1. A basis of a topological space $X$ is a family of open subsets $\mathcal{B}$ with the property that every open set $U$ in the topological space is a union of elements of $\mathcal{B}$.

(a) Show that the family of open balls forms a basis for the usual topology on $\mathbb{R}^n$.

Let $U$ be an open subset of $\mathbb{R}^n$. We want to see that $U$ can be written as a union of $\mathbb{R}^n$. Since $U$ is open in the usual topology, for each $x \in U$, there is a positive real number $r_x$ such that $B_{r_x}(x) \subseteq U$. But then

$$\bigcup_{x \in U} B_{r_x}(x) = U.$$  

In order to see this, note that each $B_{r_x}(x) \subseteq U$. Hence their union is also contained in $U$. On the other hand, each $x \in U$ is also an element of $B_{r_x}(x)$. So $U$ is contained in the union.

(b) Show that a function $f : X \to Y$ is continuous iff $f^{-1}(V)$ is open for each $V$ in a basis $\mathcal{B}$ for the topology on $Y$.

Since each element $V \in \mathcal{B}$ is an open set, if $f$ is continuous, then $f^{-1}(V)$ is also open. On the other hand, suppose that $f^{-1}(V)$ is open for each $V \in \mathcal{B}$. Also suppose that $W$ is an open subset of $Y$. Since $\mathcal{B}$ is a basis, there is a subset $\mathcal{V} \subseteq \mathcal{B}$ such that

$$\bigcup \mathcal{V} = W.$$  

Hence

$$f^{-1}(W) = f^{-1}\left(\bigcup \mathcal{V}\right) = \bigcup_{V \in \mathcal{V}} f^{-1}(V).$$

But by assumption each $f^{-1}(V)$ is open in $X$. So their union is also open and $f^{-1}(W)$ is open.

2. Suppose $X$ is a finite subset of some euclidean space and it has the relative or subspace topology. If $Y$ is any topological space and $f : X \to Y$ is any function, show that $f$ is continuous.

First observe that every subset of $X$ is open. For if $x \in X$, then, since $X$ is finite,

$$r_x = \min\{||x - y|| : y \in X - \{x\}\} > 0.$$  

Hence

$$B_{r_x}(x) \cap X = \{x\},$$

and since $B_{r_x}$ is open, $\{x\}$ is an open subset of $X$. But then any subset of $X$ is open, since $\emptyset$ is open, and any nonempty set is a union of elements of $X$, each of which is open.

So any function $f : X \to Y$ is continuous, since the inverse image of any subset of $Y$ is a subset of $X$, and hence open.
3. Example 1.7.1 on page 17 of the text. Explain your answer.

Call the three graphs $A, B,$ and $C$ from left to right, and label their vertices $a_1, a_2, \ldots, b_1, b_2, \ldots,$ and $c_1, c_2, \ldots,$ respectively, where for each graph we start with the rightmost vertex and proceed counterclockwise around the graph.

Then $A$ can’t be isomorphic to $B$ or $C.$ In order to see this, consider the vertices of degree 3 in the graph: $a_1$ and $a_4$ in $A,$ $b_3$ and $b_4$ in $B,$ $c_3$ and $c_4$ in $C.$ Observe that in $B$ and $C,$ the vertices of degree three are joined by an edge, while in $A,$ they’re not. Any isomorphism must take the degree three vertices to degree three vertices, and hence if there’s an edge directly joining two degree three vertices in one graph, the isomorphism must take this edge to an edge joining two degree three vertices in the other graph. This is impossible for either $A$ and $B$ or $A$ and $C.$

In order to see that $B$ and $C$ are isomorphic, consider the bijection $f : V(B) \rightarrow V(C)$ defined by $f(b_1) = c_6,$ $f(b_6) = c_1,$ and $f(b_i) = c_i,$ for $i = 2, 3, 4, 5.$ That this defines an isomorphism can be checked by checking that the image of each edge $(b_i, b_j)$ is, in fact, $(f(b_i), f(b_j)),$ and that every edge in $C$ is the image of an edge in $B.$

4. Example 1.7.2 on page 17 of the text. Explain your answer.

Call the three graphs in the first row $A, B,$ and $C,$ and call the two graphs in the second row $D$ and $E.$ Starting with the topmost vertex and proceeding counterclockwise, label the vertices of $A$ with $a_1, a_2, \ldots, a_7.$ Use an analogous procedure to label the vertices of the other graphs: start with the topmost vertex and proceed counterclockwise, labelling those of $B$ with $b_1, b_2, \ldots,$ those of $C$ with $c_1, c_2, \ldots,$ etc.

If we use the text’s hint and count triangular paths, we see that $E$ cannot be isomorphic to the first four, since it only has six triangular paths, whereas each of the other graphs has seven. Furthermore, the graphs $A, B, C,$ and $D$ are pairwise isomorphic. The text shows that $A$ and $B$ are isomorphic, and it can be checked that each of the following rules defines an isomorphism from $A$ onto $C$ and $D,$ respectively:

\[
\begin{array}{c|cccccccc}
A & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\
C & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\
D & d_1 & d_4 & d_7 & d_3 & d_6 & d_2 & d_5 \\
\end{array}
\]

Since graph isomorphism is an equivalence relation, the graphs $A, B, C,$ and $D$ are all pairwise isomorphic.

5. **Extra Credit.** Suppose $X$ is a topological space and $A$ is a closed subset of $X.$ Also suppose that $C$ is a subset of $A$ that is closed in the relative topology on $A.$ Show that $C$ is a closed subset of $X.$

6. **Extra Credit.** A topological space $X$ is **Hausdorff** if for any two points $x, y \in X,$ there exist open neighborhoods $U$ of $x$ and $V$ of $y$ such that $U \cap V = \emptyset.$

(a) Show that $\mathbb{R}^n$ is Hausdorff.
(b) If $X$ is a Hausdorff space and $C$ is a compact subset of $X$, show that $C$ must be closed.

c) Find an example of a topological space $X$ and a compact subset $C$ that is not closed.
   In order to receive credit, you must explain why your example is correct.