Recovering structures from their semigroups of partial automorphisms

Jennifer Chubb
George Washington University
jchubb@gwu.edu

March 16, 2006

From joint work with Valentina Harizanov, Andrei Morozov, Sarah Pingrey, and Eric Ufferman
Notation and Definitions

• We consider structures $\mathcal{M}$ for a variety of countable languages $\mathcal{L}$.

• A partial function, $p : M \rightarrow M$, is a partial automorphism if $p$ is 1-1 and for every atomic formula $\theta = \theta(x_0, \ldots, x_{n-1})$ in $\mathcal{L}$, and every $a_0, \ldots, a_{n-1} \in \text{dom}(p)$, we have

$$
\mathcal{M} \models \theta(a_0, \ldots, a_{n-1}) \iff \mathcal{M} \models \theta(p(a_0), \ldots, p(a_{n-1})).
$$

• $p$ is a finite partial automorphism if it is finite.

• $p$ is a partial computable automorphism if it is a partial computable function.
Notation and Definitions

We will be interested in the following collections of partial automorphisms of $\mathcal{M}$:

- $I_{\text{fin}}(\mathcal{M}) =_{\text{def}} \{\text{All finite partial automorphisms of } \mathcal{M}\}$,
- $I_{c}(\mathcal{M}) =_{\text{def}} \{\text{All partial computable automorphisms of } \mathcal{M}\}$, and
- $I(\mathcal{M}) =_{\text{def}} \{\text{All partial automorphisms of } \mathcal{M}\}$.

Each of these forms an inverse semigroup under function composition and function inversion.

We consider these sets as structures for the language of inverse semigroups.
Basic Question

Let $I$ be an inverse semigroup of partial automorphisms for a structure $\mathcal{M}$.

Given information about $I$, what can we deduce about $\mathcal{M}$?
Past Results

**Theorem. (A. Morozov)** If $B_0$ is a nontrivial atomic computable Boolean algebra with a computable set of atoms and $B_1$ is a computable Boolean algebra, then if the groups of computable automorphisms of $B_0$ and $B_1$ are isomorphic then the Boolean algebras are computably isomorphic.
Past Results

Theorem. (E. Lipacheva) Let $A = \langle A; P_0, \ldots, P_k \rangle$ and $B = \langle B; Q_0, \ldots, Q_l \rangle$ be arbitrary structures of finite predicate signatures. Then the following statements are equivalent:

1. $I_{fin}(A) \cong I_{fin}(B)$;

2. There exists a bijection $\lambda$ from $A$ onto $B$ such that for every predicate $P_i$, the set $\{ \lambda(x) \mid A \models P_i(x) \}$ is definable in $B$ by means of a quantifier–free formula and for every predicate $Q_j$, the set $\{ \lambda^{-1}(x) \mid B \models Q_j(x) \}$ is definable in $A$ by means of a quantifier–free formula.
Partial Orderings

**Theorem.** Let $\mathcal{M}_0 = \langle M_0, <_0 \rangle$ and $\mathcal{M}_1 = \langle M_1, <_1 \rangle$ be strict partial orders and let $I_i$ be inverse semigroups such that

$$I_{fin}(\mathcal{M}_i) \subseteq I_i \subseteq I(\mathcal{M}_i), \quad i = 0, 1.$$ 

Then

$$I_0 \equiv I_1 \Rightarrow (\mathcal{M}_0 \equiv \mathcal{M}_1 \lor \mathcal{M}_0 \equiv \mathcal{M}_1^{Rev}), \quad \text{and}$$

$$I_0 \cong I_1 \Rightarrow (\mathcal{M}_0 \cong \mathcal{M}_1 \lor \mathcal{M}_0 \cong \mathcal{M}_1^{Rev}).$$
Boolean Algebras and RCDLs

A partial ordering $\mathcal{B} = \langle B, < \rangle$ with smallest element 0 is called a *relatively complemented distributive lattice* (RCDL) if it is a distributive lattice and for all $a \leq b$ in $\mathcal{B}$, there exists the unique relative complement of $a$ in $b$, i.e., an element $a'$ such that $\sup \{a, a'\} = b$ and $\inf \{a, a'\} = 0$.

A Boolean algebra is a special case of an RCDL.
RCDLs in the language of partial orderings

**Corollary.** If $B_0$ and $B_1$ are RCDLs considered in the language $\langle<\rangle$ and $I_i$ are inverse semigroups such that

$$I_{fin}(B_i) \subseteq I_i \subseteq I(B_i), \quad i = 0, 1.$$  

Then

$$I_0 \equiv I_1 \Rightarrow B_0 \equiv B_1, \quad \text{and}$$

$$I_0 \cong I_1 \Rightarrow B_0 \cong B_1.$$
**RCDLs**

**Theorem.** Let $\mathcal{B}_0$ and $\mathcal{B}_1$ be RCDLs considered in the language $\langle \cap, \cup, \setminus, 0 \rangle$ and $I_i$ are inverse semigroups such that $I_{fin}(\mathcal{B}_i) \subseteq I_i \subseteq I(\mathcal{B}_i)$, $i = 0, 1$.

