Data Structures and Algorithms

CS245-2015S-FR

Final Review

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FR-0: Big-Oh Notation

$O(f(n))$ is the set of all functions that are bound from above by $f(n)$ p

$T(n) \in O(f(n))$ if

$$\exists c, n_0 \text{ such that } T(n) \leq c * f(n) \text{ when } n > n_0$$

FR-1: Big-Oh Examples

$$
\begin{aligned}
n &\in O(n)\ ? \\
10n &\in O(n)\ ? \\
n &\in O(10n)\ ? \\
n &\in O(n^2)\ ? \\
n^2 &\in O(n)\ ? \\
10n^2 &\in O(n^2)\ ? \\
n \lg n &\in O(n^2)\ ? \\
\ln n &\in O(2n)\ ? \\
\lg n &\in O(n)\ ? \\
3n + 4 &\in O(n)\ ? \\
5n^2 + 10n - 2 &\in O(n^3)?\ O(n^2)\ ?\ O(n)\ ?
\end{aligned}
$$

FR-2: Big-Oh Examples

$$
\begin{aligned}
n &\in O(n) \\
10n &\in O(n) \\
n &\in O(10n) \\
n &\in O(n^2) \\
n^2 &\notin O(n) \\
10n^2 &\in O(n^2) \\
n \lg n &\in O(n^2) \\
\ln n &\in O(2n) \\
\lg n &\in O(n) \\
3n + 4 &\in O(n) \\
5n^2 + 10n - 2 &\in O(n^3), \in O(n^2), \notin O(n) \text{ ?}
\end{aligned}
$$

FR-3: Big-Oh Examples II

$$\begin{aligned}
\sqrt{n} &\in O(n) \text{ ?} \\
\lg n &\in O(2^n) \text{ ?} \\
\lg n &\in O(n) \text{ ?} \\
n \lg n &\in O(n) \text{ ?} \\
n \lg n &\in O(n^2) \text{ ?} \\
\sqrt{n} &\in O(\lg n) \text{ ?} \\
\lg n &\in O(\sqrt{n}) \text{ ?} \\
n \lg n &\in O(n^{\frac{3}{2}}) \text{ ?} \\
n^3 + n \lg n + n\sqrt{n} &\in O(n \lg n) \text{ ?} \\
n^3 + n \lg n + n\sqrt{n} &\in O(n^3) \text{ ?} \\
n^3 + n \lg n + n\sqrt{n} &\in O(n^4) \text{ ?}
\end{aligned}$$

Big-Oh Examples II

$$
\begin{aligned}
\sqrt{n} &\in O(n) \\
\lg n &\in O(2^n) \\
\lg n &\in O(n) \\
n \lg n &\notin O(n) \\
n \lg n &\in O(n^2) \\
\sqrt{n} &\notin O(\lg n) \\
\lg n &\in O(\sqrt{n}) \\
n \lg n &\in O(n^{\frac{3}{2}}) \\
n^3 + n \lg n + n\sqrt{n} &\notin O(n \lg n) \\
n^3 + n \lg n + n\sqrt{n} &\in O(n^3) \\
n^3 + n \lg n + n\sqrt{n} &\in O(n^4)
\end{aligned}
$$

FR-5: Big-Oh Examples III

$$f(n) = \begin{cases} n & \text{for } n \text{ odd} \\ n^3 & \text{for } n \text{ even} \end{cases}$$

$$g(n) = n^2$$

$$\begin{aligned} f(n) &\in O(g(n)) \text{ ?} \\ g(n) &\in O(f(n)) \text{ ?} \\ n &\in O(f(n)) \text{ ?} \\ f(n) &\in O(n^3) \text{ ?} \end{aligned}$$

FR-6: Big-Oh Examples III

$$f(n) = \begin{cases} n & \text{for } n \text{ odd} \\ n^3 & \text{for } n \text{ even} \end{cases}$$

$$g(n) = n^2$$

$$\begin{aligned} f(n) &\notin O(g(n)) \\ g(n) &\notin O(f(n)) \\ n &\in O(f(n)) \\ f(n) &\in O(n^3) \end{aligned}$$

FR-7: Big-Ω Notation

$\Omega(f(n))$ is the set of all functions that are bound from *below* by $f(n)$

$T(n) \in \Omega(f(n))$ if

$\exists c, n_0$ such that $T(n) \geq c * f(n)$ when $n > n_0$

FR-8: Big-Ω Notation

$\Omega(f(n))$ is the set of all functions that are bound from *below* by $f(n)$

$T(n) \in \Omega(f(n))$ if

$\exists c, n_0$ such that $T(n) \geq c * f(n)$ when $n > n_0$

$$f(n) \in O(g(n)) \Rightarrow g(n) \in \Omega(f(n))$$

FR-9: Big-Θ Notation

$\Theta(f(n))$ is the set of all functions that are bound *both* above *and* below by $f(n)$. Θ is a *tight bound*

$T(n) \in \Theta(f(n))$ if

$$T(n) \in O(f(n)) \text{ and } T(n) \in \Omega(f(n))$$

FR-10: Big-Oh Rules

1. If $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$, then $f(n) \in O(h(n))$
2. If $f(n) \in O(kg(n)$ for any constant $k > 0$, then $f(n) \in O(g(n))$
3. If $f_1(n) \in O(g_1(n))$ and $f_2(n) \in O(g_2(n))$, then $f_1(n) + f_2(n) \in O(\max(g_1(n), g_2(n)))$
4. If $f_1(n) \in O(g_1(n))$ and $f_2(n) \in O(g_2(n))$, then $f_1(n) * f_2(n) \in O(g_1(n) * g_2(n))$

(Also work for Ω, and hence Θ)

FR-11: Big-Oh Guidelines

- Don’t include constants/low order terms in Big-Oh
- Simple statements: $\Theta(1)$
- Loops: Θ(inside) * # of iterations
  - Nested loops work the same way
- Consecutive statements: Longest Statement
- Conditional (if) statements:
  O(Test + longest branch)

FR-12: Calculating Big-Oh

```
for (i=1; i<n; i++)
  for (j=1; j < n/2; j++)
    sum++;
```

Calculating Big-Oh

```
for (i=1; i<n; i++)                 Executed n times
  for (j=1; j < n/2; j++)            Executed n/2 times
    sum++;                             O(1)
```

Running time: $O(n^2), \Omega(n^2), \Theta(n^2)$

FR-14: Calculating Big-Oh

```
for (i=1; i<n; i=i*2)
   sum++;
```

FR-15: Calculating Big-Oh

```
for (i=1; i<n; i=i*2)    Executed lg n times
  sum++;                   O(1)
```

Running Time: $O(\lg n), \Omega(\lg n), \Theta(\lg n)$

FR-16: Calculating Big-Oh

```
for (i=1; i<n; i=i*2)
  for (j=0; j < n; j = j + 1)
    sum++;
for (i=n; i >1; i = i / 2)
  for (j = 1; j < n; j = j * 2)
     for (k = 1; k < n; k = k * 3)
        sum++
```

FR-17: Recurrence Relations

$T(n)$ = Time required to solve a problem of size n

Recurrence relations are used to determine the running time of recursive programs – recurrence relations themselves are recursive

$$
\begin{aligned}
T(0) &= \text{time to solve problem of size 0} \\
&\quad \text{– Base Case} \\
T(n) &= \text{time to solve problem of size } n \\
&\quad \text{– Recursive Case}
\end{aligned}
$$

FR-18: Recurrence Relations

```
long power(long x, long n) {
  if (n == 0)
    return 1;
  else
    return x * power(x, n-1);
}
```

$T(0) = c_1$ for some constant c_1

$T(n) = c_2 + T(n-1)$ for some constant c_2

FR-19: Building a Better `Power`

```
long power(long x, long n) {
  if (n==0) return 1;
  if (n==1) return x;
  if ((n % 2) == 0)
    return power(x*x, n/2);
  else
    return power(x*x, n/2) * x;
}
```

FR-20: Building a Better `Power`

```
long power(long x, long n) {
  if (n==0) return 1;
  if (n==1) return x;
  if ((n % 2) == 0)
    return power(x*x, n/2);
  else
    return power(x*x, n/2) * x;
}
```

$T(0) = c_1$

$T(1) = c_2$

$T(n) = T(n/2) + c_3$

(Assume n is a power of 2)

