{x\to\infty} \frac{x^2-2x+1}{4x^2-3x+2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

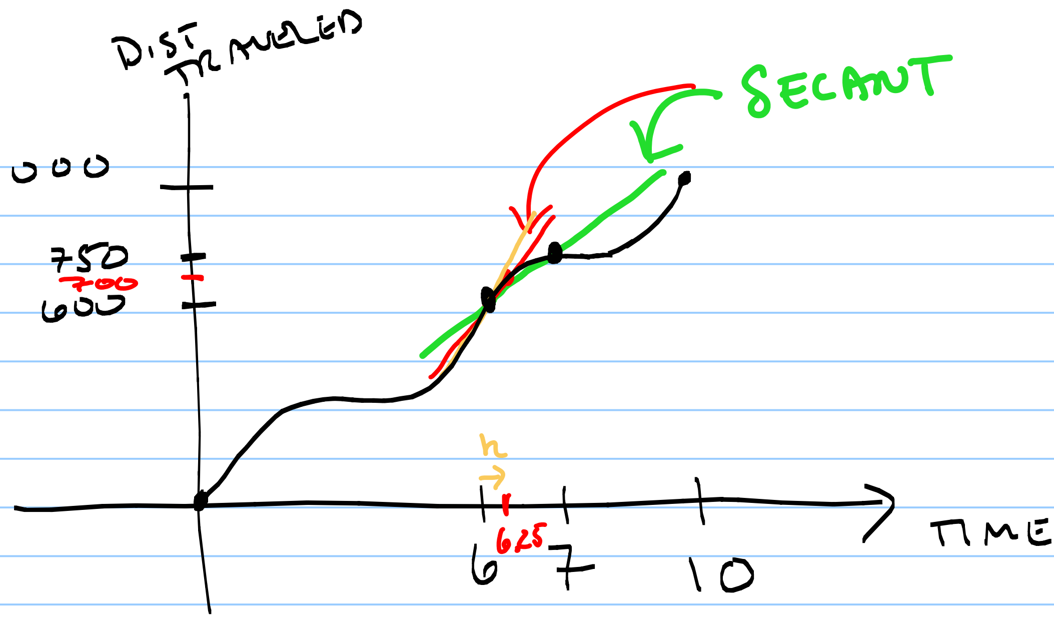
$$= \lim_{x \to \infty} \frac{1 - \frac{2}{x} + \frac{1}{x^2}}{4 - \frac{3}{x} + \frac{2}{x}} = \frac{1}{4}$$

- $\lim_{x \to \infty} \frac{3x^3 - 2x}{x^2 + 1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \to \infty} \frac{3x - \cancel{\frac{2}{x}}^{\,0}}{1 + \cancel{\frac{1}{x^2}}_{\,0}}$

$$\lim_{x \to \infty} 3x = \infty$$

$\boxed{\text{Ex } 1\text{–}14, 16, 23\text{–}25 \quad \text{MC } 1\text{–}16, 19\text{–}21, 25}$

- WHEN deg(den) > deg(num), THERE IS AN H.A. AT $y = 0$.
- WHEN deg(den) = deg(num), THERE'S AN H.A. AT $y = \frac{LC(NUM)}{LC(Den)}$
- WHEN deg(den) < deg(num), NO H.A.



$$\text{AVG VEL.} = \frac{\Delta \text{ DISTANCE}}{\Delta \text{ TIME}}$$

$$\text{AVG VEL IN } 6^{th} \text{ HOUR} = \frac{750 - 600}{7 - 6} = 150$$

SLOPE OF THE SECANT LINE CONNECTING $(x, f(x))$ TO $(x+h, f(x+h))$ $= \dfrac{f(x+h) - f(x)}{(x+h) - x} = \dfrac{f(x+h) - f(x)}{h}$

SLOPE OF THE TANGENT LINE $= \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ = derivative of $f(x)$.

$= f'(x) = \dfrac{df}{dx}$

Defn THE LINE TANGENT TO $f(x)$ AT $x = a$ IS THE LINE THAT GOES THROUGH $(a, f(a))$ AND HAS SLOPE

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Let $f(x) = x^2$. FIND THE EQUATION OF THE TANGENT TO $f(x)$ AT $x = 1$.

