1. Prove that the function \( a : \mathbb{R}^2 \to \mathbb{R} \) defined by \( a(x, y) = x + y \) is continuous. Hint: observe that the inverse image of \( z \) is \( \{(x, y) \in \mathbb{R}^2 : x + y = z\} \). That is, the inverse image of a point is a line with slope -1. From this determine what the inverse image of an open interval is, and use this to prove that the inverse image of an open set is open.

We first show that open subsets of \( \mathbb{R} \) are unions of open intervals. So suppose \( U \) is an open subset of \( \mathbb{R} \). Then for each \( x \in U \), there exists a real number \( r_x > 0 \), such that \( B_{r_x}(x) = (x - r_x, x + r_x) \subseteq U \).

Thus,
\[
U = \bigcup_{x \in U} (x - r_x, x + r_x).
\]

Since
\[
a^{-1}(U) = a^{-1} \left( \bigcup_{x \in U} (x - r_x, x + r_x) \right) = \bigcup_{x \in U} a^{-1}(x - r_x, x + r_x),
\]
in order to show that \( a \) is continuous it suffices to show that if \( p \) and \( q \) are real numbers, and \( p < q \), then \( a^{-1}([p, q]) \) is an open subset of \( \mathbb{R}^2 \). But from the observation in the hint, we have that
\[
a^{-1}([p, q]) = \{(x, y) \in \mathbb{R}^2 : p < x + y < q\}.
\]

That is, \( a^{-1}([p, q]) \) is the strip lying strictly between the lines \( x + y = p \) and \( x + y = q \). To see that this is open, suppose \((x_0, y_0)\) lies on neither \( x + y = p \) nor \( x + y = q \). To any line \( x + y = r \) is
\[
d_r = \frac{\sqrt{2}}{2}|x_0 + y_0 - r|
\]

Since by assumption \((x_0, y_0)\) lies on neither \( x + y = p \) nor \( x + y = q \), both \( d_p \) and \( d_q \) are positive. So if we let
\[
d = \min\{d_p, d_q\},
\]
then \( d > 0 \). Then neither the line \( x + y = p \) nor the line \( x + y = q \) will contain any point of \( B_d(x_0, y_0) \). Hence \( B_d(x_0, y_0) \subseteq a^{-1}([p, q]) \), and \( a^{-1}([p, q]) \) is open.

2. Suppose \( X \) is a compact topological space. If \( C \) is a closed subset of \( X \), show that \( C \) is compact.

Let \( \mathcal{U} \) be an open cover of \( C \). We need to show that there is a finite subset \( \{U_1, U_2, \ldots, U_k\} \) of \( \mathcal{U} \) such that
\[
C \subseteq \bigcup_{i=1}^k U_i.
\]
Consider

\[ V = U \cup \{ X - C \}. \]

Since \( C \) is closed, \( X - C \) is open. Furthermore,

\[ X \subseteq C \cup (X - C) \subseteq \left[ \bigcup U \right] \cup (X - C). \]

Since, by assumption \( X \) is compact, there is a finite subcover of \( U \cup \{ X - C \} \). So suppose that

\[ \{ U_1, U_2, \ldots, U_k \} \subseteq U \]

is the intersection of the finite subcover with \( U \).

Now observe that we must have that

\[ C \subseteq U_1 \cup U_2 \cup \ldots \cup U_k. \]

For

\[ C \subseteq X \subseteq \left[ \bigcup_{i=1}^{k} U_i \right] \cup \{ X - C \}, \]

and since \( X - C \) contains no point of \( C \), we have

\[ C \subseteq U_1 \cup U_2 \cup \ldots \cup U_k, \]

and hence \( C \) is compact. (Note that whether \( X - C \) is needed to cover all of \( X \) is not important here.)

3. Suppose that \( G \) is a group and \( H \) is a subgroup of \( G \).

(a) If \( h \in H \), prove that the left coset \( hH \) (or \( h + H \), if \( G \) is abelian) is the same as \( eH \) (or \( 0 + H \), if \( G \) is abelian).

