11-0: **Merge Sort – Recursive Sorting**

- Base Case:
  - A list of length 1 or length 0 is already sorted
- Recursive Case:
  - Split the list in half
  - Recursively sort two halves
  - Merge sorted halves together

Example: 5 1 8 2 6 4 3 7

11-1: **Merging**

- Merge lists into a new temporary list, T
- Maintain three pointers (indices) i, j, and n
  - i is index of left hand list
  - j is index of right hand list
  - n is index of temporary list T
- If $A[i] < A[j]$
  - $T[n] = A[i]$, increment n and i
- else
  - $T[n] = A[j]$, increment n and j

Example: 1 2 5 8 and 3 4 6 7

11-2: $\Theta()$ **for Merge Sort**

$T(0) = c_1$ for some constant c_1
$T(1) = c_2$ for some constant c_2
$T(n) = nc_3 + 2T(n/2)$ for some constant c_3

$$T(n) = nc_3 + 2T(n/2)$$

11-3: $\Theta()$ **for Merge Sort**

$T(0) = c_1$ for some constant c_1
$T(1) = c_2$ for some constant c_2
$T(n) = nc_3 + 2T(n/2)$ for some constant c_3

$$\begin{aligned} T(n) &= nc_3 + 2T(n/2) \\ &= nc_3 + 2(n/2c_3 + 2T(n/4)) \\ &= 2nc_3 + 4T(n/4) \end{aligned}$$

11-4: $\Theta()$ **for Merge Sort**

$T(0) = c_1$ for some constant c_1
$T(1) = c_2$ for some constant c_2
$T(n) = nc_3 + 2T(n/2)$ for some constant c_3

$$\begin{aligned} T(n) &= nc_3 + 2T(n/2) \\ &= nc_3 + 2(n/2c_3 + 2T(n/4)) \\ &= 2nc_3 + 4T(n/4) \\ &= 2nc_3 + 4(n/4c_3 + 2T(n/8)) \\ &= 3nc_3 + 8T(n/8)) \end{aligned}$$

11-5: $\Theta()$ **for Merge Sort**

$T(0) = c_1$ for some constant c_1
$T(1) = c_2$ for some constant c_2
$T(n) = nc_3 + 2T(n/2)$ for some constant c_3

$$
\begin{aligned}
T(n) &= nc_3 + 2T(n/2) \\
&= nc_3 + 2(n/2c_3 + 2T(n/4)) \\
&= 2nc_3 + 4T(n/4) \\
&= 2nc_3 + 4(n/4c_3 + 2T(n/8)) \\
&= 3nc_3 + 8T(n/8)) \\
&= 3nc_3 + 8(n/8c_3 + 2T(n/16)) \\
&= 4nc_3 + 16T(n/16)
\end{aligned}
$$

11-6: $\Theta()$ **for Merge Sort**

$T(0) = c_1$	for some constant c_1
$T(1) = c_2$	for some constant c_2
$T(n) = nc_3 + 2T(n/2)$	for some constant c_3

$$
\begin{aligned}
T(n) &= nc_3 + 2T(n/2) \\
&= nc_3 + 2(n/2c_3 + 2T(n/4)) \\
&= 2nc_3 + 4T(n/4) \\
&= 2nc_3 + 4(n/4c_3 + 2T(n/8)) \\
&= 3nc_3 + 8T(n/8)) \\
&= 3nc_3 + 8(n/8c_3 + 2T(n/16)) \\
&= 4nc_3 + 16T(n/16) \\
&= 5nc_3 + 32T(n/32)
\end{aligned}
$$

11-7: $\Theta()$ **for Merge Sort**

$T(0) = c_1$	for some constant c_1
$T(1) = c_2$	for some constant c_2
$T(n) = nc_3 + 2T(n/2)$	for some constant c_3

$$
\begin{aligned}
T(n) &= nc_3 + 2T(n/2) \\
&= nc_3 + 2(n/2c_3 + 2T(n/4)) \\
&= 2nc_3 + 4T(n/4) \\
&= 2nc_3 + 4(n/4c_3 + 2T(n/8)) \\
&= 3nc_3 + 8T(n/8)) \\
&= 3nc_3 + 8(n/8c_3 + 2T(n/16)) \\
&= 4nc_3 + 16T(n/16) \\
&= 5nc_3 + 32T(n/32) \\
&= knc_3 + 2^kT(n/2^k)
\end{aligned}
$$

