Calibrating Spread Options using a Seasonal Forward Model

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Workshop on Computational Methods for Pricing and Hedging Exotic Options, July the 12th, ’08
1. Outline the pricing of spread options
2. Review a two-factor seasonal commodities model
3. Describe a calibration algorithm based on principal components
4. Present a numerical example of a heating rate option
Background

- Recent surge of interest in commodity derivatives
- In many cases, only the forward prices of the commodity assets are market observables
- Manage exposure to loss through holding commodity derivatives
- Credit risk models must accurately predict the underlying correlated dynamics of the term structure
- Interest rate derivative modeling techniques are useful but limited, e.g. seasonality
Overview of Approaches

1. Ribeiro and Hodges \(^1\) and Barlow et al. \(^2\) respectively apply Kalman filters to calibrate commodity spot prices.
2. Cortazar and Schwartz \(^3\) perform a least squares regression on \(\ln F_t(T)\).
3. Borovkova and Geman \(^4\) propose a seasonality model for a wider class of seasonal commodity futures.
4. Borovkova and Geman \(^5\) apply principal component analysis to deseasonalized futures prices under the real-world measure.

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\(^1\) Diana Ribeiro and Stewart Hodges [2004], A Two-Factor Model for Commodity Prices and Futures Valuation, Technical report, Financial Options Research Center, Warwick Business School.


\(^3\) G. Cortazar and E.S. Schwartz [2003], Implementing a stochastic model for oil futures, Energy Economics 25, pp. 215â238.


The risk neutral price $V(t)$ of a time $T$ expiring option on the spread of two forward contracts is

$$V(t) = \exp\{-r\tau\} \mathbb{E}_t^*(|F_1(T, T_1) - F_2(T, T_2) - K|^+)$$

The risk neutral conditional expectation $\mathbb{E}_t^*(\cdot) := \mathbb{E}^*(\cdot | \mathcal{F}_t)$

$F_i(t, T)$ denotes the time $t$ value of a forward contract to deliver the underlying $S(T)$ at time $T$

$F_1(t, T_1)$ and $F_2(t, T_2)$ may reference difference underlyings $S_1(t)$ and $S_2(t)$
Examples of Spread Options (I)

Heating rate Option

- The risk neutral price of a heating rate/spark-spread (call) option $a$ is

$$ V(t) = \exp\{-r\tau\} \mathbb{E}_t^* \left( |F_p(T, T_p) - H_{\text{eff}} F_g(T, T_g) - K|^+ \right) $$

- $F_p(t, T_p)$ is the time $T_p \geq T$ expiring forward power contract
- $F_g(t, T_g)$ is the time $T_g \geq T$ expiring forward natural gas contract
- $H_{\text{eff}}$ is a fixed energy efficiency factor
- $K$ is the strike of the option expiring at time $T$

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$^a$Daily strip of heating rate options:

$$ V_t = \exp\{-r\tau\} \sum_{m=1}^{N_m} \sum_{d=1}^{N_d} h_d^m \mathbb{E}_t^* \left( |F_p(T, T_p) - H_{\text{eff}} F_g(T, T_g) - K|^+ \right) $$
Examples of Spread Options (II)

Calender Spread Option

- The risk neutral price of a calender spread (call) option is

\[ V(t) = \exp\{-r\tau\} \mathbb{E}^*_t (F(T, T_1) - F(T, T_2) - K^+) \]

- \( F(t, T_i) \) is the time \( T_i \geq T \) expiring forward contract
- \( K \) is the strike of the option expiring at time \( T \)
Kirk’s Formula

- There exists a closed form expression for $V(t)$
  $$V(t) = \exp\{-r\tau\}(F_2(t, T_2) + K)(F(t, T)N(d_+) - N(d_-)),$$
  where
  $$F(t, T_1, T_2) := \frac{F_1(t, T_1)}{F_2(t, T_2) + K},$$
  $$d_+ = \frac{\ln F(t, T)}{\sigma \sqrt{\tau}} + \frac{\sigma \sqrt{\tau}}{2},$$
  $$d_- = d_1 - \sigma \sqrt{\tau}.$$

- $F(t, T)$ is assumed to be a Martingale w.r.t. the risk neutral measure
  $$dF(t, T) = \sigma F(t, T) dW_t^*.$$
The Story so Far

- The closed form expression for pricing spread options on forward contracts assumes that $F(t, T) = \frac{F_1(t, T_1)}{F_2(t, T_2) + K}$ is log normal under the pricing measure.