FR-21: Solving Recurrence Relations

$$
\begin{aligned}
T(n) &= T(n/2) + c_3 \\
&= T(n/4) + c_3 + c_3 \\
&= T(n/4)2c_3 \\
&= T(n/8) + c_3 + 2c_3 \\
&= T(n/8)3c_3 \\
&= T(n/16) + c_3 + 3c_3 \\
&= T(n/16) + 4c_3 \\
&= T(n/32) + c_3 + 4c_3 \\
&= T(n/32) + 5c_3 \\
&= \ldots \\
&= T(n/2^k) + kc_3
\end{aligned}
$$

$$T(n/2) = T(n/4) + c_3$$

$$T(n/4) = T(n/8) + c_3$$

$$T(n/8) = T(n/16) + c_3$$

$$T(n/16) = T(n/32) + c_3$$

FR-22: Solving Recurrence Relations

$$T(0) = c_1$$
$$T(1) = c_2$$
$$T(n) = T(n/2) + c_3$$

$$T(n) = T(n/2^k) + kc_3$$

We want to get rid of $T(n/2^k)$. Since we know $T(1)$ ...

$$\begin{aligned} n/2^k &= 1 \\ n &= 2^k \\ \lg n &= k \end{aligned}$$

FR-23: Solving Recurrence Relations

$$T(1) = c_2$$
$$T(n) = T(n/2^k) + kc_3$$

$$\begin{aligned} T(n) &= T(n/2^{\lg n}) + \lg n c_3 \\ &= T(1) + c_3 \lg n \\ &= c_2 + c_3 \lg n \\ &\in \Theta(\lg n) \end{aligned}$$

FR-24: Abstract Data Types

- An Abstract Data Type is a definition of a type based on the operations that can be performed on it.
- An ADT is an *interface*
- Data in an ADT cannot be manipulated directly – only through operations defined in the interface

FR-25: Stack

A Stack is a Last-In, First-Out (LIFO) data structure.

Stack Operations:

- Add an element to the top of the stack
- Remove the top element
- Check if the stack is empty

FR-26: Stack Implementation

Array:

- Stack elements are stored in an array
- Top of the stack is the *end* of the array
  - If the top of the stack was the beginning of the array, a push or pop would require moving all elements in the array
- Push: data[top++] = elem
- Pop: elem = data[--top]

FR-27: Stack Implementation

Linked List:

- Stack elements are stored in a linked list
- Top of the stack is the *front* of the linked list
- push: top = new Link(elem, top)
- pop: elem = top.element(); top = top.next()

FR-28: Queue

A Queue is a Last-In, First-Out (FIFO) data structure.

Queue Operations:

- Add an element to the end (tail) of the Queue
- Remove an element from the front (head) of the Queue
- Check if the Queue is empty

FR-29: Queue Implementation

Linked List:

- Maintain a pointer to the first and last element in the Linked List
- Add elements to the back of the Linked List
- Remove elements from the front of the linked list
- Enqueue: tail.setNext(new link(elem,null));
  tail = tail.next()
- Dequeue: elem = head.element();
  head = head.next();

FR-30: Queue Implementation

Array:

- Store queue elements in a circular array
- Maintain the index of the first element (head) and the next location to be inserted (tail)
- Enqueue: data[tail] = elem;
  tail = (tail + 1) % size
- Dequeue: elem = data[head];
  head = (head + 1) % size

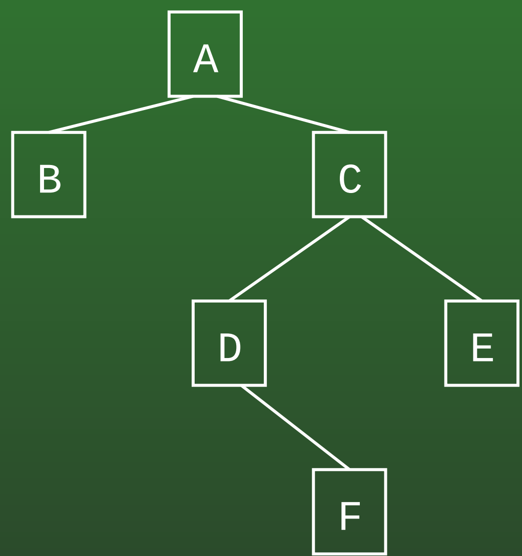
FR-31: Binary Trees

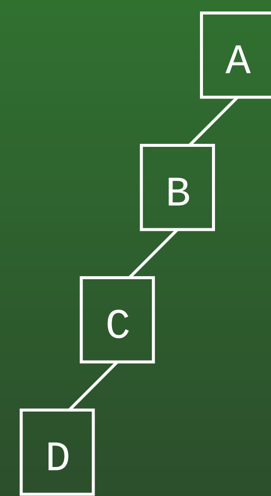
Binary Trees are Recursive Data Structures

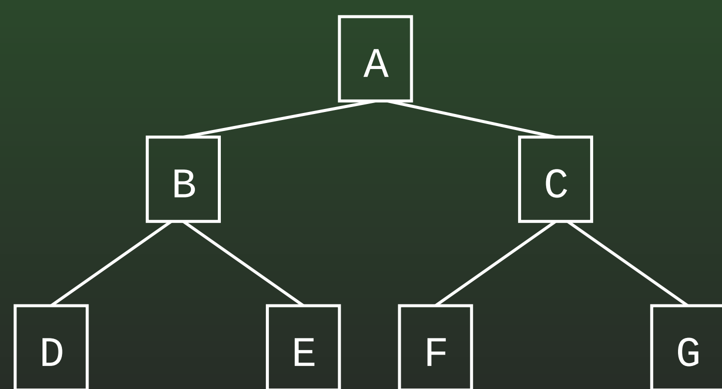
- Base Case: Empty Tree
- Recursive Case: Node, consiting of:
  - Left Child (Tree)
  - Right Child (Tree)
  - Data

FR-32: Binary Tree Examples

The following are all Binary Trees (Though not Binary *Search* Trees)







Tree Terminology

- Parent / Child
- Leaf node
- Root node
- Edge (between nodes)
- Path
- Ancestor / Descendant
- Depth of a node n
  - Length of path from root to n
- Height of a tree
  - (Depth of deepest node) + 1

FR-34: Binary Search Trees

- Binary Trees
- For each node n, (value stored at node n) $>$ (value stored in left subtree)
- For each node n, (value stored at node n) $<$ (value stored in right subtree)

FR-35: Writing a Recursive Algorithm

- Determine a small version of the problem, which can be solved immediately. This is the *base case*
- Determine how to make the problem smaller
- Once the problem has been made smaller, we can assume that the function that we are writing *will work correctly on the smaller problem* (Recursive Leap of Faith)
  - Determine how to use the solution to the smaller problem to solve the larger problem

FR-36: Finding an Element in a BST

- First, the Base Case – when is it easy to determine if an element is stored in a Binary Search Tree?
  - If the tree is empty, then the element can't be there
  - If the element is stored at the root, then the element is there

Finding an Element in a BST

- Next, the Recursive Case – how do we make the problem smaller?
  - Both the left and right subtrees are smaller versions of the problem. Which one do we use?
  - If the element we are trying to find is $<$ the element stored at the root, use the left subtree. Otherwise, use the right subtree.
- How do we use the solution to the subproblem to solve the original problem?
  - The solution to the subproblem *is* the solution to the original problem (this is not always the case in recursive algorithms)

Printing out a BST

To print out all element in a BST:

- Print all elements in the left subtree, in order
- Print out the element at the root of the tree
- Print all elements in the right subtree, in order
  - Each subproblem is a smaller version of the original problem – we can assume that a recursive call will work!

FR-39: Printing out a BST

```
void print(Node tree) {
  if (tree != null) {
    print(tree.left());
    System.out.prinln(tree.element());
    print(tree.right());
  }
}
```

FR-40: Inserting e into BST T

- Base case – T is empty:
  - Create a new tree, containing the element e
- Recursive Case:
  - If e is less than the element at the root of T, insert e into left subtree
  - If e is greater than the element at the root of T, insert e into the right subtree

FR-41: Inserting e into BST T

```
Node insert(Node tree, Comparable elem) {
  if (tree == null) {
    return new Node(elem);
  if (elem.compareTo(tree.element() < 0)) {
    tree.setLeft(insert(tree.left(), elem));
    return tree;
  } else {
    tree.setRight(insert(tree.right(), elem));
    return tree;
  }
}
```

FR-42: Deleting From a BST

- Removing a leaf:
  - Remove element immediately
- Removing a node with one child:
  - Just like removing from a linked list
  - Make parent point to child
- Removing a node with two children:
  - Replace node with largest element in left subtree, or the smallest element in the right subtree

FR-43: Priority Queue ADT

Operations

- Add an element / priority pair
- Return (and remove) element with highest priority

Implementation:

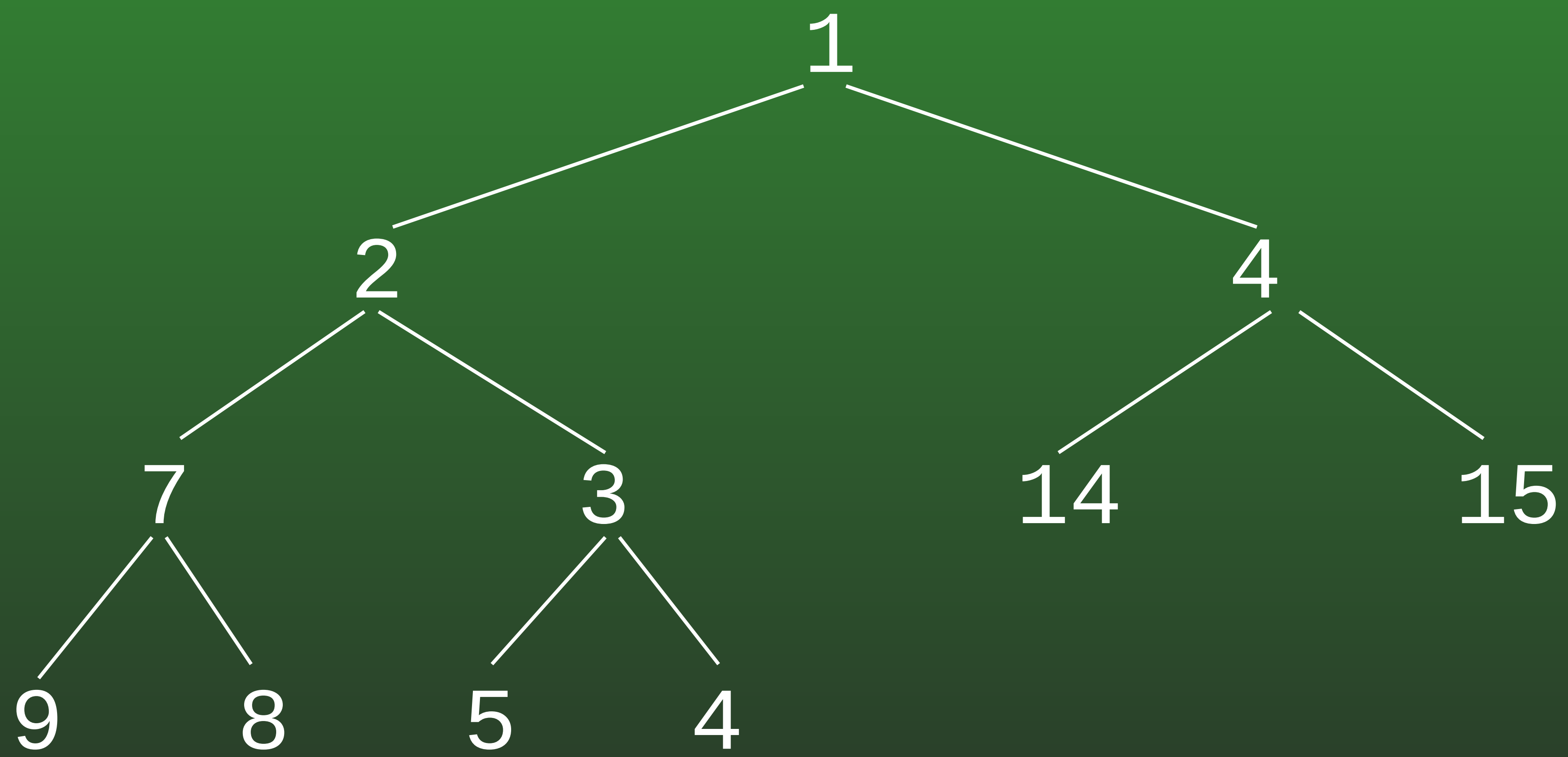
- Heap

Add Element	$O(\lg n)$
Remove Higest Priority	$O(\lg n)$

FR-44: Heap Definition

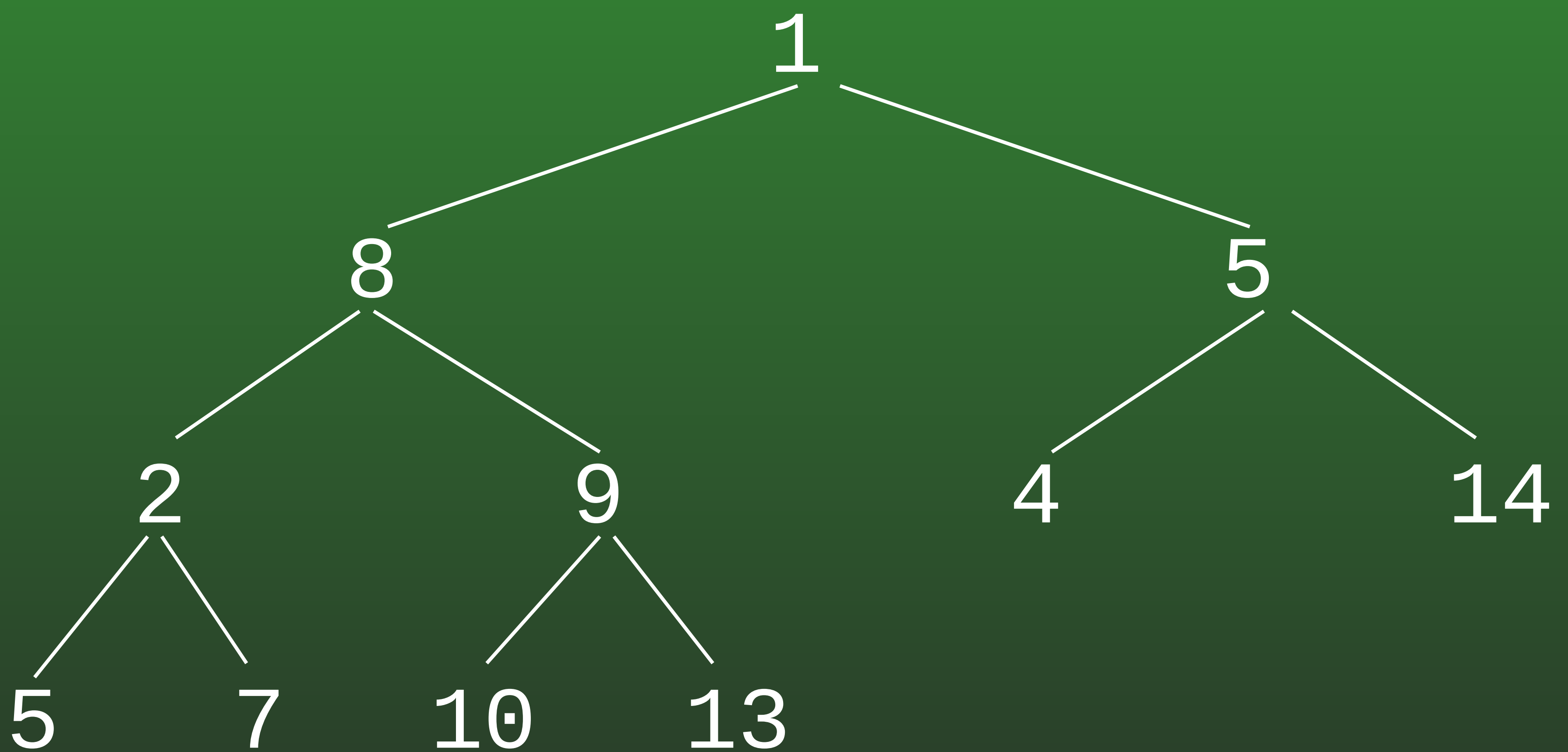
- Complete Binary Tree
- Heap Property
  - For every subtree in a tree, each value in the subtree is <= value stored at the root of the subtree