11-8: $\Theta()$ **for Merge Sort**

$T(0) = c_1$
$T(1) = c_2$
$T(n) = knc_3 + 2^kT(n/2^k)$

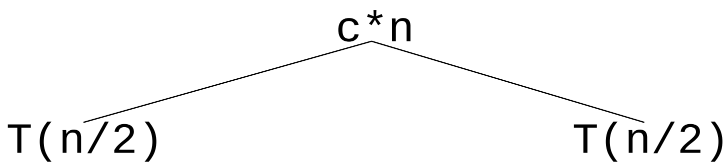
Pick a value for k such that $n/2^k = 1$:

$$
\begin{aligned}
n/2^k &= 1 \\
n &= 2^k \\
\lg n &= k \\
T(n) &= (\lg n)nc_3 + 2^{\lg n}T(n/2^{\lg n}) \\
&= c_3 n \lg n + nT(n/n) \\
&= c_3 n \lg n + nT(1) \\
&= c_3 n \lg n + c_2 n \\
&\in O(n \lg n)
\end{aligned}
$$

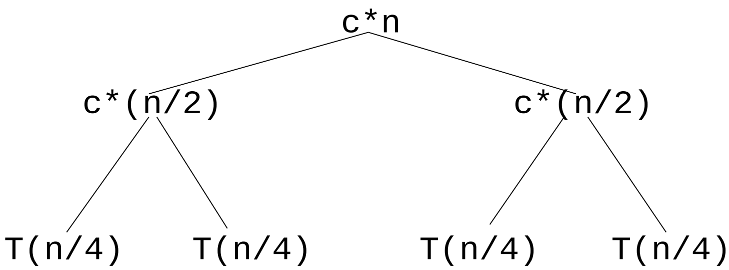
11-9: $\Theta()$ **for Merge Sort**

T(n)