$f(1+h) = (1+h)^2$

POINT: $(1, 1)$

SLOPE: FIND $f'(1) = 2$

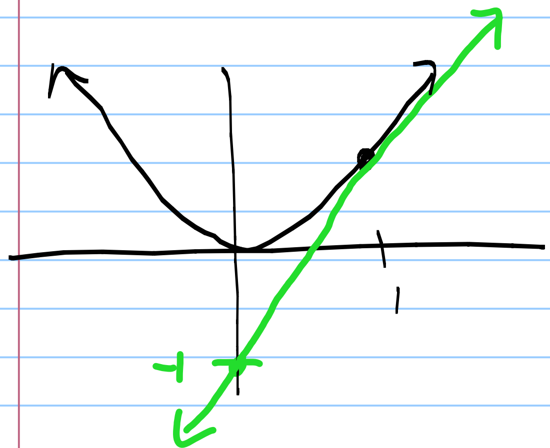
$y - 1 = 2(x-1)$

$y = 2x - 1$

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^2 - 1}{h}$$

$$= \lim_{h \to 0} \frac{1+2h+h^2 - 1}{h} = \lim_{h \to 0} \frac{2h+h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(2+h)}{\cancel{h}}$$

$$= 2$$



DERIVATIVE RULES

- $f(x) = c \longrightarrow f'(x) = 0$
- $f(x) = x^p \longrightarrow f'(x) = p x^{p-1}$
- $f(x) = k \cdot g(x) \longrightarrow f'(x) = k\, g'(x)$
- $f(x) = g(x) \pm h(x) \longrightarrow f'(x) = g'(x) \pm h'(x)$

- PRODUCT RULE

$$h(x) = f(x) \cdot g(x) \rightarrow h'(x) = f'g + g'f$$

- QUOTIENT RULE

$$h(x) = \frac{f(x)}{g(x)} \rightarrow h'(x) = \frac{f'g - g'f}{(g)^2}$$

Ex

1. $f(x) = 3x^2 - \sqrt{x} + \frac{1}{x^2} = 3x^2 - x^{1/2} + x^{-2}$

   $f'(x) = 6x - \frac{1}{2}x^{-1/2} + (-2x^{-3})$

2. $g(x) = \frac{3x^2 - 2}{x^2 + 2x}$

$$g'(x) = \frac{(6x)(x^2+2x) - (2x+2)(3x^2-2)}{(x^2+2x)^2}$$

CHAIN RULE

$$h(x) = f(g(x))$$

$$h'(x) = f'(g(x))\, g'(x)$$

$$h(x) = (3x^2-2)^7$$

$$g(x) = 3x^2-2$$

$$f(x) = x^7$$

$$f'(x) = 7x^6$$

$$g'(x) = 6x$$

$$h'(x) = 7(3x^2-2)^6 \cdot 6x$$

Derivatives of TRANSCENDENTAL FUNCTIONS.

EXPONENTIAL FUNCTIONS

$$f(x) = e^x \longrightarrow f'(x) = e^x$$

$$f(x) = b^x \longrightarrow f'(x) = b^x \cdot \ln b$$

LOGARITHMIC FUNCTIONS

$$f(x) = \ln x \longrightarrow f'(x) = \frac{1}{x}$$

$$f(x) = \log_b x \longrightarrow f'(x) = \frac{1}{(\ln b)\, x}$$

Ex $f(x) = e^{(x^2+3x)}$ $\quad f'(x) = e^{x^2+3x} \cdot (2x+3)$

$g(x) = e^x \qquad g'(x) = e^x$

$h(x) = x^2+3x \quad h'(x) = 2x+3$

Ex $\quad f(x) = \ln(2x^3 - e^x) \qquad f'(x) = \dfrac{1}{2x^3-e^x} \cdot (6x^2 - e^x)$