- We have not yet specified the dynamics of each forward contract $F_i(t, T_i)$. 
Figure: (Left) Expectation and (right) std. dev. of the Tet M3 natural gas forward curve as a function of monthly maturity date $T$ traded in the month of July.
A Seasonal Forward Model [GEMAN  BOROVKOVA]

- Borovkova and Geman⁷ express forward prices in component form

\[ F_t(T) = \bar{F}_t \exp(s(T) - \gamma_t(\tau)\tau), \]

- \( \bar{F}_t \) denotes the mean value of the curve \( T \rightarrow F(t, T) \) at each time \( t \)
- \( \gamma_t(\tau) \) is the stochastic convenience yield
- \( s(T) \) is the seasonality function

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⁷ S. Borovkova and H. Geman, Seasonal and stochastic effects in commodity forward curves, Rev Deriv Res (9), 2006, pp. 167-186
**A Seasonal Forward Model**

**Figure:** The historical Tet M3 natural gas forward curve at the start of the time series (where $T = \tau$) is separated into its constitutive components, the seasonality $s(T)$ and the convenience yield $\gamma_0(\tau)$. 
A Seasonal Forward Model

Intrinsic Dynamics

\[ d\ln \bar{F}_t = \alpha (m - \ln \bar{F}_t) dt + \sigma dW_t^{[1]} \]
\[ d\gamma_t(\tau) = -a(\tau)\gamma_t(\tau) + \eta(\tau) dW_t^{[2]} \]

Two-Factor Forward Model

\[ a'(\tau) = a(\tau) + 1 \]

\[ d\ln F_t(T) = \left[ \alpha (m - \ln \bar{F}_t) + \gamma_t(\tau)a'(\tau) \right] dt + \sigma dW_t^{[1]} - \eta(\tau)\tau dW_t^{[2]} \]

- \( W_t^{[1]} \) and \( W_t^{[2]} \) are two independent Wiener processes under the \textit{real-world} measure
A Seasonal Forward Model in the Pricing Measure

Risk Neutral Intrinsic Dynamics

\[ d\ln \tilde{F}_t = \sigma dW_t^{[1]} \]
\[ d\gamma_t(\tau) = \eta(\tau) dW_t^{[2]} \]

Risk Neutral Two-Factor Forward Model

\[ d\ln F_t(T) = \sigma dW_t^{[1]} - \eta(\tau) \tau dW_t^{[2]}, \]

- \( W_t^{[1]} \) and \( W_t^{[2]} \) are two independent Wiener processes under the risk neutral measure
Review of Methodology

1. Compute the geometric average of the futures price

\[ \ln \bar{F}_t = \frac{1}{N} \sum_{i=1}^{N} \ln F_t(T_i) \]

2. Estimate the seasonality function from the historical futures price series

\[ \hat{s}(T) = \frac{1}{n} \sum_{i=1}^{n} \ln F_{t_i}(T) - \ln \bar{F}_{t_i} \]

3. Imply the convenience yield time series from the seasonal forward model

\[ \gamma_t(\tau) = \frac{\ln \bar{F}_t(T) - \hat{s}(T)}{\frac{\tau}{T}} \]

\[^8N \text{ is assumed to be a multiple of 12.}\]
The Covariance Matrix

The theoretical covariance matrix takes the form

\[ V_{ij}^{Th} = \int_{t_1}^{t_2} d\gamma_t(T_i)d\gamma_t(T_j) \]

The implied (empirical) covariance matrix is

\[ V_{ij}^{Imp} = \sum_{t=t_1}^{t_2-1} \Delta\gamma_t(T_i)\Delta\gamma_t(T_j) \]
Calibration of the Forward Model