FR-45: Heap Examples



Valid Heap

FR-46: Heap Examples



Invalid Heap

FR-47: Heap Insert

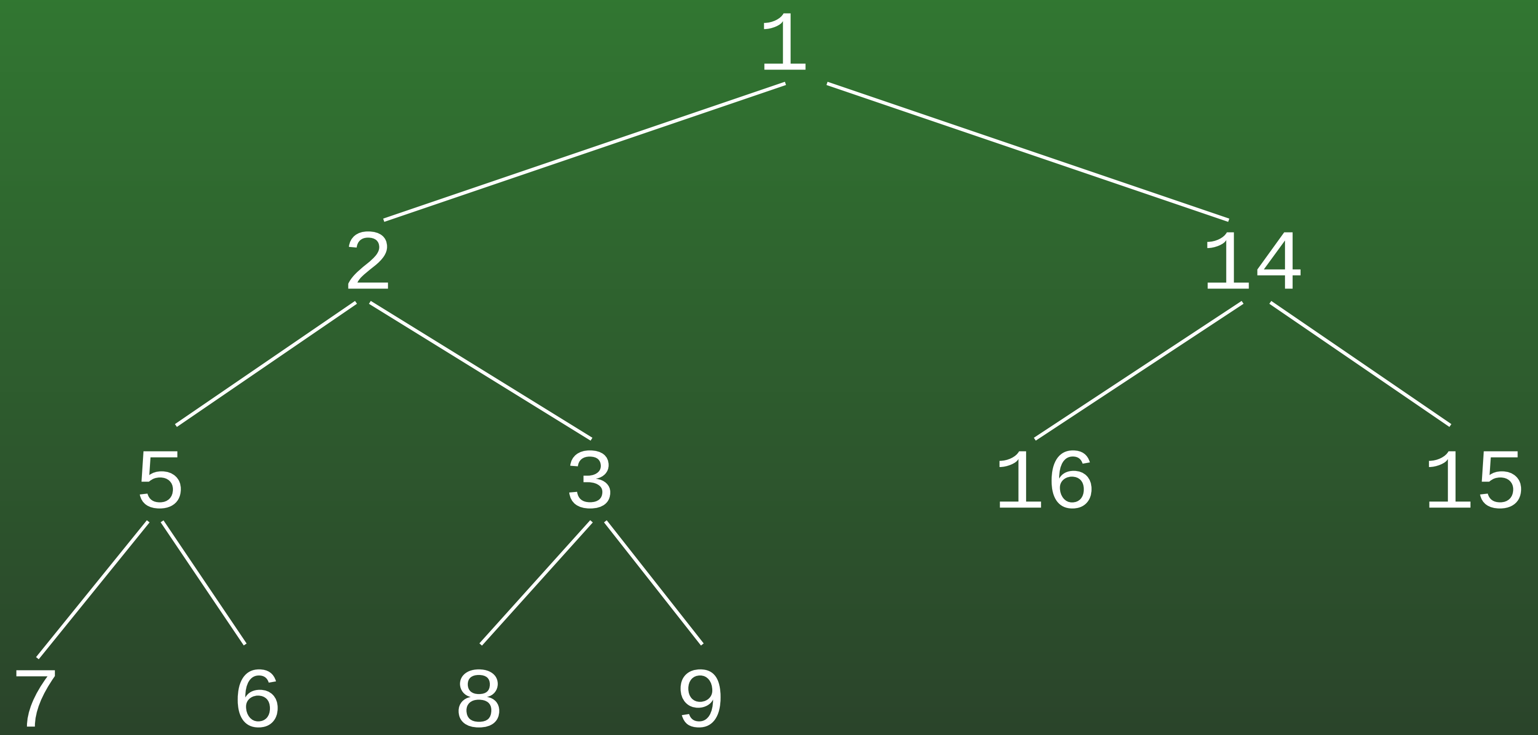
- There is only one place we can insert an element into a heap, so that the heap remains a complete binary tree
- Inserting an element at the “end” of the heap might break the heap property
  - Swap the inserted value up the tree

FR-48: Heap Remove Largest

- Removing the Root of the heap is hard
- Removing the element at the “end” of the heap is easy
  - Move last element into root
    - Shift the root down, until heap property is satisfied

Representing Heaps

A Complete Binary Tree can be stored in an array:



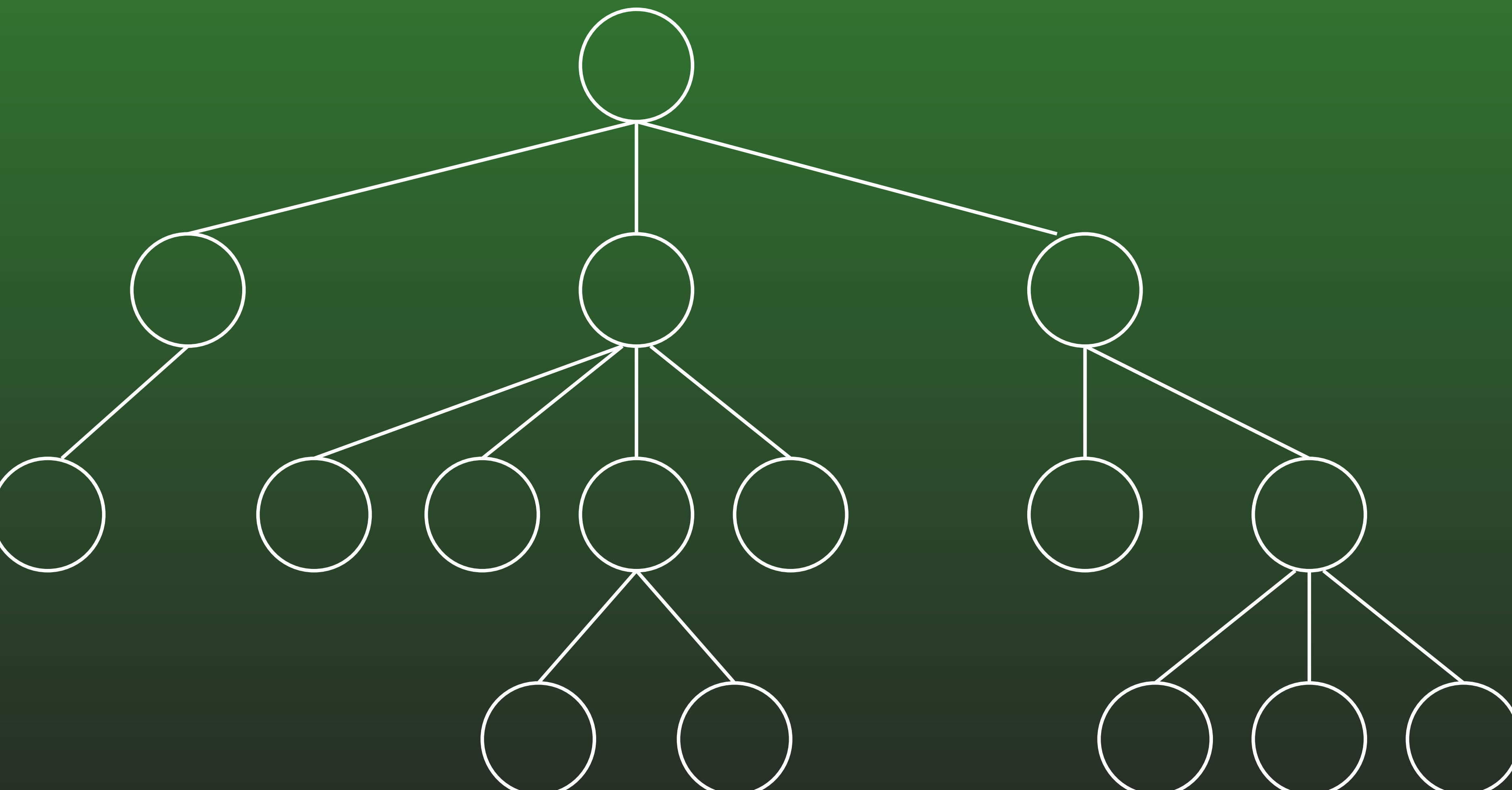
	1	2	14	5	3	16	15	7	6	8	9		
0	1	2	3	4	5	6	7	8	9	10	11	12	13

FR-50: CBTs as Arrays

- The root is stored at index 0
- For the node stored at index i:
  - Left child is stored at index $2 * i + 1$
  - Right child is stored at index $2 * i + 2$
  - Parent is stored at index $\lfloor (i - 1)/2 \rfloor$