TRIG FUNCTIONS

$f(x) = \sin x \qquad f'(x) = \cos x$

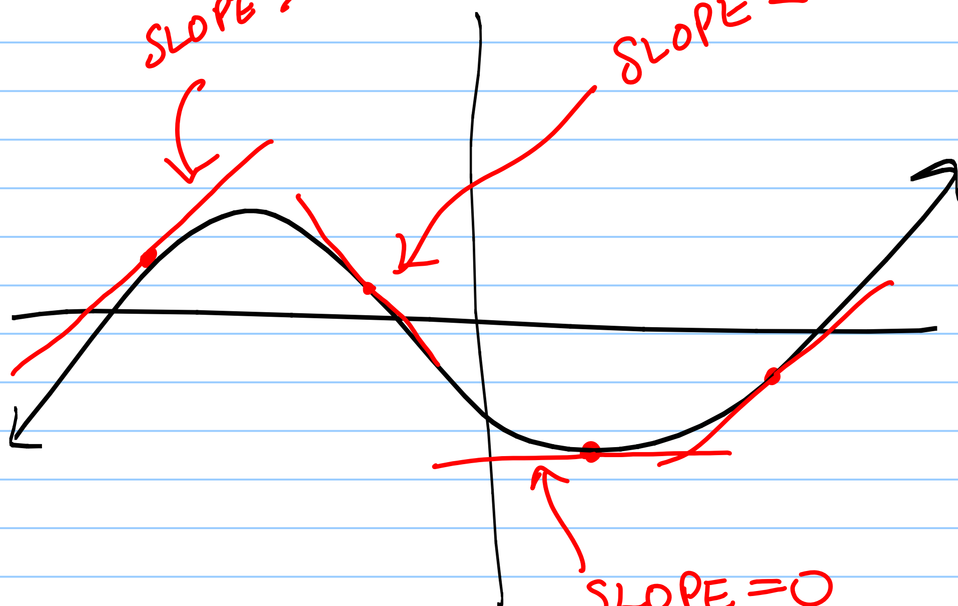
$f(x) = \cos x \qquad f'(x) = -\sin x$

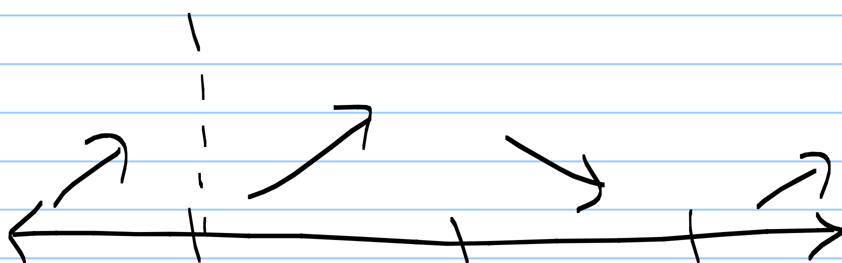
$f(x) = \tan x \qquad f'(x) = \sec^2 x$

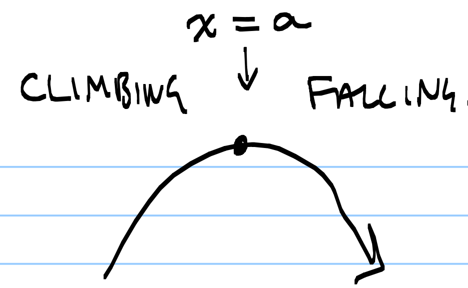
$f(x) = \sec x \qquad f'(x) = \sec x \tan x$