Then

$I_0 \equiv I_1 \Rightarrow \mathcal{B}_0 \equiv \mathcal{B}_1$, and

$I_0 \simeq I_1 \Rightarrow \mathcal{B}_0 \simeq \mathcal{B}_1$. 
RCDLs

Let $\mathcal{F}$ denote the (unique) computable nontrivial atomless RCDL with no greatest element.

**Theorem.** Assume that $\mathcal{B}_0$ and $\mathcal{B}_1$ are computable RCDLs in the language $\langle \cap, \cup, \setminus, 0 \rangle$. Suppose that there exists a computable isomorphic embedding of $\mathcal{F}$ into $\mathcal{B}_0$ and that $I_c(\mathcal{B}_0) \cong I_c(\mathcal{B}_1)$. Then $\mathcal{B}_0 \cong_c \mathcal{B}_1$. 
Equivalence Structures

**Theorem.** Let \( \mathcal{M}_0 = \langle M_0, E_0 \rangle \) and \( \mathcal{M}_1 = \langle M_1, E_1 \rangle \) be nontrivial equivalence structures and let \( I_i \) be inverse semigroups such that

\[
I_{fin}(\mathcal{M}_i) \subseteq I_i \subseteq I(\mathcal{M}_i), \quad i = 0, 1.
\]

Then

1. \( I_0 \cong I_1 \iff \mathcal{M}_0 \cong \mathcal{M}_1 \);

2. \( I_0 \equiv I_1 \Rightarrow \mathcal{M}_0 \equiv \mathcal{M}_1 \); and

3. if both the structures \( \mathcal{M}_0 \) and \( \mathcal{M}_1 \) are countable then

\[
I_{fin}(\mathcal{M}_0) \equiv I_{fin}(\mathcal{M}_1) \iff \mathcal{M}_0 \cong \mathcal{M}_1.
\]
Equivalence Structures

**Theorem.** Let $\mathcal{M}$ be a nontrivial computable equivalence structure. Then there exists a first order sentence $\varphi$ in the language of inverse semigroups such that for any nontrivial computable equivalence structure $\mathcal{N}$,

$$I_c(\mathcal{N}) \models \varphi \Rightarrow \mathcal{N} \simeq_c \mathcal{M}.$$
Strategy

Our general approach is to interpret as much of the structure $\mathcal{M}$ into $I$ as possible.
Basic Interpretations

Our first goal is to interpret the universe of \( \mathcal{M} \) in \( I \), where \( I \) is any inverse semigroup so that \( I_{fin}(\mathcal{M}) \subseteq I \subseteq I(\mathcal{M}) \).

1. Interpret (some) subsets of \( \mathcal{M} \) in \( I \).

   • Let \( \text{Id}(x) \) be the formula \( x^2 = x \), a first-order formula requiring \( x \) to be idempotent.

   • Functions satisfying \( \text{Id}(x) \) are the identity on their domain.

   • They can be identified with subsets of \( \mathcal{M} \).
Basic Interpretations

2. Define the notion of “subset” in $I$.
   - $\text{Id}(x) \& \text{Id}(y) \& xy = x$ holds in $I$ exactly when $x \subseteq y$ in $M$.

3. Interpret the empty set, $\emptyset$, as the (unique) function contained in all other functions.

4. Define $A(M) = \{(a, a) | a \in M\}$, the interpretation of the universe of $M$ in $I$.
   - $x \in I$ is in $A(M)$ if $x \neq \emptyset \& \neg \exists u (\emptyset \subset u \subset x)$.
   - We identify $x \in M$ with the partial automorphism $\{(x, x)\} \in I$. 
Basic Interpretations

The second goal is to interpret the natural action of elements of $I$ on elements $A(M) \cup \{\emptyset\}$.

For $g \in I$ and $x, y \in M$, $g(x) = y$ exactly when $I \models gxg^{-1} = y$. 
Equivalence structures

Here we consider structures of kind $\mathcal{M} = \langle M; E \rangle$, where $E$ is an equivalence relation on $M$.

We say an equivalence structure is nontrivial if $E$ is not the same as equality.
Interpreting the equivalence relation in the semigroup

We’ll need to interpret $E$ into $I$ where $I_{fin}(\mathcal{M}) \subseteq I \subseteq I(\mathcal{M})$.

