Strong degree spectra of initial segments of scattered linear orders

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Linear orders and initial segments

A linear order may be highly computable, but have complicated initial segments.

Example

$(\mathbb{Q}, <)$ is computable, but has initial segments of every Turing degree.

Many important notions arise in our study of these objects...

- Π^0_1 classes and ranked sets,
- Ideas from algorithmic complexity theory,
- Array non-computable degrees and totally ω-c.e. sets.

Starting point

The *(Turing) degree spectrum of a relation R on a structure* $\mathcal{M}$, $DgSp_{\mathcal{M}}(R)$, is the collection of all Turing degrees of images of R in computable structures $\mathcal{N} \cong \mathcal{M}$.

Example (Harizanov, 1998)

Let $\mathcal{L}$ be a computable linear order isomorphic to $\omega + \omega^*$, and $\omega_{\mathcal{L}}$ the ω-part of $\mathcal{L}$.
Then, the degree spectrum of $\omega_{\mathcal{L}}$ is exactly the Δ^0_2-degrees.

Strong degree spectra

$A \leq_{wtt} B$ if there is an $e \in \omega$ and a computable function $g(x)$ so that for every x,

$$A(x) = \varphi_e^{B \upharpoonright g(x)}(x).$$

$$<_{tt} \implies <_{wtt} \implies <_T$$

The *wtt-spectrum a of relation R on a structure* $\mathcal{M}$, $DgSp_{\mathcal{M}}^{wtt}(R)$, is the collection of all wtt-degrees of images of R in computable structures $\mathcal{N} \cong \mathcal{M}$.

Question

What is the wtt-spectrum of the ω part of $\omega + \omega^*$?

Is it something analogous to the Turing spectrum case?

Partway...

Theorem

For every Δ^0_2 set A, there is a computable linear order $\mathcal{L}$ of order type $\omega + \omega^*$ with $\omega_{\mathcal{L}} \equiv_T A$ and $\omega_{\mathcal{L}} \leq_{wtt} A$.

In fact, this is true with wtt replaced with tt; and when A is c.e., we can ensure $\omega_{\mathcal{L}}$ will be too.

Initial segments of scattered linear orders, and ranked sets

- The collection of initial segments of a computable linear ordering forms a Π^0_1 *class* (the paths through a computable tree).
- A linear ordering is *scattered* (does not contain a copy of $(\mathbb{Q}, <)$) iff it has only countably many initial segments.
- Every initial segment of a scattered computable linear ordering (SCLO) is a *ranked set* (a member of a countable Π^0_1 class).

Π^0_1 classes and complex sets

A set A is *complex* if there is a computable, nondecreasing, unbounded function g so that $\forall n\, [C(A \upharpoonright n) \geq g(n)]$, where $C(\sigma)$ denotes the plain Kolmogorov complexity of the string σ.

Theorem (Kjos-Hanssen, Merkle, and Stephan, 2006)

A set A is complex iff there is a diagonally non-computable function $f \leq_{wtt} A$.

Note:

- Complex sets are not computable.
- The complex sets are closed up with respect to $\leq_{wtt}$.
- The halting set is complex.

A c.e. wtt-degree not in $DgSp^{wtt}_{\omega_{\mathcal{L}}}(\omega + \omega^*)$

Theorem

If a Π^0_1 class $\mathcal{P}$ contains a complex set A, then $\mathcal{P}$ has a perfect Π^0_1 subclass.

Proof.
Define the Π^0_1 subclass $\{B \in \mathcal{P} \mid \forall n[C(B \upharpoonright n) \geq g(n)]\}$.

□

Corollary

The halting set, $0'$, is not wtt-reducible to any initial segment of any computable scattered linear ordering, or to any ranked set, nor is anything in the wtt-cone above $0'$.

Superlow degrees

A set A is *superlow* if $A' \leq_{wtt} \emptyset'$.

Theorem

There is a superlow set that is not wtt-reducible to any ranked set.

Proof.
The collection of $\{0, 1\}$-valued diagonally non-computable functions forms a Π^0_1 class. The proof of the Low Basis Theorem yields a superlow element of this class. □

However,

- Every superlow c.e. set wtt-reduces to some ranked set.
- If a set is complex and c.e., it is wtt-complete.

Are there incomplete c.e. degrees that can't be wtt-reduced to ranked sets?

Low c.e. degrees

A degree **d** is *array non-computable* (ANC) if for each $f \leq_{wtt} \emptyset'$ there is a **d**-computable function g so that f does not dominate g.

d is *uniformly ANC* if there is a fixed **d**-computable function g that is not dominated by any $f \leq_{wtt} \emptyset'$.

Theorem

For every uniformly ANC degree **d**, *there is a* **d**-*computable set D so that D is not wtt-reducible to any ranked set.*

Additionally, when **d** *is c.e., D can be made c.e.*

Low c.e. degrees

Theorem

There is a low c.e. uniformly ANC degree.

From these, we have a *low* c.e set D so that the wtt-cone above D is disjoint from every countable Π^0_1 class, and every Π^0_1 class that intersects that cone has a perfect Π^0_1 subclass.

(This set, D, is cannot wtt-reduce to any initial segment of any computable scattered linear ordering.)

Related results of Downey & Greenberg

A degree **d** is *totally* ω*-c.e.* if every **d**-computable function is $\leq_{wtt} \emptyset'$.

- A c.e. degree is totally ω-c.e. iff it is not uniformly ANC.
- Every c.e. totally ω-c.e. set wtt-reduces to a c.e. set that is the ω part of a computable linear order of order type $\omega + \omega^*$.

Thus, for the c.e. degrees, we have that **d** contains a c.e. set that is not wtt-reducible to a ranked set if and only if **d** is uniformly ANC.

References

- Chisholm, Chubb, Harizanov, Hirschfeldt, Jockusch, McNicholl, Pingrey. Π^0_1 classes and strong degree spectra of relations, to appear in the *Journal of Symbolic Logic*.
- Harizanov. Turing degrees of certain isomorphic images of recursive relations, *Annals of Pure and Applied Logic* 93 (1998), 103 – 113.
- Kjos-Hanssen, Merkle, Stephan. Kolmogorov complexity and the recursion theorem, STACS 2006: *Twenty-Third Annual Symposium on Theoretical Aspects of Computer Science* (Marseille, France, February 23-25, 2006, Proceedings, Springer LNCS 3884), 149-161.