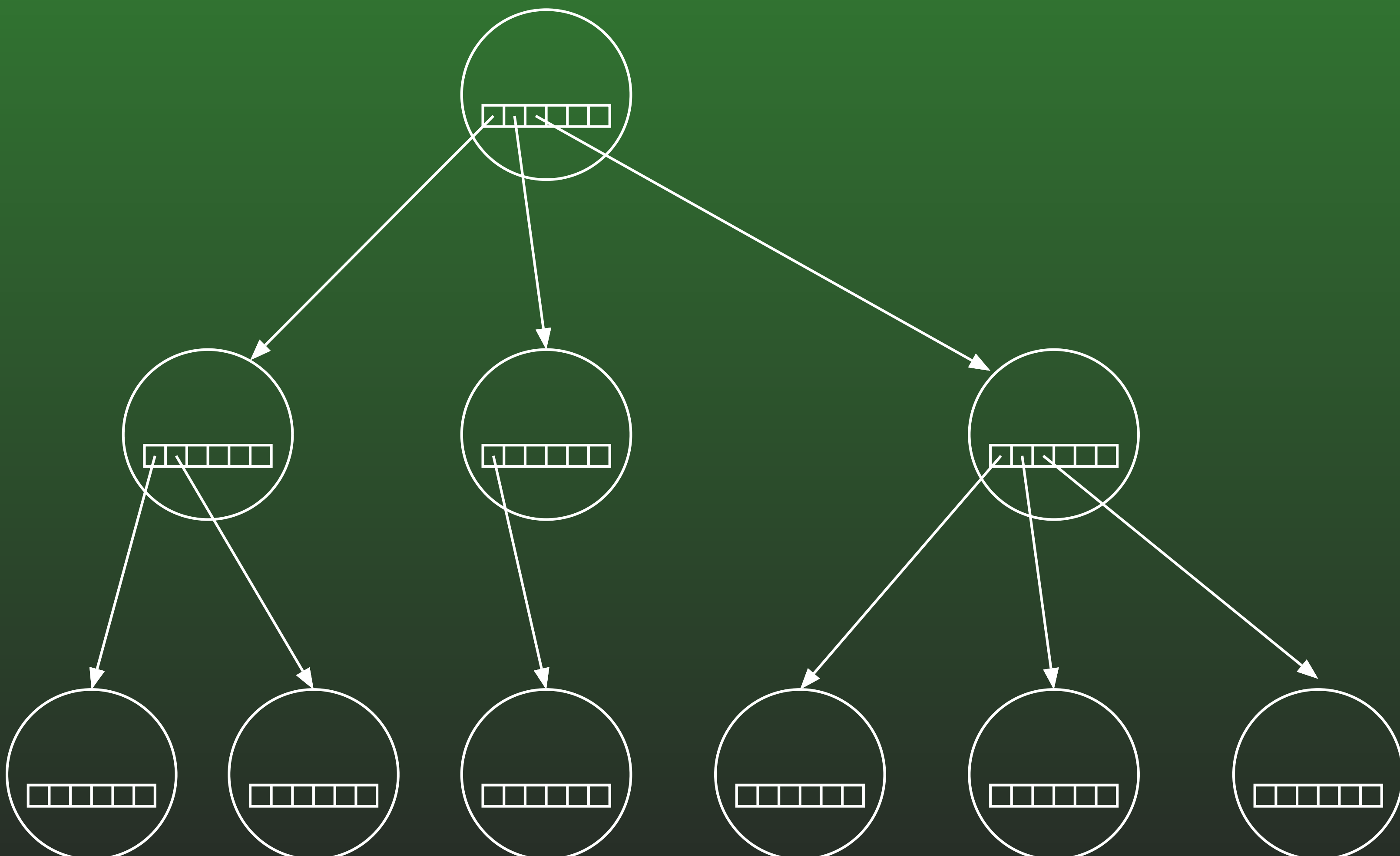
FR-51: Trees with $>$ 2 children

How can we implement trees with nodes that have $>$ 2 children?