Let \( g \in hH \). We want to first show that \( g \in eH = H \). Since \( g \in hH \), \( g = h\bar{h} \) for some \( \bar{h} \in H \). But by assumption \( H \) is a subgroup of \( G \). So it’s closed under multiplication. That is, \( g = h\bar{h} \in H = eH \).

Conversely suppose \( g \in eH = H \). We want to see that \( g \in hH \). Consider \( h(h^{-1}g) \). By assumption \( g \in H \), and since \( H \) is a subgroup and \( h \in H \), we also have that \( h^{-1} \in H \), and \( h^{-1}g \in H \). Thus, \( h(h^{-1}g) \in hH \). But by associativity, \( h(h^{-1}g) = (hh^{-1})g = eg = g \). That is, \( g \in hH \).

(b) If \( G \) is abelian, prove that the addition of left cosets defined by

\[ (g_1 + H) + (g_2 + H) = (g_1 + g_2) + H \]

is well-defined. That is, if \( g_1' + H = g_1 + H \) and \( g_2' + H = g_2 + H \), then \( (g_1' + g_2') + H = (g_1 + g_2) + H \).

Suppose that \( g_1 + H = g_1' + H \), and that \( g_2' + H = g_2 + H \). We want to show that \( (g_1 + g_2 + H = (g_1' + g_2') + H \). By a problem on homework assignment 4, we have that

\[ g + H = g' + H \] iff \( g - g' \in H \).
So we show that
\[(g_1 + g_2) - (g'_1 + g'_2) \in H.\]

But using the homework 4, result, we must have that
\[g_1 - g'_1 \in H \text{ and } g_2 - g'_2 \in H.\]

Since \(H\) is a subgroup, we have that
\[(g_1 - g'_1) + (g_2 - g'_2) = (g_1 + g_2) - (g'_1 + g'_2) \in H,\]

and hence addition of cosets is well-defined.

Note that commutativity of addition was essential in our argument. In fact, as the next problem shows, the result is false if \(G\) is nonabelian. However, if \(G\) is nonabelian the result is true if \(H\) is the kernel of a homomorphism. This turns out to be equivalent to
\[gHg^{-1} = \{ghg^{-1} : h \in H\} = H\]

for all \(g \in G\). Subgroups that satisfy this are called normal subgroups.

(c) Find an example of a nonabelian group \(G\) and a subgroup \(H < G\) for which multiplication of left cosets is not well-defined. That is, find \(g_1, g'_1, g_2, g'_2 \in G\), such that \(g_1H = g'_1H\) and \(g_2H = g'_2H\), but \(g_1g_2H \neq g'_1g'_2H\). Hint: Look at \(G = S_3\), the symmetric group on 3 symbols.

Consider \(S_3\), the group of bijections from a 3-element to itself. Using the notation we developed in class, we have
\[
\begin{align*}
\left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \right\}.
\end{align*}
\]

Since
\[
\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}
\]

has order 2,
\[H = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \right\}\]

is a subgroup. However, if
\[
\begin{align*}
g_1 &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \\
g_2 &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \\
g'_1 &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \\
g'_2 &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},
\end{align*}
\]

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then we have
\[ g_1H = g'_1H = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}, \]

and
\[ g_2H = g'_2H = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \right\}, \]

However,
\[ g_1g_2H = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \right\}, \]

and
\[ g'_1g'_2H = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \right\}. \]

4. Suppose that \( G \) and \( H \) are abelian groups, and \( \phi : G \rightarrow H \), is a homomorphism. Prove that the construction \( \bar{\phi}(g + H) = \phi(g) \) specifies a well-defined isomorphism
\[ \bar{\phi} : G/\ker(\phi) \rightarrow \text{im}(\phi). \]

Note: this theorem is true even when \( G \) and \( H \) are nonabelian groups. In particular, when \( H < G \) is the kernel of a homomorphism, multiplication of left cosets of \( H \) is well-defined.

This problem was deferred to homework assignment 4.