1. Let $p, q \sim r, s$ abbreviate $\exists f (f(p) = r \& f(q) = s)$.

2. Let

$$\tilde{E}(x, y) \overset{\text{def}}{=} (x \neq \emptyset) \& (y \neq \emptyset) \& \forall a \forall b \forall c ((x, y \sim a, b \& x, y \sim b, c) \rightarrow x, y \sim a, c).$$

Note that the following holds,

$$\mathcal{M} \models E(x, y) \iff I \models \tilde{E}(x, y).$$
Equivalence structures

Theorem. Let $\mathcal{M}_0 = \langle M_0, E_0 \rangle$ and $\mathcal{M}_1 = \langle M_1, E_1 \rangle$ be nontrivial equivalence structures and let $I_i$ be inverse semigroups such that

$$I_{fin}(\mathcal{M}_i) \subseteq I_i \subseteq I(\mathcal{M}_i), \quad i = 0, 1.$$ 

Then

1. $I_0 \cong I_1 \iff \mathcal{M}_0 \cong \mathcal{M}_1$;

2. $I_0 \equiv I_1 \Rightarrow \mathcal{M}_0 \equiv \mathcal{M}_1$; and

3. if both the structures $\mathcal{M}_0$ and $\mathcal{M}_1$ are countable then

$$I_{fin}(\mathcal{M}_0) \equiv I_{fin}(\mathcal{M}_1) \iff \mathcal{M}_0 \cong \mathcal{M}_1.$$
Equivalence structures

Sketch of proof for (3).

• $\mathcal{M}_0$ and $\mathcal{M}_1$ are isomorphic iff they have exactly the same number of $n$-element equivalence classes for $n \in \omega \cup \{\omega\}$.

• Let $\varphi_{m,n}$ say “$E$ has at least $m$ $n$-element equivalence classes.”
  – For finite $n$, it is easy to find such a formula.
  – For the infinite case, we need only see how to say “$x$ is a member of an infinite equivalence class.”
  – Note that this is the case exactly when
    $$\neg \exists f (\forall y (\tilde{E}(x,y) \rightarrow y \in \text{dom}(f))).$$
Theorem. Let $\mathcal{M}$ be a nontrivial computable equivalence structure. Then there exists a first order sentence $\varphi$ in the language of inverse semigroups such that for any nontrivial computable equivalence structure $\mathcal{N}$,

$$I_c(\mathcal{N}) \models \varphi \Rightarrow \mathcal{N} \cong_c \mathcal{M}.$$
Proof idea:

Divide the proof into cases based on three scenarios:

Case 1. \( \mathcal{M} \) has finitely many equivalence classes.

Case 2. \( \mathcal{M} \) has infinitely many equivalence classes.

\hspace{1cm} \textbf{Subcase 1.} The set of cardinalities of the equivalence classes of \( \mathcal{M} \) is finite, that is, \( \mathcal{M} \) has \textit{bounded character}.

\hspace{1cm} \textbf{Subcase 2.} This set is infinite, or \( \mathcal{M} \) has \textit{unbounded character}.
Case 1 versus Case 2

There is a first order formula $\pi(p)$ in the language of semigroups requiring that the function $p$ has, among other properties, an infinite domain consisting of exactly one representative of each equivalence class.

*The sentence “$\exists p \, \pi(p)$” will distinguish Case 1 from Case 2.*
Subcase 1 versus Subcase 2

There is a first order sentence, $\gamma$, in the language of semigroups asserting the existence of a finite set $F$ so that for any $x \in A(M)$, there are $y \in F$ and $g \in I_c(M)$ so that $g$ is a bijection from $[x]_E$ onto $[y]_E$.

The existence of such an $F$ will distinguish Subcase 1 from Subcase 2.
Case 1.

\( \mathcal{M} \) has \( m \) equivalence classes having cardinalities \( k_0, k_1, \ldots, k_{m-1} \), where \( k_i \in \omega \cup \{\omega\} \).

- This property can be expressed in the language of \( I_c(\mathcal{M}) \) by

\[
\exists x_0, \ldots, x_{m-1} \left[ \bigwedge_{i<j<m} (x_i, x_j) \notin E \; \& \; \forall x \left( \bigvee_{i<m} (x, x_i) \in E \right) \; \& \; \bigwedge_{i<m} [x_i]_E \text{ contains } k_i \text{ elements} \right].
\]

- If a computable equivalence structure \( \mathcal{N} \) satisfies this formula, it is computably isomorphic to \( \mathcal{M} \).
Case 2

\(\mathcal{M}\) has infinitely many equivalence classes – so there is a partial computable automorphism \(p\) satisfying \(\pi(p)\).

- We’ll use this \(p\) as a list of the distinct equivalence classes of \(\mathcal{M}\), and describe their cardinalities along this list.

We give the idea for Subcase 1.
Case 2, Subcase 1.

$\mathcal{M}$ has infinitely many equivalence classes and the set of their cardinalities is finite.

Let $K = \{k_0 < k_1 < \ldots < k_{m-1}\} \subset \omega \cup \{\omega\}$ be this set.

- Use $p$ as a list of the equivalence classes in $\mathcal{M}$, and we can describe the cardinalities along this list by a formula in the language of $I_c(\mathcal{M})$:

$$\forall t \in \text{dom}(p) \bigvee_{i < m} (\phi_i(t) \& \psi_i(t)).$$
Conclusions

• Even when we know nothing about a structure’s global symmetries, we can learn about it by looking at local symmetries.