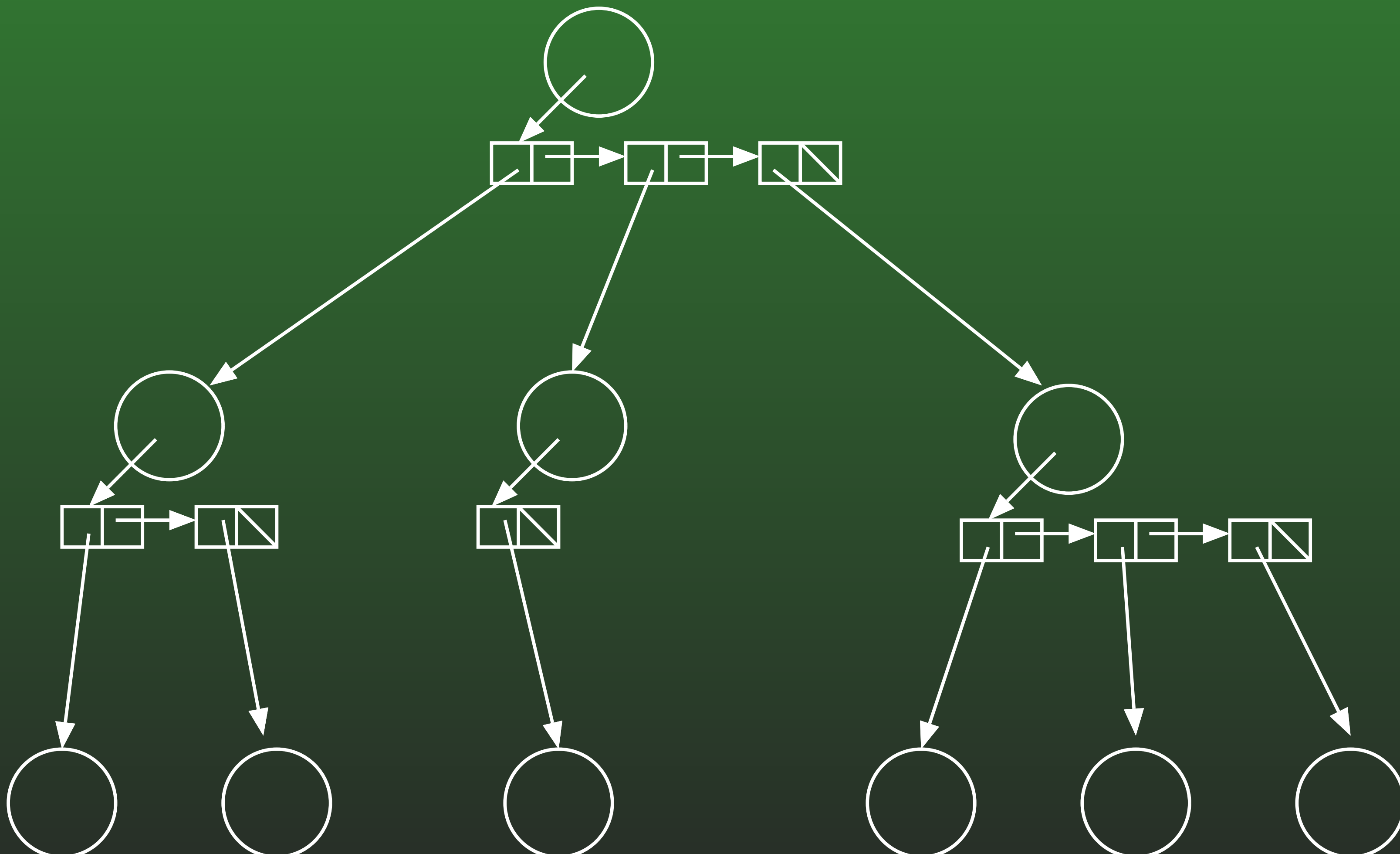
FR-52: Trees with $>$ 2 children

- Array of Children



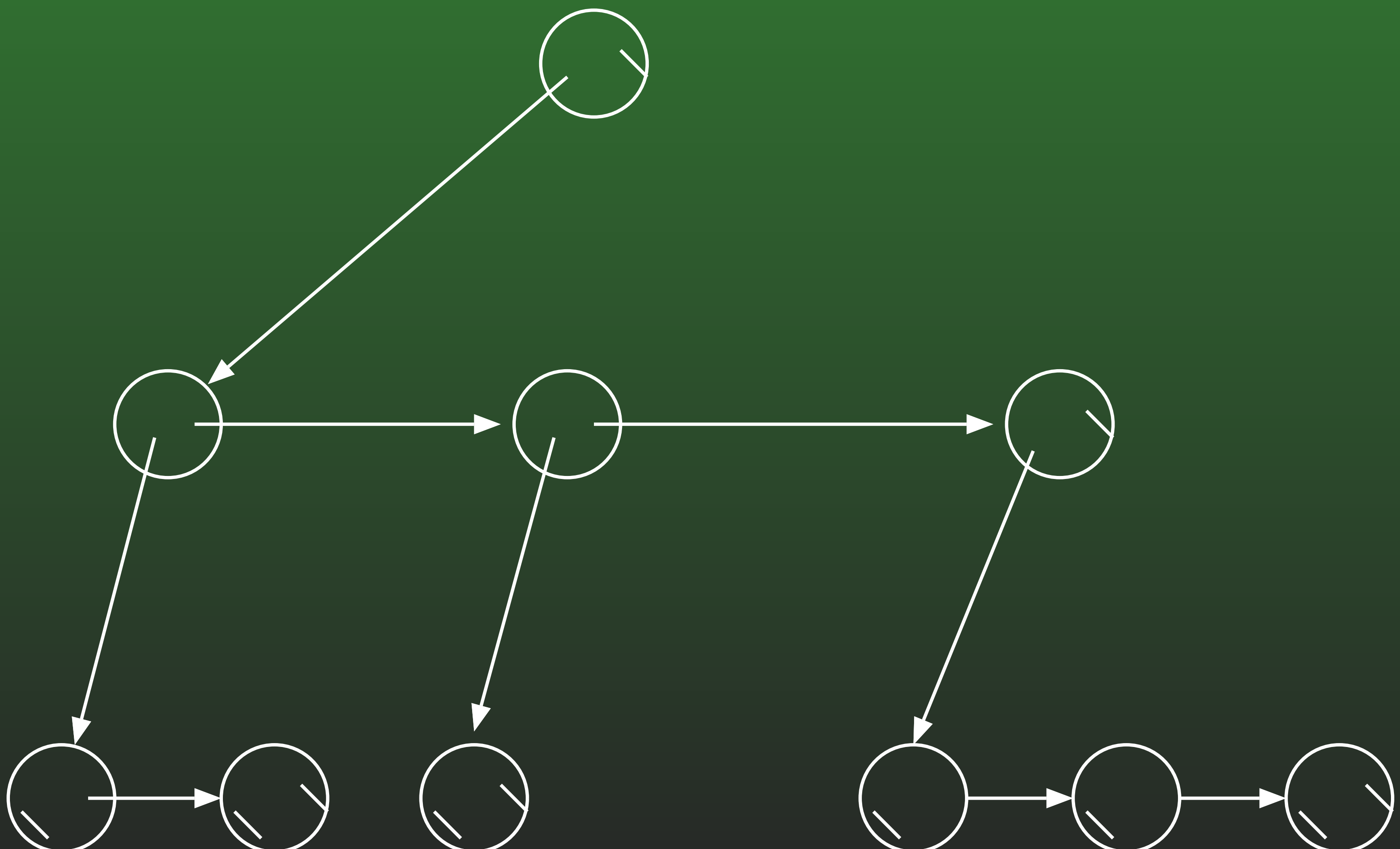
FR-53: Trees with $>$ 2 children

- Linked List of Children



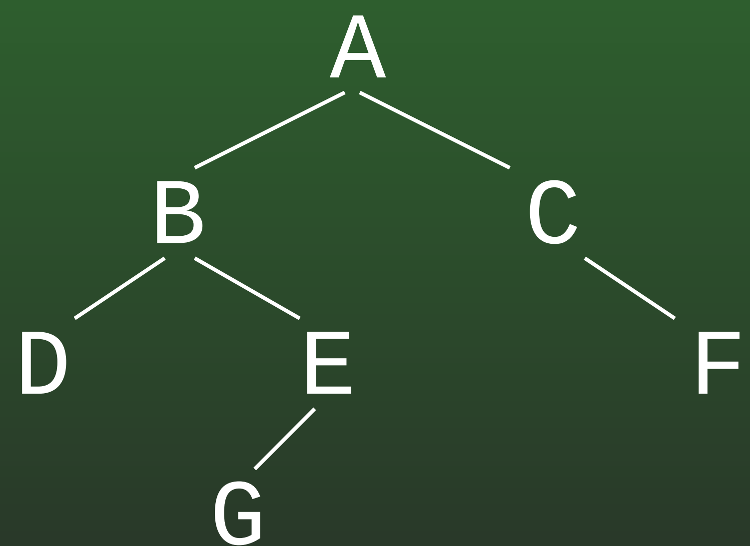
FR-54: Left Child / Right Sibling

- We can integrate the linked lists with the nodes themselves:



FR-55: Serializing Binary Trees

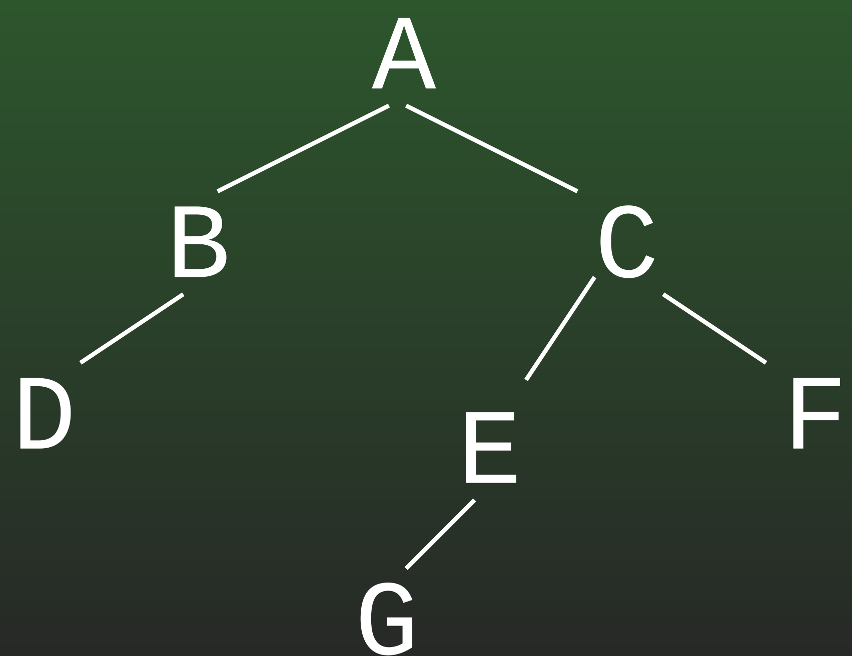
- Printing out nodes, in order that they would appear in a PREORDER traversal does not work, because we don't know when we've hit a null pointer
- Store null pointers, too!



ABD//EG///C/F//

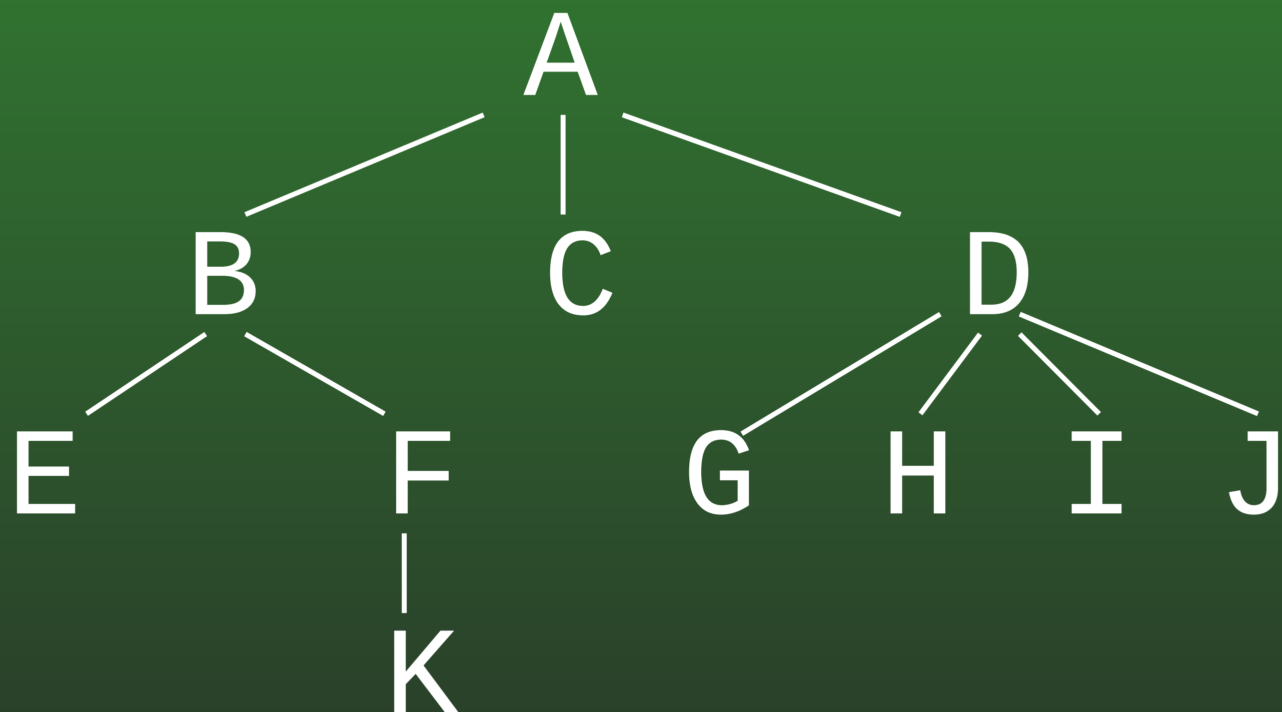
FR-56: Serializing Binary Trees

- In most trees, more null pointers than internal nodes
- Instead of marking null pointers, mark internal nodes
- Still need to mark some nulls, for nodes with 1 child



FR-57: Serializing General Trees

- Store an “end of children” marker



FR-58: Main Memory Sorting

- All data elements can be stored in memory at the same time
- Data stored in an array, indexed from $0 \ldots n-1$, where n is the number of elements
- Each element has a key value (accessed with a `key()` method)
- We can compare keys for <, >, =
- For illustration, we will use arrays of integers – though often keys will be strings, other Comparable types

FR-59: Stable Sorting

- A sorting algorithm is *Stable* if the relative order of duplicates is preserved
- The order of duplicates matters if the *keys* are duplicated, but the *records* are not.

