Section 2.3

1. (a) Find the number of positive integer divisors of 642 = 2^33^4.
   (b) Find the sum of all positive integer divisors of 642.

Solution

(a) Each divisor must have the form 2^i3^j where 0 ≤ i ≤ 3 and 0 ≤ j ≤ 4. Hence, there are 3 · 4 = 12 divisors of 642.

(b) The sum of the divisors is

\[
\sum_{i=0}^{3} \sum_{j=0}^{4} 2^i3^j = 2^0 \sum_{j=0}^{4} 3^j + 2^1 \sum_{j=0}^{4} 3^j + 2^2 \sum_{j=0}^{4} 3^j + 2^3 \sum_{j=0}^{4} 3^j
\]

\[
= (2^0 + 2^1 + 2^2 + 2^3)121
\]

\[
= 1815.
\]

2. Find a formula for the sum of all divisors of integers of the form 2^m3^n (m, n ≥ 0).

Solution Each divisor must have the form 2^i3^j where 0 ≤ i ≤ m and 0 ≤ j ≤ n. Hence, the sum of the divisors is

\[
\sum_{i=0}^{m} \sum_{j=0}^{n} 2^i3^j = 2^0 \sum_{j=0}^{n} 3^j + 2^1 \sum_{j=0}^{n} 3^j + \cdots + 2^m \sum_{j=0}^{n} 3^j
\]

\[
= (2^0 + 2^1 + 2^2 + \cdots + 2^m) \sum_{j=0}^{n} 3^j
\]

\[
= (2^{m+1} - 1) \frac{3^{n+1} - 1}{2}.
\]

Prove that 101 is the only number in this sequence that is prime. (Hint: Use place value to write each number in terms of the sum of its digits; for example, abcd = a10^4+b10^3+c10^2+d10+e. Then examine how the sum might be factored.)
Solution It is easily checked that 101 is prime. Given any number of the form 10101...01 greater than 101, there is an integer \( n \geq 2 \) such that

\[
10101...01 = 10^{2n} + 10^{2n-2} + \cdots + 10^4 + 10^2 + 1
\]

\[
= \frac{10^{2n+2} - 1}{99}
\]

\[
= \frac{(10^{n+1})^2 - 1}{99}
\]

\[
= \frac{(10^{n+1} - 1)(10^{n+1} + 1)}{99}
\]

\[
= \frac{a_n(10^{n+1} + 1)}{11}
\]

where \( a_n \) is the integer that is a string of \( n + 1 \) 1s. If \( n \) is odd, then 11\( | a_n \). If \( n \) is even, then 11\( | (10^{n+1} + 1) \). In either case, 10101...01 is a product of two integers, each greater than 1. Therefore 10101...01 is not prime if \( n > 1 \).