Computability and Vaughtian Models

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Outline

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Assumptions

Everything will be countable; in particular, all languages, theories, and models are countable.

Theories will be complete (except when they're not), and taken to have only infinite models.

Types

Let $p(\bar{x})$ be a collection of $\mathcal{L}$-formulas having variables among $x_0, \ldots, x_{n-1}$ for some fixed n.

- $p(\bar{x})$ is an *n-type of T* if there is a model $\mathcal{M}$ of T, and an element $\bar{a}$ in the universe of that model so that every formula in $p(\bar{x})$ is true of $\bar{a}$ in $\mathcal{M}$. We say $p(\bar{x})$ *is realized by $\bar{a}$ in $\mathcal{M}$*.
- $p(\bar{x})$ is a *complete n-type of T* if it is a maximal consistent set of n-ary formulas.
- Given an $\mathcal{L}$-structure $\mathcal{M} \models T$ and an element $\bar{a}$ of its universe, the *type of $\bar{a}$ in $\mathcal{M}$*, $\mathrm{tp}_{\mathcal{M}}(\bar{a}) = \{\theta(\bar{x}) | \theta(\bar{a}) \text{ is true in } \mathcal{M}.\}$.

Type Space

- The collection of all complete n-types is $S_n(T)$.
- We can put a topology on this space... the basic open sets are given by the n-ary $\mathcal{L}$-formulas, that is, for an n-ary $\mathcal{L}$-formula $\varphi(\bar{x})$, we have the basic open set

$$\{p \in S_n(T) | \varphi(\bar{x}) \in p\}.$$

- With this topology, $S_n(T)$ is a totally disconnected space, compact, and Hausdorff. (Such a space is called Boolean.)

Type Space

$S_n(T)$ can be viewed as the set of all paths in a tree:

- Let $\{\theta_i(\bar{x})\}_{i\in\omega}$ be an enumeration of all n-ary formulas of $\mathcal{L}$.
- Let $\theta^1 = \theta$ and $\theta^0 = \neg\theta$.
- For $\alpha \in 2^{<\omega}$, let $\theta_\alpha(\bar{x}) = \bigwedge_{i<|\alpha|}\{\theta_i^{\alpha(i)}(\bar{x})\}$.
- Define the tree of n-ary formulas consistent with T as

$$\mathcal{T}_n(T) = \{\theta_\alpha(\bar{x})|\alpha \in 2^{<\omega} \; \& \; (\exists \bar{x})\theta(\bar{x}) \in T\}.$$

- Paths in $\mathcal{T}_n(T)$ are complete n-types of T.

Type Space

Note that we identify formulas with their indices when convenient.

- A node, α is an *atom* if it does not split. Paths passing through atoms atoms *isolated* or *principle* paths. These correspond to formulas which generate *principle types*.
- $\mathcal{T}$ is *atomic* if every node in $\mathcal{T}$ is extended by an atom, equivalently, the isolated paths are dense in $[\mathcal{T}]$.
- A complete theory, T, is *atomic* if $\mathcal{T}_n(T)$ is atomic for every $n \geq 1$.
- A node β that cannot be extended to an atom is called *atomless*.

Warm up example

Let T be the theory of the rationals as a *DLO* without endpoints, and consider the structure $\mathcal{Q} = \langle \mathbb{Q}; <, c_q \rangle_{q \in \mathbb{Q}}$.

- Countably many isolated types. (Corresponding to generators of the form $x = c_q$ for $q \in \mathbb{Q}$.)
- Uncountably many non-principal types. (Corresponding to the cuts of the rationals.)
- T is atomic as the principal types are dense.

Homogeneous models

Definition

An $\mathcal{L}$-structure $\mathcal{M}$ is called *homogeneous* if for any two finite tuples, $\bar{a}$ and $\bar{b}$, in M we have

$$\langle \mathcal{M}, \bar{a} \rangle \equiv \langle \mathcal{M}, \bar{b} \rangle \implies (\forall c \in M)(\exists d \in M)[\langle \mathcal{M}, \bar{a}, c \rangle \equiv \langle \mathcal{M}, \bar{b}, d \rangle].$$

Facts

- It is equivalent to say that any finite elementary map can be extended to an automorphism.
- Any two homogeneous models of the same cardinality that realize the same types are isomorphic.
- Any countable theory has a homogeneous model.

Prime and Atomic models

Definition

- $\mathcal{M} \models T$ is *prime* if $\mathcal{M}$ can be elementarily embedded in any other model of T.
- $\mathcal{M}$ is *atomic* if all the types realized by $\mathcal{M}$ are principle.

