Topology
Quotient Space Example

Problem 4 on this week’s homework says, in part:

Suppose that \( X \) is a topological space and \( \sim \) is an equivalence relation on \( X \). The set of equivalence classes, often denoted \( X/\sim \), can be made into a topological space by defining a subset \( U \) of \( X/\sim \) to be open iff \( \cup U \) is an open subset of \( X \). Since the elements of \( X/\sim \) are equivalence classes of elements of \( X \), the notation \( \cup U \) just means “form the union in \( X \) of the equivalence classes that belong to \( U \).” Intuitively, \( X/\sim \) is formed from \( X \) by collapsing the equivalence classes of \( X \) to points.

1. Show that with this definition \( X/\sim \) is a topological space.

2. If \( p : X \to X/\sim \) is the projection function, \( p(x) = [x] \), show that a subset \( U \) of \( X/\sim \) is open iff \( p^{-1}(U) \) is open. Thus, the projection function is continuous. (Note that this projection is different from the projection functions defined on product spaces.)

I know a lot of you are a bit rusty on equivalence relations. So I thought a full-blown example might help. Be careful, though. Don’t read too much into the example.

Let \( X \) be \( \mathbb{R}^2 \) with the usual topology. Define an equivalence relation \( \sim \) on \( \mathbb{R}^2 \) as follows:

\[
(x, y) \sim (x', y') \text{ iff } x + y = x' + y'.
\]

Then \( \sim \) is an equivalence relation because

1. \((x, y) \sim (x, y)\) for all points \((x, y) \in \mathbb{R}^2\).
2. If \((x, y) \sim (x', y')\) then \((x', y') \sim (x, y)\).
3. If \((x, y) \sim (x', y')\) and \((x', y') \sim (x'', y'')\), then \((x, y) \sim (x'', y'')\).

(You should make sure that you know why these statements are true.)

Observe that the condition

\[
(x, y) \sim (x', y') \text{ iff } x + y = x' + y'
\]

implies that \((u, v) \sim (u', v')\) iff both points lie on the same line \( x + y = k \), for some real constant \( k \). Thus, the equivalence class of a point \((u, v)\) is

\[
[(u, v)] = \{(x, y) \in \mathbb{R}^2 : x + y = u + v\}.
\]

That is, \([u, v]\) is the line with slope -1 passing through \((u, v)\). Let’s denote this line by \( L_{u+v} \), and if \( u + v = a \in \mathbb{R} \), we’ll also denote it by \( L_a \).

So the quotient space \( \mathbb{R}^2/\sim \) is the set of lines with slope -1, and, given a collection of lines,

\[
U = \{L_a : a \in A \subseteq \mathbb{R}\},
\]
the collection is open in $X/\sim$ iff
\[ \bigcup U = \bigcup_{a \in A} L_a \]
is an open subset of $\mathbb{R}^2$. For example, if $A$ is the open interval $(p, q)$, then
\[ \bigcup U = \bigcup_{a \in (p, q)} L_a \]
which is the open strip consisting of lines with slope -1 crossing the $x$-axis in the interval $(p, q)$. (Draw a picture!)

In fact, any collection of lines
\[ U = \{ L_a : a \in A \} \]
will be an open subset of $\mathbb{R}^2/\sim$ precisely when $A$ is an open subset of $\mathbb{R}$. (How would you prove this?)

The function $p : \mathbb{R}^2 \rightarrow \mathbb{R}^2/\sim$ just sends the point $(x, y)$ to its equivalence class, the line, $L_{x+y}$. So if
\[ U = \{ L_a : a \in A \subseteq \mathbb{R} \}, \]
then $p^{-1}(U)$ will be all points $(x, y)$ that belong to some line $L_a$. But this is just
\[ \bigcup U = \bigcup_{a \in A} L_a. \]

So $p$ is continuous.

The observation that
\[ U = \{ L_a : a \in A \subseteq \mathbb{R} \} \]
is open in $\mathbb{R}^2/\sim$ precisely when $A$ is an open subset of $\mathbb{R}$ is the key to seeing what $\mathbb{R}^2/\sim$ is. For we can define $f : \mathbb{R} \rightarrow \mathbb{R}^2$ by $f(x) = L_x$. Then $f$ is clearly one-to-one and onto. It’s also open, because
\[ f(V) = \{ L_x : x \in V \}, \]
which is open iff $V$ is open. But this also says that $f$ is continuous, because
\[ f^{-1}(U) = f^{-1}[\{ L_a : a \in A \}] = A. \]