Recovering structures from their semigroups of partial automorphisms

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Notation and Definitions

- We consider structures $\mathcal{M}$ for a variety of countable languages $\mathcal{L}$.

- A partial function, $p : M \to M$, is a *partial automorphism* if p is 1-1 and for every atomic formula $\theta = \theta(x_0, \ldots, x_{n-1})$ in $\mathcal{L}$, and every $a_0, \ldots, a_{n-1} \in \mathtt{dom}\,(p)$, we have

$$\mathcal{M} \models \theta(a_0, \ldots, a_{n-1}) \Leftrightarrow \mathcal{M} \models \theta(p(a_0), \ldots, p(a_{n-1})).$$

- p is a *finite partial automorphism* if it is finite.

- p is a *partial computable automorphism* if it is a partial computable function.

Notation and Definitions

We will be interested in the following collections of partial automorphisms of $\mathcal{M}$:

- $I_{fin}(\mathcal{M}) =_{def}$ {All finite partial automorphisms of $\mathcal{M}$},
- $I_c(\mathcal{M}) =_{def}$ {All partial computable automorphisms of $\mathcal{M}$}, and
- $I(\mathcal{M}) =_{def}$ {All partial automorphisms of $\mathcal{M}$}.

Each of these forms an inverse semigroup under function composition and function inversion.

We consider these sets as structures for the language of inverse semigroups.

Basic Question

Let I be an inverse semigroup of partial automorphisms for a structure $\mathcal{M}$.

Given information about I, what can we deduce about $\mathcal{M}$?

Past Results

Theorem. (A. Morozov) *If $\mathcal{B}_0$ is a nontrivial atomic computable Boolean algebra with a computable set of atoms and $\mathcal{B}_1$ is a computable Boolean algebra, then if the groups of computable automorphisms of $\mathcal{B}_0$ and $\mathcal{B}_1$ are isomorphic then the Boolean algebras are computably isomorphic.*

Past Results

Theorem. (E. Lipacheva) *Let* $\mathcal{A} = \langle A; P_0, \ldots, P_k \rangle$ *and* $\mathcal{B} = \langle B; Q_0, \ldots, Q_l \rangle$ *be arbitrary structures of finite predicate signatures. Then the following statements are equivalent:*

1. $I_{fin}(\mathcal{A}) \cong I_{fin}(\mathcal{B})$;

2. *There exists a bijection* λ *from* A *onto* B *such that for every predicate* P_i*, the set* $\{\lambda(\overline{x}) \mid \mathcal{A} \models P_i(\overline{x})\}$ *is definable in* $\mathcal{B}$ *by means of a quantifier–free formula and for every predicate* Q_j*, the set* $\{\lambda^{-1}(\overline{x}) \mid \mathcal{B} \models Q_j(\overline{x})\}$ *is definable in* $\mathcal{A}$ *by means of a quantifier–free formula.*

Partial Orderings

Theorem. *Let* $\mathcal{M}_0 = \langle M_0, <_0 \rangle$ *and* $\mathcal{M}_1 = \langle M_1, <_1 \rangle$ *be strict partial orders and let* I_i *be inverse semigroups such that*

$$I_{fin}(\mathcal{M}_i) \subseteq I_i \subseteq I(\mathcal{M}_i), \quad i = 0, 1.$$

Then

$$I_0 \equiv I_1 \Rightarrow (\mathcal{M}_0 \equiv \mathcal{M}_1 \vee \mathcal{M}_0 \equiv \mathcal{M}_1^{Rev}), \text{ and}$$

$$I_0 \cong I_1 \Rightarrow (\mathcal{M}_0 \cong \mathcal{M}_1 \vee \mathcal{M}_0 \cong \mathcal{M}_1^{Rev}).$$

Boolean Algebras and RCDLs

A partial ordering $\mathcal{B} = \langle B, < \rangle$ with smallest element 0 is called a *relatively complemented distributive lattice* (RCDL) if it is a distributive lattice and for all $a \leqslant b$ in $\mathcal{B}$, there exists the unique relative complement of a in b, i.e., an element a' such that $\sup\{a, a'\} = b$ and $\inf\{a, a'\} = 0$.

A Boolean algebra is a special case of an RCDL.

RCDLs in the language of partial orderings

Corollary. *If* $\mathcal{B}_0$ *and* $\mathcal{B}_1$ *are RCDLs considered in the language* $\langle < \rangle$ *and* I_i *are inverse semigroups such that*

$$I_{fin}(\mathcal{B}_i) \subseteq I_i \subseteq I(\mathcal{B}_i), \quad i = 0, 1.$$

Then

$$I_0 \equiv I_1 \Rightarrow \mathcal{B}_0 \equiv \mathcal{B}_1, \text{ and}$$

$$I_0 \cong I_1 \Rightarrow \mathcal{B}_0 \cong \mathcal{B}_1.$$

RCDLs

Theorem. *Let* $\mathcal{B}_0$ *and* $\mathcal{B}_1$ *be RCDLs considered in the language* $\langle \cap, \cup, \setminus, 0 \rangle$ *and* I_i *are inverse semigroups such that*

$$I_{fin}(\mathcal{B}_i) \subseteq I_i \subseteq I(\mathcal{B}_i), \quad i = 0, 1.$$

Then

$$I_0 \equiv I_1 \Rightarrow \mathcal{B}_0 \equiv \mathcal{B}_1, \text{ and}$$

$$I_0 \cong I_1 \Rightarrow \mathcal{B}_0 \cong \mathcal{B}_1.$$

RCDLs

Let $\mathcal{F}$ denote the (unique) computable nontrivial atomless RCDL with no greatest element.