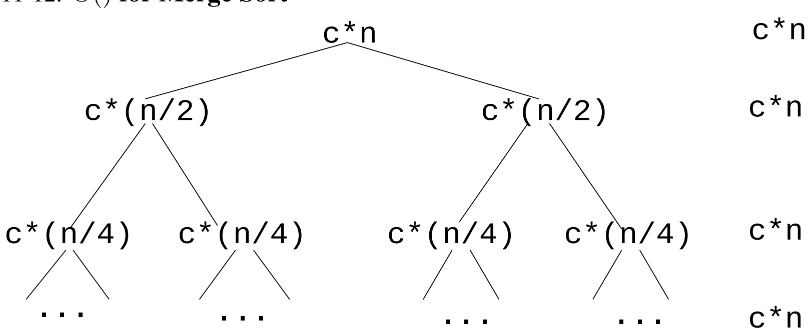
11-10: $\Theta()$ **for Merge Sort**



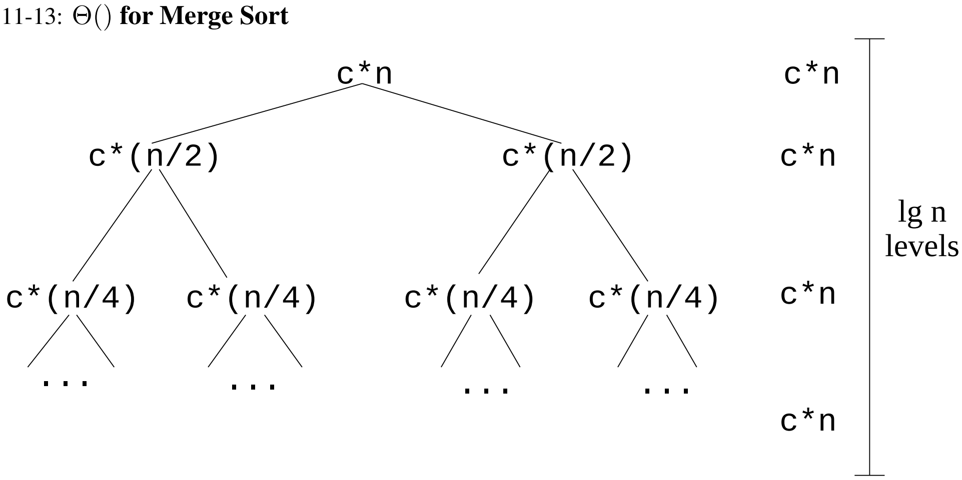
11-11: $\Theta()$ **for Merge Sort**



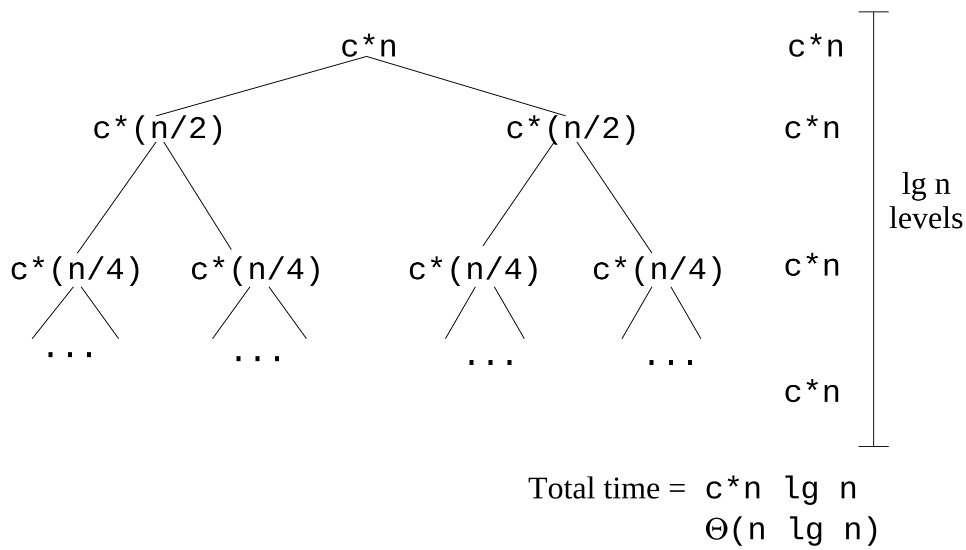
11-12: $\Theta()$ **for Merge Sort**



11-13: $\Theta()$ **for Merge Sort**



11-14: $\Theta()$ **for Merge Sort**



11-15: $\Theta()$ **for Merge Sort**

$T(0) = c_1$ for some constant c_1
$T(1) = c_2$ for some constant c_2
$T(n) = nc_3 + 2T(n/2)$ for some constant c_3

$T(n) = aT(n/b) + f(n)$
$a = 2, b = 2, f(n) = n$
$n^{\log_b a} = n^{\log_2 2} = n \in \Theta(n)$
By second case of the Master Method, $T(n) \in \Theta(n \lg n)$

11-16: **Divide & Conquer**

Merge Sort:

- Divide the list two parts
  - No work required – just calculate midpoint
- Recursively sort two parts
- Combine sorted lists into one list
  - Some work required – need to merge lists

11-17: **Divide & Conquer**

Quick Sort:

- Divide the list two parts
  - Some work required – Small elements in left sublist, large elements in right sublist
- Recursively sort two parts
- Combine sorted lists into one list
  - No work required!

11-18: **Quick Sort**

- Pick a pivot element

- Reorder the list:
  - All elements < pivot
  - Pivot element
  - All elements > pivot
- Recursively sort elements < pivot
- Recursively sort elements > pivot

Example: 3 7 2 8 1 4 6

11-19: **Quick Sort - Partitioning**

Basic Idea:

- Swap pivot elememt out of the way (we'll swap it back later)
- Maintain two pointers, i and j
  - i points to the beginning of the list
  - j points to the end of the list
- Move i and j in to the middle of the list – ensuring that all elements to the left of i are < the pivot, and all elememnts to the right of j are greater than the pivot
- Swap pivot element back to middle of list

11-20: **Quick Sort - Partitioning**

Pseudocode:

- Pick a pivot index
- Swap A[pivotindex] and A[high]
- Set $i \leftarrow low$, $j \leftarrow$ high-1
- while $(i <= j)$
  - while $A[i] < A$[pivot], increment i
  - while $A[j] > A$[pivot], decrement i
  - swap $A[i]$ and $A[j]$
  - increment i, decrement j
- swap $A[i]$ and A[pivot]

11-21: $\Theta()$ **for Quick Sort**

- Coming up with a recurrence relation for quicksort is harder than mergesort
- How the problem is divided depends upon the data

- Break list into:

  size 0, size $n-1$
  size 1, size $n-2$
  ...
  size $\lfloor (n-1)/2 \rfloor$, size $\lceil (n-1)/2 \rceil$
  ...
  size $n-2$, size 1
  size $n-1$, size 0

