

Strong Reducibilities, Scattered Linear Orders, Ranked Sets, and Kolmogorov Complexity

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Preliminaries

- $A \leq_T B$ if there is an algorithm using B as an oracle that will compute the characteristic function of A .
- $A \leq_{wtt} B$ if there's an algorithm like before, but also a computable function that limits how much of the oracle B the algorithm can use.
- The *Turing degree of the set A* , $\text{deg}(A)$ is the collection of all sets \equiv_T to A .
- The *wtt-degree of the set A* , $\text{deg}_{wtt}(A)$ is the collection of all sets \equiv_{wtt} to A .

Background

We consider computable linear orderings (CLOs) $\mathcal{L} = \langle L, <_{\mathcal{L}} \rangle$, and think about an additional relation R on the structure.

Example

$\mathcal{L} \cong \omega + \omega^*$ with additional relation $R = \omega_{\mathcal{L}}$.



- The *degree spectrum of relation R on a computable structure \mathcal{M}* , $DgSp_{\mathcal{M}}(R)$, is the collection of all Turing degrees of images of R in computable structures $\mathcal{N} \cong \mathcal{M}$.
- The *wtt-spectrum of relation R on a computable structure \mathcal{M}* , $DgSp_{\mathcal{M}}^{wtt}(R)$, is the collection of all wtt-degrees of images of R in computable structures $\mathcal{N} \cong \mathcal{M}$.

Context and some facts about $\omega + \omega^*$

Let \mathcal{L} be a CLO isomorphic to $\omega + \omega^*$, and $\omega_{\mathcal{L}}$ the ω -part of \mathcal{L} .

- (Harizanov, 1998) The (Turing) degree spectrum of $\omega_{\mathcal{L}}$ is exactly the Δ_2^0 -degrees.
- Is the same true of the wtt-spectrum? Does it consist of all wtt-degrees that are wtt-computable from the halting set?

No.

This is what we *can* say:

Theorem

For every Δ_2^0 set A , there is a CLO \mathcal{L} of order type $\omega + \omega^*$ with $A \leq_T \omega_{\mathcal{L}} \leq_{wtt} A$.

We'll see that this is the best we can do: \leq_T can't be replaced with \leq_{wtt} in the Theorem.

A stronger statement

Theorem

There is a c.e. set D that is not wtt-reducible to any initial segment of any computable scattered linear ordering.

(A linear ordering is *scattered* just in case it fails to contain a copy of $\mathcal{Q} = \langle \mathbb{Q}, <_{\mathbb{Q}} \rangle$. For example, $\omega + \omega^*$.)

The punchline: **The halting set, $0'$, itself will be this set.**

We will see that if $0'$ is wtt-reducible to an initial segment of a CLO, then that linear ordering is not scattered.

Though $0'$ is at the top of the Δ_2^0 sets, we can find a *low* c.e. set that does the same thing.

A nice fact about scattered linear orderings

Let \mathcal{L} be a countable linear ordering. Then \mathcal{L} is scattered iff \mathcal{L} has only countably many initial segments.

If \mathcal{L} is a CLO, then \mathcal{L} is scattered iff each of its initial segments is *ranked* – an element of a countable Π_1^0 class.

(A set of sets of natural numbers is a Π_1^0 class if it is the collection of paths through a computable tree.)

Fact

Let \mathcal{L} be a countable linear ordering. Then \mathcal{L} is scattered iff \mathcal{L} has only countably many initial segments.

Proof. \leftarrow . If \mathcal{L} has a copy of \mathcal{Q} , it has as many initial segments as \mathcal{Q} does... uncountably many.

\rightarrow . Suppose \mathcal{L} has uncountably many initial segments... then it has a copy of \mathcal{Q} :

- Let \mathcal{I} be the collection of initial segments of \mathcal{L} (view these as paths through a subtree of $2^{<\omega}$).
- \mathcal{I} is a closed uncountable set in Cantor space 2^ω , and so has a perfect subset \mathcal{J} . Take T to be the perfect subtree of $2^{<\omega}$ with $[T] = \mathcal{J}$.
- For each branching node of T , take a_σ to be an element of \mathcal{L} that the extending nodes disagree on.
- It's easy to check that these a_σ 's form a copy of \mathcal{Q} .

So, we need to show that if an initial segment of a CLO wtt-computes $0'$, then that CLO has uncountably many initial segments.

Equivalently, the collection of initial segments has a (nonempty) perfect subset.

To do this, we'll use facts about Π_1^0 classes and their members since the collection of initial segments forms a Π_1^0 class.

Some definitions.

- For finite strings σ , the *Kolmogorov complexity* of σ , $C(\sigma)$, is the length of the shortest program you can write that will output σ .
- An *order* is a computable, nondecreasing, unbounded function.
- A set A is *complex* if there is an order g so that

$$\forall n C(A \upharpoonright n) \geq g(n).$$

- A function f is *diagonally non-computable* (DNC) if for each $e \in \omega$, the value of $f(e)$ is different from $\varphi_e(e)$ whenever $\varphi_e(e) \downarrow$.

Some facts.

Theorem (Kjos-Hanssen, Merkle and Stephan)

A set A is complex iff there is a DNC function $f \leq_{wtt} A$.

So...

- If $A \leq_{wtt} B$ and A is complex, so is B . (\leq_{wtt} is transitive.)
- $0'$ is complex. Why? $0'$ wtt-computes

$$f(\mathbf{e}) = \begin{cases} \varphi_{\mathbf{e}}(\mathbf{e}) + 1 & \text{if } \varphi_{\mathbf{e}}(\mathbf{e}) \downarrow \\ 0 & \text{if } \varphi_{\mathbf{e}}(\mathbf{e}) \uparrow \end{cases} .$$

A theorem about Π_1^0 classes

Theorem

Let P be a Π_1^0 class with a complex element A . Then P has a perfect Π_1^0 subclass Q with $A \in Q$.

Proof. Let g be an order witnessing that A is complex:

$$\forall n C(A \upharpoonright n) \geq g(n).$$

Set $Q = \{X \in P \mid \forall n C(X \upharpoonright n) \geq g(n)\}$, and note that Q is a Π_1^0 subclass of P and that it is nonempty. (A is in it.)

By definition, every element in Q is complex, and so can't have any isolated elements (such an element would be computable). So Q has to be perfect.

$0'$ is not wtt-reducible to any initial segment of any scattered CLO

Take a CLO \mathcal{L} with an initial segment A that wtt-computes $0'$.
Let P be the (Π_1^0) class of initial segments of \mathcal{L} .

A is complex since $0'$ is, and is an element of P , so P has a nonempty perfect Π_1^0 subclass by the Theorem we just proved, and so \mathcal{L} must have uncountably many initial segments.

By the earlier lemma, we see that \mathcal{L} contains a copy of the rationals, and so is not scattered.

References

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