Definition (Filtered box constrained calibration problem)

\[
\min_{\eta \in S \subset \mathbb{R}_+^N} \hat{z} = |R^T V^{Th} R - \Lambda|^2 = \sum_{k,l}^{d \leq N} \left( R_{ki} V_{ij}^{Th} R_{jl} - \delta_{kk} \lambda^k \right)^2
\]

- The columns of \( R \) and principal diagonal elements of \( \Lambda \) are the eigenvectors and eigenvalues of \( V^{imp} \)
Calibration Algorithm

- \( \hat{\eta} = R^T \eta \) is the solution vector projected onto the principal component basis
- Express the gradient \( \nabla_\eta \hat{z} = R \nabla_{\hat{\eta}} \hat{z} \)
- \( \hat{z} \) is everywhere differentiable w.r.t. the projected solution vector
- Specify bounds on the solution vector (not on the projected solution vector)
- Use a gradient-based constrained non-linear optimization algorithm (e.g. projected gradient methods with Armijo rule.)
Overview

- Tet M3 and Conn NE natural gas and peak electricity futures prices (USD)
- Monthly increments up to two year futures contracts with full historical data over the period Nov-04 to Sep-07
- Perform Shapiro-Wilks and Box-Ljung tests on log returns to measure normality and stationarity
- Compare the performance of numerous constrained optimization algorithms \(^9\) provided in the opensource c++ library Opt++ \(^10\).

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\(^9\) C.T. Kelley [1999], Iterative Methods for Optimization, Frontiers in Applied Mathematics 18, SIAM.

\(^10\) http://csmr.ca.sandia.gov/opt++
Time series analysis

Time Series Analysis: Natural Gas Futures

![Graphs showing time series analysis of Natural Gas Futures](image-url)
Time Series Analysis: Electricity Futures
Parameter estimation

**Estimated Seasonality**

**Figure:** The seasonality of Tet M3 natural gas and Conn NE peak electricity forwards.
Parameter estimation

**Correlation between Natural gas and Electricity Convenience Yields**
Parameter estimation

**Convenience Yield Volatility Term-Structure**

**Figure:** The calibrated convenience yield volatility term-structure of Tet M3 natural gas and Conn NE peak electricity forwards.
**Figure:** The gradient projection method.
Constrained optimization algorithms

**Performance Comparison of Constrained Optimization Algorithms**

*Figure:* The projected BFGS method.
Constrained optimization algorithms

**Performance Comparison of Constrained Optimization Algorithms**

![Graph showing the performance comparison of constrained optimization algorithms.](image)

**Figure:** The interior reflective Newton method.
Constrained optimization algorithms

**Performance Comparison of Constrained Optimization Algorithms**

![Graph showing performance comparison of constrained optimization algorithms](image)

**Figure:** The finite difference interior point method.
Constrained optimization algorithms

Performance Comparison of Constrained Optimization Algorithms

Figure: The quasi-newton interior point method.
Constrained optimization algorithms

**Performance Comparison of Constrained Optimization Algorithms**

*Figure:* The BC quasi-newton method.
Constrained optimization algorithms

**Performance Comparison of Constrained Optimization Algorithms**

Numerical Experiments: Heating Rate Option
Principal Component Analysis: Natural Gas and Electricity Futures
Constrained optimization algorithms

**Heating Rate Call Price (USD)**
Constrained optimization algorithms

Error in Heating Rate Call Price (USD) with 3 PCs

![Graph showing error in Heating Rate Call Price with 3 PCs.](image)
Constrained optimization algorithms

**Error in Heating Rate Call Price (USD) with 5 PCs**
Constrained optimization algorithms

**Error in Heating Rate Call Price (USD) with 10 PCs**
Constrained optimization algorithms

**Summary**

- The accurate calibration of non-storable spread options to the observed underlying forward contracts is challenging.
- First deseasonalize historical time series of log returns and perform PCA on the correlated convenience yield returns.
- The volatility term-structure can be captured with only a few principal components.
- Preliminary results suggest that the combination of a seasonal forward model, PCA and a gradient based constrained optimization algorithm is efficient and robust (avoid simulated annealing/genetic algorithms).
- **Future directions:**
  - Automate the selection of the number of principal components according to errors in the greeks.
  - Fit uncorrelated GARCH processes for the volatility w.r.t. each of the principal components.