11-22: $\Theta()$ **for Quick Sort**

Worst case performance occurs when break list into size $n-1$ and size 0

$T(0) = c_1$ for some constant c_1
$T(1) = c_2$ for some constant c_2
$T(n) = nc_3 + T(n-1) + T(0)$ for some constant c_3

$$\begin{aligned} T(n) &= nc_3 + T(n-1) + T(0) \\ &= T(n-1) + nc_3 + c_2 \end{aligned}$$

11-23: $\Theta()$ **for Quick Sort** Worst case: $T(n) = T(n-1) + nc_3 + c_2$

$$\begin{aligned} &T(n) \\ &= T(n-1) + nc_3 + c_2 \end{aligned}$$

11-24: $\Theta()$ **for Quick Sort** Worst case: $T(n) = T(n-1) + nc_3 + c_2$

$$\begin{aligned} &T(n) \\ &= T(n-1) + nc_3 + c_2 \\ &= [T(n-2) + (n-1)c_3 + c_2] + nc_3 + c_2 \\ &= T(n-2) + (n + (n-1))c_3 + 2c_2 \end{aligned}$$

11-25: $\Theta()$ **for Quick Sort** Worst case: $T(n) = T(n-1) + nc_3 + c_2$

$$\begin{aligned} &T(n) \\ &= T(n-1) + nc_3 + c_2 \\ &= [T(n-2) + (n-1)c_3 + c_2] + nc_3 + c_2 \\ &= T(n-2) + (n + (n-1))c_3 + 2c_2 \\ &= [T(n-3) + (n-2)c_3 + c_2] + (n + (n-1))c_3 + 2c_2 \\ &= T(n-3) + (n + (n-1) + (n-2))c_3 + 3c_2 \end{aligned}$$

11-26: $\Theta()$ **for Quick Sort** Worst case: $T(n) = T(n-1) + nc_3 + c_2$

$$\begin{aligned} &T(n) \\ &= T(n-1) + nc_3 + c_2 \\ &= [T(n-2) + (n-1)c_3 + c_2] + nc_3 + c_2 \\ &= T(n-2) + (n + (n-1))c_3 + 2c_2 \\ &= [T(n-3) + (n-2)c_3 + c_2] + (n + (n-1))c_3 + 2c_2 \\ &= T(n-3) + (n + (n-1) + (n-2))c_3 + 3c_2 \\ &= T(n-4) + (n + (n-1) + (n-2) + (n-3))c_3 + 4c_2 \end{aligned}$$

11-27: $\Theta()$ **for Quick Sort** Worst case: $T(n) = T(n-1) + nc_3 + c_2$

$$
\begin{aligned}
&T(n) \\
&= T(n-1) + nc_3 + c_2 \\
&= [T(n-2) + (n-1)c_3 + c_2] + nc_3 + c_2 \\
&= T(n-2) + (n + (n-1))c_3 + 2c_2 \\
&= [T(n-3) + (n-2)c_3 + c_2] + (n + (n-1))c_3 + 2c_2 \\
&= T(n-3) + (n + (n-1) + (n-2))c_3 + 3c_2 \\
&= T(n-4) + (n + (n-1) + (n-2) + (n-3))c_3 + 4c_2 \\
&\ldots \\
&= T(n-k) + (\textstyle\sum_{i=0}^{k-1}(n-i)c_3) + kc_2
\end{aligned}
$$

11-28: $\Theta()$ **for Quick Sort** Worst case:

$T(n) = T(n-k) + (\sum_{i=0}^{k-1}(n-i)c_3) + kc_2$

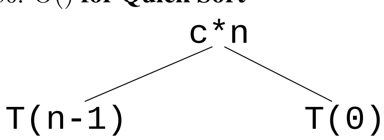
Set $k = n$:

$$
\begin{aligned}
T(n) &= T(n-k) + (\textstyle\sum_{i=0}^{k-1}(n-i)c_3) + kc_2 \\
&= T(n-n) + (\textstyle\sum_{i=0}^{n-1}(n-i)c_3) + kc_2 \\
&= T(0) + (\textstyle\sum_{i=0}^{n-1}(n-i)c_3) + kc_2 \\
&= T(0) + (\textstyle\sum_{i=0}^{n-1} ic_3) + kc_2 \\
&= c_1 + c_3 n(n+1)/2 + kc_2 \\
&\in \Theta(n^2)
\end{aligned}
$$