ex $\quad f(x) = \sin(\ln x) \qquad f'(x) = \cos(\ln x) \cdot \dfrac{1}{x}$

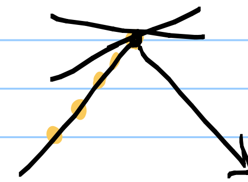
DERIVATIVES & GEOMETRY



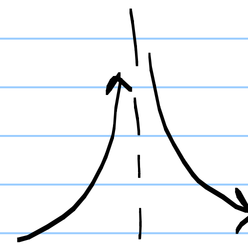




$f'(a) = 0$

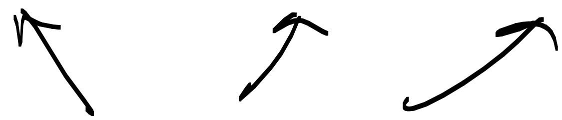


$f'(a)$ DNE



$f'(a)$ DNE

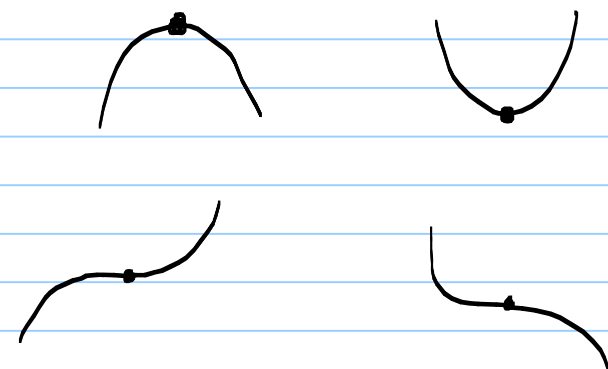
CRITICAL NUMBERS ARE x-VALUES WHERE $f' = 0$ OR DNE.



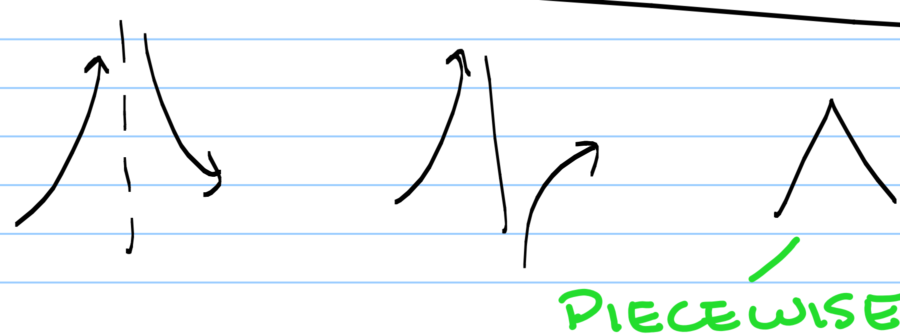
IF THESE ARE ALL THE CRITICAL #'S, THEN FUNCTION IS MONOTONIC ON THE INTERVALS.

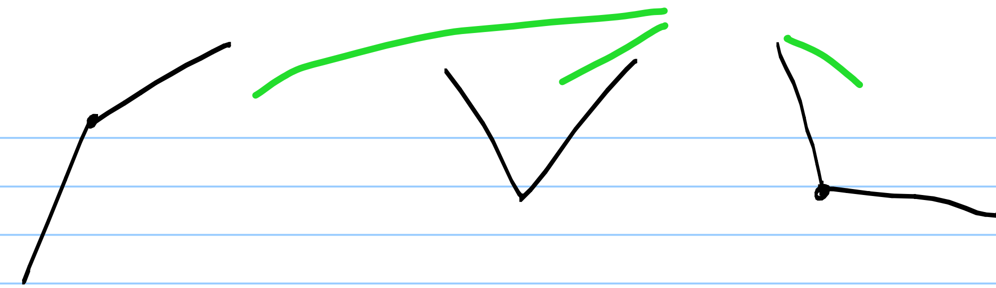
CRITICAL NUMBERS

$f'(a) = 0$



$f'(a)$ DNE





MAKE SIGN CHART

1. FIND C.N.
2. DRAW # LINE & ONLY LABEL C.N.
3. CHECK ↗ ↘ ON EACH INTERVAL

Ex $f(x) = \dfrac{x+1}{x^2-3}$

$$f'(x) = \frac{1\cdot(x^2-3) - 2x(x+1)}{(x^2-3)^2} = \frac{x^2-3-2x^2-2x}{(x^2-3)^2} = \frac{-x^2-2x-3}{(x^2-3)^2}$$

$$f'(x) = -\left(\frac{x^2+2x+3}{(x^2-3)^2}\right)$$

f' DNE AT $x = \pm\sqrt{3}$

f' NEVER $= 0$ (DISCRIMINANT of NUM < 0)