Theorem. *Assume that $\mathcal{B}_0$ and $\mathcal{B}_1$ are computable RCDLs in the language $\langle \cap, \cup, \setminus, 0 \rangle$. Suppose that there exists a computable isomorphic embedding of $\mathcal{F}$ into $\mathcal{B}_0$ and that $I_c(\mathcal{B}_0) \cong I_c(\mathcal{B}_1)$. Then $\mathcal{B}_0 \cong_c \mathcal{B}_1$.*

Equivalence Structures

Theorem. *Let* $\mathcal{M}_0 = \langle M_0, E_0 \rangle$ *and* $\mathcal{M}_1 = \langle M_1, E_1 \rangle$ *be nontrivial equivalence structures and let* I_i *be inverse semigroups such that*

$$I_{fin}(\mathcal{M}_i) \subseteq I_i \subseteq I(\mathcal{M}_i), \quad i = 0, 1.$$

Then

1. $I_0 \cong I_1 \;\Leftrightarrow\; \mathcal{M}_0 \cong \mathcal{M}_1$;

2. $I_0 \equiv I_1 \;\Rightarrow\; \mathcal{M}_0 \equiv \mathcal{M}_1$; *and*

3. *if both the structures* $\mathcal{M}_0$ *and* $\mathcal{M}_1$ *are countable then*

$$I_{fin}(\mathcal{M}_0) \equiv I_{fin}(\mathcal{M}_1) \;\Leftrightarrow\; \mathcal{M}_0 \cong \mathcal{M}_1.$$

Equivalence Structures

Theorem. *Let $\mathcal{M}$ be a nontrivial computable equivalence structure. Then there exists a first order sentence φ in the language of inverse semigroups such that for any nontrivial computable equivalence structure $\mathcal{N}$,*

$$I_c(\mathcal{N}) \models \varphi \;\Rightarrow\; \mathcal{N} \cong_c \mathcal{M}.$$

Strategy

Our general approach is to interpret as much of the structure $\mathcal{M}$ into I as possible.

Basic Interpretations

Our first goal is to interpret the universe of $\mathcal{M}$ in I, where I is any inverse semigroup so that $I_{fin}(\mathcal{M}) \subseteq I \subseteq I(\mathcal{M})$.

1. Interpret (some) subsets of $\mathcal{M}$ in I.

   - Let $\mathtt{Id}(x)$ be the formula $x^2 = x$, a first-order formula requiring x to be idempotent.

   - Functions satisfying $\mathtt{Id}(x)$ are the identity on their domain.

   - They can be identified with subsets of $\mathcal{M}$.

Basic Interpretations

2. Define the notion of "subset" in I.

- $\texttt{Id}(x)$ & $\texttt{Id}(y)$ & $xy = x$ holds in I exactly when $x \subseteq y$ in $\mathcal{M}$.

3. Interpret the empty set, $\emptyset$, as the (unique) function contained in all other functions.

4. Define $A(\mathcal{M}) = \left\{\{(a,a)\} | a \in \mathcal{M}\right\}$, the interpretation of the universe of $\mathcal{M}$ in I.

- $x \in I$ is in $A(\mathcal{M})$ if $x \neq \emptyset$ & $\neg\exists u(\emptyset \subset u \subset x)$.
- We identify $x \in M$ with the partial automorphism $\{(x,x)\} \in I$.

Basic Interpretations

The second goal is to interpret the natural action of elements of I on elements $A(\mathcal{M}) \cup \{\emptyset\}$.

For $g \in I$ and $x, y \in M$, $g(x) = y$ exactly when $I \models gxg^{-1} = y$.

Equivalence structures

Here we consider structures of kind $\mathcal{M} = \langle M; E \rangle$, where E is an equivalence relation on M.

We say an equivalence structure is nontrivial if E is not the same as equality.

Interpreting the equivalence relation in the semigroup

We'll need to interpret E into I where $I_{fin}(\mathcal{M}) \subseteq I \subseteq I(\mathcal{M})$.