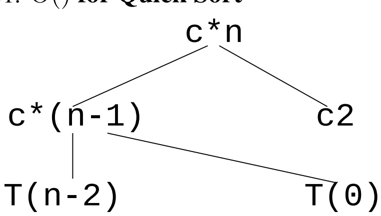
11-29: $\Theta()$ **for Quick Sort**

T(n)

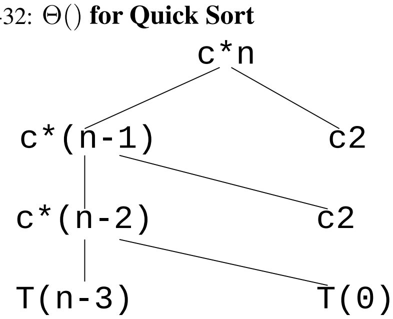
11-30: $\Theta()$ **for Quick Sort**



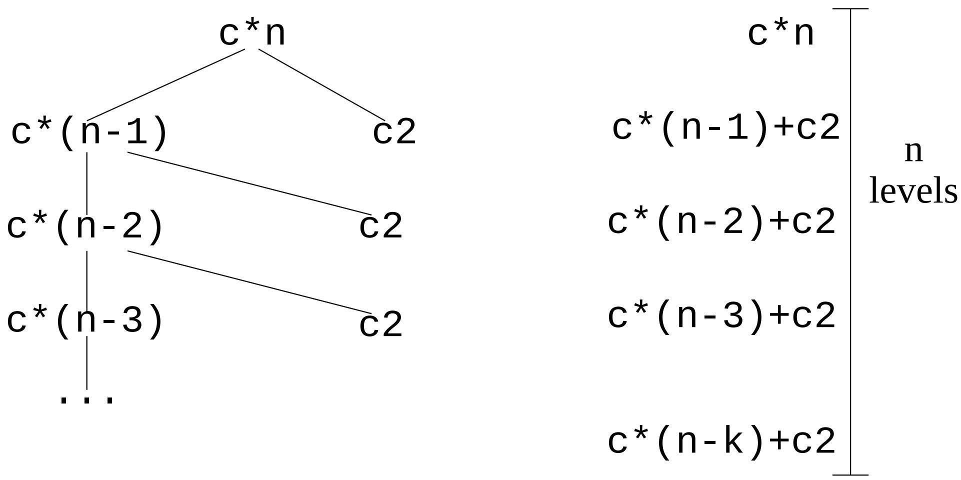
11-31: $\Theta()$ **for Quick Sort**



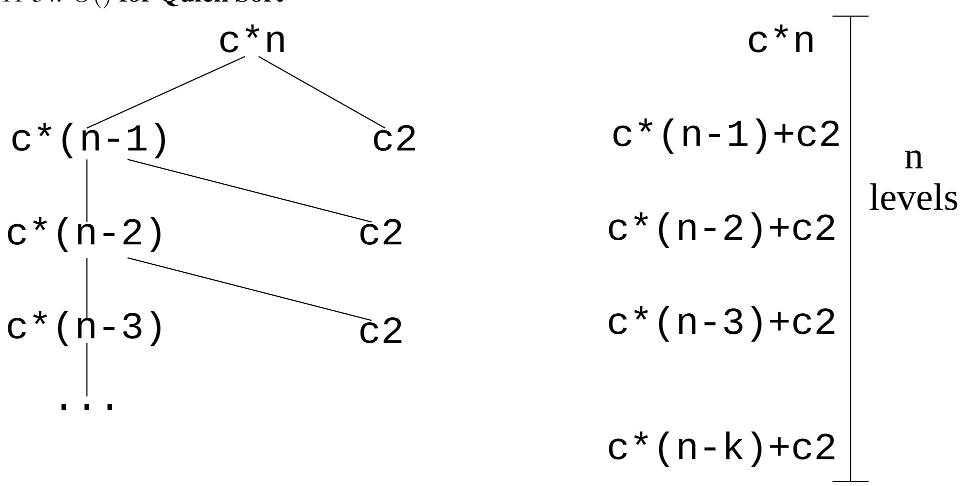
11-32: $\Theta()$ **for Quick Sort**



11-33: $\Theta()$ **for Quick Sort**



11-34: $\Theta()$ **for Quick Sort**



Total time = c*n*(n+1)/2 + nc2

$\Theta(n^2)$

11-35: $\Theta()$ **for Quick Sort**

Best case performance occurs when break list into size $\lfloor (n-1)/2 \rfloor$ and size $\lceil (n-1)/2 \rceil$

$T(0) = c_1$	for some constant c_1
$T(1) = c_2$	for some constant c_2
$T(n) = nc_3 + 2T(n/2)$	for some constant c_3

This is the same as Merge Sort: $\Theta(n \lg n)$

11-36: ***Quick* Sort?**

If Quicksort is $\Theta(n^2)$ on some lists, why is it called *quick*?