3	1	2	1	1	2	3	Key
Bob	Joe	Ed	Amy	Sue	Al	Bud	Data

1	1	1	2	2	3	3	Key
Amy	Joe	Sue	Ed	Al	Bob	Bud	Data

A *non*-Stable sort

FR-60: Insertion Sort

- Separate list into sorted portion, and unsorted portion
- Initially, sorted portion contains first element in the list, unsorted portion is the rest of the list
  - (A list of one element is always sorted)
- Repeatedly insert an element from the unsorted list into the sorted list, until the list is sorted

FR-61: Bubble Sort

- Scan list from the last index to index 0, swapping the smallest element to the front of the list
- Scan the list from the last index to index 1, swapping the second smallest element to index 1
- Scan the list from the last index to index 2, swapping the third smallest element to index 2

  $\ldots$
- Swap the second largest element into position $(n-2)$

FR-62: Selection Sort

- Scan through the list, and find the smallest element
- Swap smallest element into position 0
- Scan through the list, and find the second smallest element
- Swap second smallest element into position 1

  $\ldots$

- Scan through the list, and find the second largest element
- Swap smallest largest into position $n - 2$

FR-63: Shell Sort

- Sort $n/2$ sublists of length 2, using insertion sort
- Sort $n/4$ sublists of length 4, using insertion sort
- Sort $n/8$ sublists of length 8, using insertion sort

  $\ldots$
- Sort 2 sublists of length $n/2$, using insertion sort
- Sort 1 sublist of length n, using insertion sort

FR-64: Merge Sort

- Base Case:
  - A list of length 1 or length 0 is already sorted
- Recursive Case:
  - Split the list in half
  - Recursively sort two halves
  - Merge sorted halves together

Example: 5 1 8 2 6 4 3 7

Divide & Conquer

Quick Sort:

- Divide the list two parts
  - Some work required – Small elements in left sublist, large elements in right sublist
- Recursively sort two parts
- Combine sorted lists into one list
  - No work required!

FR-66: Quick Sort

- Pick a pivot element
- Reorder the list:
  - All elements $<$ pivot
  - Pivot element
  - All elements $>$ pivot
- Recursively sort elements $<$ pivot
- Recursively sort elements $>$ pivot

Example: 3 7 2 8 1 4 6

FR-67: Comparison Sorting

- Comparison sorts work by comparing elements
  - Can only compare 2 elements at a time
  - Check for $<$, $>$, $=$.
- All the sorts we have seen so far (Insertion, Quick, Merge, Heap, etc.) are comparison sorts
- If we know nothing about the list to be sorted, we need to use a comparison sort

FR-68: Sorting Lower Bound

- All comparison sorting algorithms can be represented by a decision tree with $n!$ leaves
- Worst-case number of comparisons required by a sorting algorithm represented by a decision tree is the height of the tree
- A decision tree with $n!$ leaves must have a height of at least $n \lg n$
- All comparison sorting algorithms have worst-case running time $\Omega(n \lg n)$

FR-69: Binsort

- Sort n elements, in the range $1 \ldots m$
- Keep a list of m linked lists
- Insert each element into the appropriate linked lists
- Collect the lists together

FR-70: Bucket Sort

- Modify binsort so thtat each list can hold a range of values
- Need to keep each bucket sorted

FR-71: Counting Sort

```
for(i=0; i<A.length; i++)
   C[A[i].key()]++;
for(i=1; i<C.length; i++)
  C[i] = C[i] + C[i-1];

for (i=A.length - 1; i>=0; i++) {
   B[C[A[i].key()]] = A[i];
   C[A[i].key()]--;
}

for (i=0; i<A.length; i++)
   A[i] = B[i];
```

FR-72: Radix Sort

- Sort a list of numbers one digit at a time
  - Sort by 1st digit, then 2nd digit, etc
- Each sort can be done in linear time, using counting sort
- First Try: Sort by most significant digit, then the next most significant digit, and so on
  - Need to keep track of a lot of sublists

FR-73: Radix Sort

Second Try:

- Sort by *least significant* digit first
- Then sort by next-least significant digit, using a Stable sort

  . . .
- Sort by most significant digit, using a Stable sort

At the end, the list will be completely sorted.

FR-74: Searching & Selecting

- Maintian a Database (keys and associated data)
- Operations:
  - **Add** a key / value pair to the database
  - **Remove** a key (and associated value) from the database
  - **Find** the value associated with a key

FR-75: Hash Function

- What if we had a “magic function” –
  - Takes a key as input
  - Returns the index in the array where the key can be found, if the key is in the array
- To add an element
  - Put the key through the magic function, to get a location
  - Store element in that location
- To find an element
  - Put the key through the magic function, to get a location
  - See if the key is stored in that location

FR-76: Hash Function

- The “magic function” is called a *Hash function*
- If `hash(key) = i`, we say that the `key` hashes to the value `i`
- We’d like to ensure that different keys will always hash to different values.
- Not possible – too many possible keys

FR-77: Integer Hash Function

- When two keys hash to the same value, a *collision* occurs.
- We cannot avoid collisions, but we can minimize them by picking a hash function that distributes keys evenly through the array.
- Example: Keys are integers
  - Keys are in range $1 \ldots m$
  - Array indices are in range $1 \ldots n$
  - $n << m$
- $\texttt{hash}(k) = k \texttt{mod} n$

FR-78: String Hash Function

- Hash tables are usually used to store string values
- If we can convert a string into an integer, we can use the integer hash function
- How can we convert a string into an integer?
  - Concatenate ASCII digits together

$$\sum_{k=0}^{keysize-1} key[k] * 256^{keysize-k-1}$$

String Hash Function

- Concatenating digits does not work, since numbers get big too fast. Solutions:
  - Overlap digits a little (use base of 32 instead of 256)
  - Ignore early characters (shift them off the left side of the string)

```
static long hash(String key, int tablesize) {
  long h = 0;
  int i;
  for (i=0; i<key.length(); i++)
    h = (h << 4) + (int) key.charAt(i);
    return h % tablesize;
}
```

FR-80: ElfHash

- For each new character, the hash value is shifted to the left, and the new character is added to the accumulated value.
- If the string is long, the early characters will "fall off" the end of the hash value when it is shifted
  - Early characters will not affect the hash value of large strings
- Instead of falling off the end of the string, the most significant bits can be shifted to the middle of the string, and XOR'ed.
- Every character will influence the value of the hash function.

FR-81: Collisions

- When two keys hash to the same value, a *collision* occurs
- A collision strategy tells us what to do when a collision occurs
- Two basic collision strategies:
  - Open Hashing (Closed Addressing, Separate Chaining)
  - Closed Hashing (Open Addressing)

FR-82: Closed Hashing

- To add element X to a closed hash table:
  - Find the smallest i, such that Array[hash(x) + f(i)] is empty (wrap around if necessary)
  - Add X to Array[hash(x) + f(i)]
  - If f(i) = i, linear probing

FR-83: Closed Hashing

- Quadradic probing
  - Find the smallest i, such that Array[hash(x) + f(i)] is empty
  - Add X to Array[hash(x) + f(i)]
  - f(i) = i^2

FR-84: Closed Hashing

- Multiple keys hash to the same element
  - Secondary clustering
- Double Hashing
  - Use a secondary hash function to determine how far ahead to look
  - f(i) = i * hash2(key)

Disjoint Sets

- Elements will be integers (for now)
- Operations:
  - CreateSets(n) – Create n sets, for integers 0..(n-1)
  - Union(x,y) – merge the set containing x and the set containing y
  - Find(x) – return a representation of x’s set
    - Find(x) = Find(y) iff x,y are in the same set

FR-86: Implementing Disjoint Sets

- Find: (pseudo-Java)

```
int Find(x) {
   while (Parent[x] > 0)
      x = Parent[x]
   return x
}
```

FR-87: Implementing Disjoint Sets

- Union(x,y) (pseudo-Java)

```
void Union(x,y) {
    rootx = Find(x);
    rooty = Find(y);
    Parent[rootx] = Parent[rooty];
}
```

FR-88: Union by Rank

- When we merge two sets:
  - Have the shorter tree point to the taller tree
  - Height of taller tree does not change
  - If trees have the same height, choose arbitrarily

FR-89: Path Compression

- After each call to `Find(x)`, change `x`'s parent pointer to point directly at root
- Also, change all parent pointers on path from `x` to root

Graphs

- A graph consists of:
  - A set of **nodes** or **vertices** (terms are interchangable)
  - A set of **edges** or **arcs** (terms are interchangable)
- Edges in graph can be either directed or undirected