1. Let $p, q \sim r, s$ abbreviate $\exists f(f(p) = r \ \& \ f(q) = s)$.

2. Let

$$\begin{aligned}\widetilde{E}(x, y) \quad =_{\mathsf{def}} \quad & (x \neq \emptyset) \ \& \ (y \neq \emptyset) \ \& \\ & \forall a \, \forall b \, \forall c \left((x, y \sim a, b \ \& \ x, y \sim b, c) \rightarrow x, y \sim a, c\right).\end{aligned}$$

Note that the following holds,

$$\mathcal{M} \models E(x, y) \Leftrightarrow I \models \widetilde{E}(x, y).$$

Equivalence structures

Theorem. *Let $\mathcal{M}_0 = \langle M_0, E_0 \rangle$ and $\mathcal{M}_1 = \langle M_1, E_1 \rangle$ be nontrivial equivalence structures and let I_i be inverse semigroups such that*

$$I_{fin}(\mathcal{M}_i) \subseteq I_i \subseteq I(\mathcal{M}_i), \quad i = 0, 1.$$

Then

1. $I_0 \cong I_1 \;\Leftrightarrow\; \mathcal{M}_0 \cong \mathcal{M}_1$;
2. $I_0 \equiv I_1 \;\Rightarrow\; \mathcal{M}_0 \equiv \mathcal{M}_1$; *and*
3. *if both the structures $\mathcal{M}_0$ and $\mathcal{M}_1$ are countable then*

$$I_{fin}(\mathcal{M}_0) \equiv I_{fin}(\mathcal{M}_1) \;\Leftrightarrow\; \mathcal{M}_0 \cong \mathcal{M}_1.$$

Equivalence structures

Sketch of proof for (3).

- $\mathcal{M}_0$ and $\mathcal{M}_1$ are isomorphic iff they have exactly the same number of n-element equivalence classes for $n \in \omega \cup \{\omega\}$.

- Let $\varphi_{m,n}$ say *"E has at least m n-element equivalence classes."*
  - *For finite n, it is easy to find such a formula.*
  - *For the infinite case, we need only see how to say "x* is a member of an infinite equivalence class."
  - *Note that this is the case exactly when*

$$\neg \exists f (\forall y (\widetilde{E}(x, y) \to y \in \mathtt{dom}\,(f))).$$

Characterization of computable equivalence structures

Theorem. *Let $\mathcal{M}$ be a nontrivial computable equivalence structure. Then there exists a first order sentence φ in the language of inverse semigroups such that for any nontrivial computable equivalence structure $\mathcal{N}$,*

$$I_c(\mathcal{N}) \models \varphi \;\Rightarrow\; \mathcal{N} \cong_c \mathcal{M}.$$

Proof idea:

Divide the proof into cases based on three scenarios:

Case 1. $\mathcal{M}$ has finitely many equivalence classes.

Case 2. $\mathcal{M}$ has infinitely many equivalence classes.

- **Subcase 1.** The set of cardinalities of the equivalence classes of $\mathcal{M}$ is finite, that is, $\mathcal{M}$ has *bounded character*.

- **Subcase 2.** This set is infinite, or $\mathcal{M}$ has *unbounded character*.

Case 1 versus Case 2

There is a first order formula $\pi(p)$ in the language of semigroups requiring that the function p has, among other properties, an infinite domain consisting of exactly one representative of each equivalence class.

The sentence "$\exists p\ \pi(p)$" will distinguish Case 1 from Case 2.

Subcase 1 versus Subcase 2

There is a first order sentence, γ, in the language of semigroups asserting the existence of a finite set F so that for any $x \in A(\mathcal{M})$, there are $y \in F$ and $g \in I_c(\mathcal{M})$ so that g is a bijection from $[x]_E$ onto $[y]_E$.

The existence of such an F will distinguish Subcase 1 from Subcase 2.

Case 1.

$\mathcal{M}$ has m equivalence classes having cardinalities $k_0, k_1, \ldots, k_{m-1}$, where $k_i \in \omega \cup \{\omega\}$.

- This property can be expressed in the language of $I_c(\mathcal{M})$ by

$$\exists x_0, \ldots, x_{m-1}\Big[\bigwedge_{i<j<m}(x_i, x_j) \notin E \ \& \ \forall x \Big(\bigvee_{i<m}(x, x_i) \in E\Big) \ \& \ \bigwedge_{i<m}[x_i]_E \textit{ contains } k_i \textit{ elements})\Big].$$

- If a computable equivalence structure $\mathcal{N}$ satisfies this formula, it is computably isomorphic to $\mathcal{M}$.

Case 2

$\mathcal{M}$ has infinitely many equivalence classes – so there is a partial computable automorphism p satisfying $\pi(p)$.

- We'll use this p as a list of the distinct equivalence classes of $\mathcal{M}$, and describe their cardinalities along this list.

We give the idea for Subcase 1.

Case 2, Subcase 1.

$\mathcal{M}$ has infinitely many equivalence classes and the set of their cardinalities is finite.

Let $K = \{k_0 < k_1 < \ldots < k_{m-1}\} \subset \omega \cup \{\omega\}$ be this set.

- Use p as a list of the equivalence classes in $\mathcal{M}$, and we can describe the cardinalities along this list by a formula in the language of $I_c(\mathcal{M})$:

$$\forall t \in \mathtt{dom}\,(p) \quad \bigvee_{i<m} \Big(\varphi_i(t) \ \& \ \psi_i(t)\Big).$$

Conclusions

- Even when we know nothing about a structure's global symmetries, we can learn about it by looking at local symmetries.