Facts

- $\mathcal{M}$ is prime iff it is countable and atomic.
- If $\mathcal{M}$ is prime (and hence atomic), it is homogeneous.
- If $\mathcal{M}_1$ and $\mathcal{M}_2$ are both prime models of T, they are isomorphic.
- If T is countable, complete, has infinite models, and is atomic, then it has a prime model.

Saturated models

Definition

Let $\mathcal{M}$ be a countable model of T.

1. $\mathcal{M}$ is *saturated* if every 1-type $p(\bar{a}, x)$ over a finite set of elements $\bar{a} \in M$ is realized in $\mathcal{M}$.
2. $\mathcal{M}$ is *weakly saturated* if every n-type of T is realized in $\mathcal{M}$.
3. $\mathcal{M}$ is ω*-universal* if $\mathcal{N} \preceq \mathcal{M}$ for every countable model $\mathcal{N}$ of T.

Saturated models

Facts

- The following are equivalent:
  - $\mathcal{M}$ is saturated.
  - $\mathcal{M}$ is weakly saturated and homogeneous.
  - $\mathcal{M}$ is ω-universal and homogeneous.
- If $\mathcal{M}_1$ and $\mathcal{M}_2$ are countable and saturated, they are isomorphic.
- A theory has a countable saturated model iff $S_n(T)$ is countable for all n.

Outline

Let T be a complete, atomic, decidable (CAD) theory.

Theorem (Millar)

There is CAD theory with no decidable prime model.

Theorem (Goncharov-Nurtazin, Harrington; 1973, 1974)

The following are equivalent:

- T has a decidable prime model.
- The collection of principal types, $S^P(T)$, has a $\mathbf{0}$-basis.
- $(\exists g \leq_T \mathbf{0})(\forall \theta_\alpha \in \mathcal{T}_n(T))[\theta_\alpha \subset g_\alpha \in S_n^P(T)]$, where $g_\alpha(y) = g(\alpha, y)$ is an element of $[\mathcal{T}_n(T)]$.

Undecidable prime models

Theorem

If T is *CAD*, then it has a prime model decidable in $\mathbf{0}'$.

Theorem (Csima)

If T is a *CAD* then it has a low prime model.

A theorem about trees...

Theorem (Hirschfeldt, 2006)

If $\mathcal{T}$ is an extendible 'paths all computable' (*PAC*) tree, and $D >_T \emptyset$, then there is a D-computable listing of all the isolated paths in $[\mathcal{T}]$.

Consequences for prime models

Corollary

If T is *CAD* and all its types are computable (*TAC*), and $D >_T \emptyset$, then T has a D-decidable prime model.

Corollary

If $\mathbf{0} \notin dgSp(\mathcal{M})$, and $\mathcal{M}$ is prime, then $dgSp(\mathcal{M}) = \{\mathbf{d} | \mathbf{d} > \mathbf{0}\}$.

Corollary (Slaman, Wehner)

There is a structure with presentations of every non-zero degree, but no computable presentation.

More reminders...

- The function g *dominates* f ($f <^* g$) if $(\forall^\infty x)[f(x) < g(x)]$.
- f *escapes* g if $f \not<^* g$, that is, $(\exists^\infty x)[g(x) \leq f(x)]$.
- f is *dominant* if f dominates every computable function.

These definitions extend naturally to degrees.

Theorem (Martin)

A degree $\mathbf{d}$ is high ($\mathbf{d}' = \mathbf{0}''$) iff $\exists$ dominant $g \leq_T \mathbf{d}$.

Relativizing yields a characterization of the nonlow$_2$ sets:

Theorem (Nonlow$_2$ escape theorem)

Degree $\mathbf{a} \leq \mathbf{0}'$ is not low$_2$, ($\mathbf{a}'' > \mathbf{0}''$) iff $\mathbf{0}'$ does not dominate $\mathbf{a}$.

More reminders...

- A set X is said to have the *escape property* if

$$(\forall g \leq_T 0')(\exists f \leq_T X)(\exists^\infty x)[g(x) \leq f(x)],$$

  that is, for any Δ^0_2 function, we can find an X-computable function f that escapes it.
- X (or the degree of X) has the *prime bounding property* if every *CAD* theory has an X-decidable prime model.

Back to the matter at hand.

Theorem (Csima, Hirschfeldt, Knight, Soare; 2004)

For $X \leq_T 0'$, the following are equivalent:

- X has the escape property.
- X is not low_2.
- X is prime bounding.