- Most lists give running time of $\Theta(n \lg n)$: The average case running time (assuming all permutations are equall likely) is $\Theta(n \lg n)$
  - We could prove this by finding the running time for each permutation of a list of length n, and averaging them
    - Math required to do this is a little beyond the prerequisites for this class

  - Consider what happens when the list is always partitioned into a list of length $n/9$ and a list of lenth $8n/9$ (recursion tree, on whiteboard)
  - Consider what happenswhen the list is always partitioned into a list of length n/k and a list of length $(k-1)n/k$, for any k

11-37: ***Quick* Sort?**

If Quicksort is $\Theta(n^2)$ on some lists, why is it called *quick*?

- Most lists give running time of $\Theta(n \lg n)$
  - Average case running time is $\Theta(n \lg n)$
- Constants are very small
  - Constants don't matter when complexity is different
  - Constants *do* matter when complexity is the same

What lists will cause Quick Sort to have $\Theta(n^2)$ performance?

11-38: **Quick Sort - Worst Case**

- Quick Sort has worst-case performance when:
  - The list is sorted (or almost sorted)
  - The list is inverse sorted (or almost inverse sorted)
- Many lists we want to sort are almost sorted!
- How can we fix Quick Sort?

11-39: **Better Partitions**

- Pick the middle element as the pivot
  - Sorted and reverse sorted lists give good performance
- Pick a random element as the pivot
  - No single list always gives bad performance
- Pick the median of 3 elements
  - First, Middle, Last
  - 3 Random Elements

11-40: **Improving Quick Sort**

- Insertion Sort runs faster than Quick Sort on small lists
  - Why?
- We can combine Quick Sort & Insertion Sort
  - When lists get small, run Insertion Sort instead of a recursive call to Quick Sort
  - When lists get small, stop! After call to Quick Sort, list will be almost sorted – finish the job with a single call to Insertion Sort

11-41: **Heap Sort**

- Copy the data into a new array (except leave out element at index 0)
- Build a heap out of the new array
- Repeat:
  - Remove the smallest element from the heap, add it to the original array
- Until all elements have been removed from the heap
- The original array is now sorted

Example: 3 1 7 2 5 4

11-42: **Heap Sort**

- This requires $\Theta(n)$ extra space
- We can modify heapsort so that it does not use extra space
- Build a heap out of the original array, with two differences:
  - Consider element 0 to be the root of the tree
    - for element i, children are at 2*i +1 and 2*i+2, and parent is at $(i-1)/2$
    - (examples)
  - Max-heap instead of a standard min-heap
    - For each subtree, element stored at root $\geq$ element stored in that subtree (instead of $\leq$, as in a standard heap)

11-43: **Heap Sort**

- Build a heap out of the original array, with two differences:
  - Consider element 0 to be the root of the tree
    - for element i, children are at 2*i +1 and 2*i+2, and parent is at $(i-1)/2$
    - (examples)
  - Max-heap instead of a standard min-heap
    - For each subtree, element stored at root $\geq$ element stored in that subtree (instead of $\leq$, as in a standard heap)
- Repeatedly remove the largest element, and insert it in the back of the heap

Example: 3 1 7 2 5 4

11-44: $\Theta()$ **for Heap Sort**

- Building the heap takes time $\Theta(n)$
- Each of the n RemoveMax calls takes time $O(\lg n)$
- Total time: $\emptyset(n \lg n)$ (also $\Theta(n \lg n)$)

11-45: **Stability**

Sorting Algorithm	Stable?
Insertion Sort	
Selection Sort	
Bubble Sort	
Shell Sort	
Merge Sort	
Quick Sort	
Heap Sort	

11-46: **Stability**

Sorting Algorithm	Stable?
Insertion Sort	Yes
Selection Sort	No
Bubble Sort	Yes
Shell Sort	No
Merge Sort	Yes
Quick Sort	No
Heap Sort	No