Outline

Let T be complete decidable (CD) with all its types computable (TAC).

Theorem (Millar)

There is a CD, TAC theory having no decidable saturated model.

Theorem (Morley, Millar; 1978)

The following are equivalent:

- T has a decidable saturated model.
- There is a computable listing of the types of T, $S(T)$, that is, $S(T)$ has a $\mathbf{0}$-basis.

Undecidable saturated models

Theorem (Harris)

There is a *CD*, *TAC* theory having no low saturated model.

Theorem

T has a saturated model computable in $\emptyset'$.

Positive results

A set X (or its degree) is called *saturated bounding* if every CD, TAC theory has an X-decidable saturated model.

If $\mathbf{d}$ is the degree of a complete extension of Peano Arithmetic, it is a *PA* degree.

Theorem (Macintyre, Marker; 1984)

Every *PA* degree is saturated bounding.

Theorem (Harris; to appear)

Every high degree is saturated bounding.

Proof that high degrees are saturated bounding.

Another tree theorem:

Theorem

Let $\mathcal{T}$ be an extendible *PAC* tree, and $\mathbf{d}$ a high degree. There is a $\mathbf{d}$-uniform listing of $[\mathcal{T}]$.

To show this, we need a theorem from computability:

Theorem (Jockusch)

If $\mathbf{d}$ is a high degree, there is a $\mathbf{d}$-uniform listing of the computable functions.

Negative results

Theorem (Harris)

No low$_n$ degree is saturated bounding.

Outline

1. Review
   - Type Space, $S(T)$
   - Vaughtian models, definitions and basic facts
2. Prime Models
   - Decidable prime models
   - Undecidable prime models
   - Degrees bounding prime models
3. Saturated Models
   - Decidable saturated models
   - Undecidable saturated models
   - Degrees bounding saturated models
4. Homogeneous Models
   - Decidable copies of homogeneous models
   - Undecidable copies of homogeneous models
   - Degrees bounding homogeneous models

Theorem (Goncharov, Peretyat'kin, Millar; 1978, 1978, 1980)

There is a *CD* theory having a homogeneous model with a **0**-basis, but no decidable copy.

Theorem (Goncahrov, Peretyat'kin; 1978, 1978)

If $\mathcal{M}$ is a homogeneous model of a *CD* theory that has a **0**-basis $X = \{p_i(\bar{x})\}_{i \in \omega}$ and an *effective extension function* for X then $\mathcal{M}$ has a decidable copy.

An *effective extension function* is a computable binary function f taking the n-type $p_i(\bar{x})$ and a (consistent) $(n+1)$-ary formula $\theta_j(\bar{x}, x_n)$ to an $(n+1)$-type, $p_{f(i,j)}$ that extends both, that is $p_i(\bar{x}) \cup \{\theta_j(\bar{x}, x_n)\} \subseteq p_{f(i,j)}(\bar{x}, x_n)$.

Undecidable copies of homogeneous models

Theorem (Lange)

If $\mathcal{M}$ is a homogeneous model of a *CD* theory T, and $X = S(\mathcal{M})$ is a $\mathbf{0}'$-basis for the types of $\mathcal{M}$, then $\mathcal{M}$ has a copy $\mathcal{N}$ that is low ($D^e(\mathcal{N})' \equiv_T 0'$).

Theorem (Lange)

Let T be a CD theory with homogeneous model $\mathcal{M}$ having $\mathbf{0}$-basis X. If $\mathbf{d} \leq \mathbf{0}'$ is nonlow$_2$, then there is a $\mathbf{d}$-decidable copy of $\mathcal{M}$.

Undecidable copies of homogeneous models

Theorem (Lange)

Let T be a *CD*, *TAC* theory. Let $\mathcal{M}$ be a homogeneous model of T with a **0**-basis. Then

$$\{\mathbf{d} | \mathbf{0} < \mathbf{d}\} \subseteq \{deg(\mathcal{N}) | \mathcal{N} \cong \mathcal{M}\}.$$

Degrees bounding homogeneous models

A degree **d** is *homogeneous bounding* if every *CD* theory has a **d**-decidable homogeneous model.

Theorem (Csima, Harizanov, Hirschfeldt, Soare; to appear)

A degree is homogeneous bounding iff it is a *PA* degree.

References

- Soare, R. "Short Course on The Computable Content of Vaughtian Models," at Leeds MATHLOGAPS Summer School, August 21-25, 2006.