FR-91: Graphs & Edges

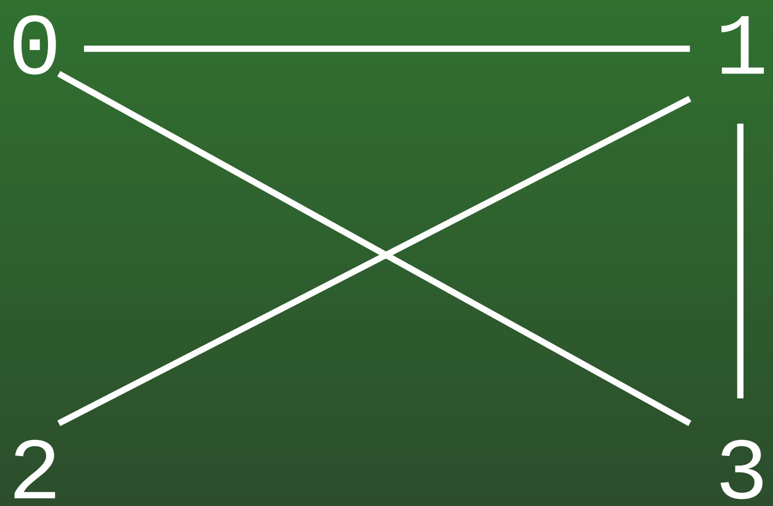
- Edges can be labeled or unlabeled
  - Edge labels are typically the *cost* assoctiated with an edge
  - e.g., Nodes are cities, edges are roads between cities, edge label is the length of road

FR-92: Graph Representations

- Adjacency Matrix
- Represent a graph with a two-dimensional array G
  - $G[i][j] = 1$ if there is an edge from node i to node j
  - $G[i][j] = 0$ if there is no edge from node i to node j
- If graph is undirected, matrix is symmetric
- Can represent edges labeled with a cost as well:
  - $G[i][j] =$ cost of link between i and j
  - If there is no direct link, $G[i][j] = \infty$

FR-93: Adjacency Matrix

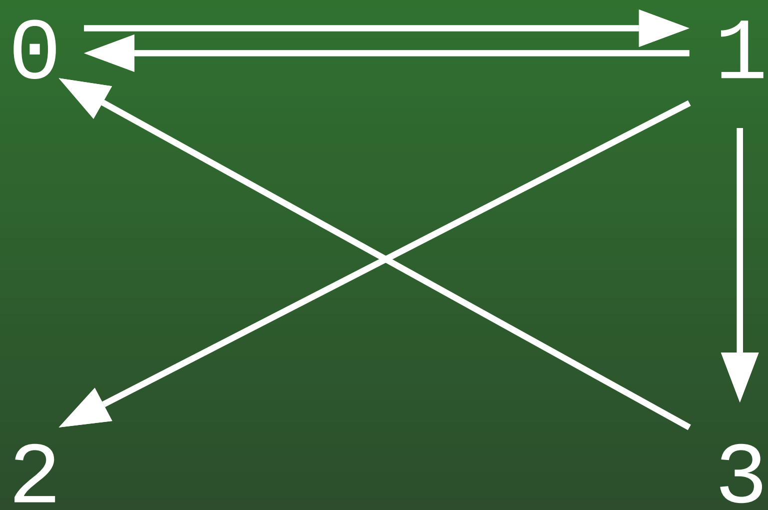
- Examples:



	0	1	2	3
0	0	1	0	1
1	1	0	1	1
2	0	1	0	0
3	1	1	0	0

FR-94: Adjacency Matrix

- Examples:



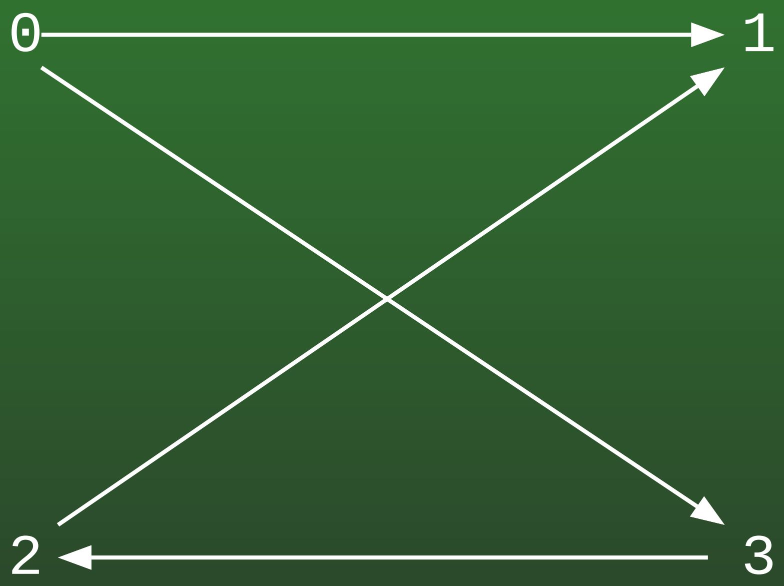
	0	1	2	3
0	0	1	0	0
1	1	0	1	1
2	0	0	0	0
3	1	0	0	0

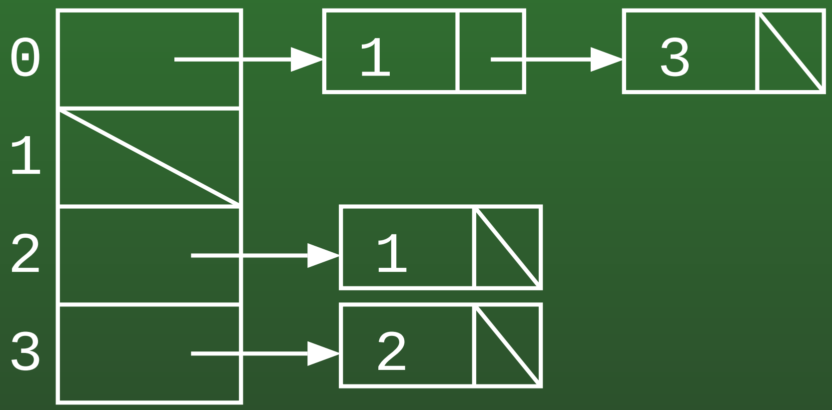
Graph Representations

- Adjacency List
- Maintain a linked-list of the neighbors of every vertex.
  - n vertices
  - Array of n lists, one per vertex
  - Each list i contains a list of all vertices adjacent to i.

FR-96: Adjacency List

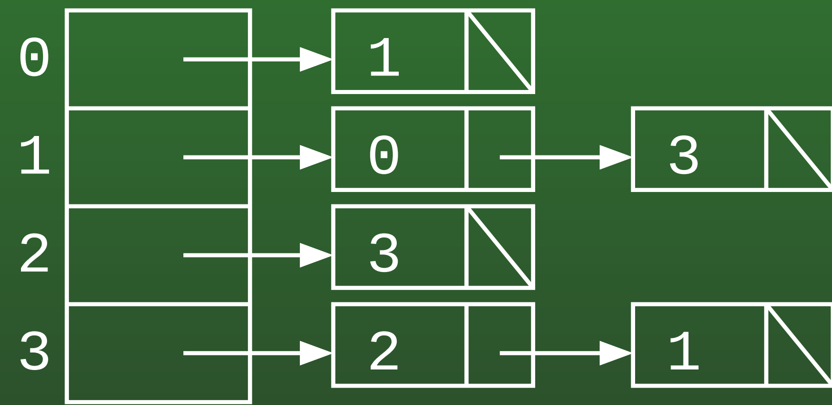
- Examples:





FR-97: Adjacency List

- Examples:



- Note – lists are not always sorted

FR-98: Topological Sort

- Directed Acyclic Graph, Vertices $v_1 \ldots v_n$
- Create an ordering of the vertices
  - If there a path from v_i to v_j, then v_i appears before v_j in the ordering
- Example: Prerequisite chains

FR-99: Topological Sort

- Pick a node v_k with no incident edges
- Add v_k to the ordering
- Remove v_k and all edges from v_k from the graph
- Repeat until all nodes are picked.

FR-100: Graph Traversals

- Visit every vertex, in an order defined by the topololgy of the graph.
- Two major traversals:
  - Depth First Search
  - Breadth First Search

FR-101: Depth First Search

- Starting from a specific node (pseudo-code):

```
DFS(Edge G[], int vertex, boolean Visited[]) {
  Visited[vertex] = true;
  for each node w adajcent to vertex:
    if (!Visited[w])
      DFS(G, w, Visited);
}
```

FR-102: Depth First Search

```
class Edge {
   public int neighbor;
   public int next;
}

void DFS(Edge G[], int vertex, boolean Visited[]) {
  Edge tmp;
  Visited[vertex] = true;
  for (tmp = G[vertex]; tmp != null; tmp = tmp.next) {
    if (!Visited[tmp.neighbor])
      DFS(G, tmp.neighbor, Visited);
  }
}
```

FR-103: Breadth First Search

- DFS: Look as *Deep* as possible, before looking wide
  - Examine all descendants of a node, before looking at siblings
- BFS: Look as *Wide* as possible, before looking deep
  - Visit all nodes 1 away, then 2 away, then three away, and so on

FR-104: Search Trees

- Describes the order that nodes are examined in a traversal
- Directed Tree
  - Directed edge from v_1 to v_2 if the edge (v_1, v_2) was followed during the traversal

FR-105: Computing Shortest Path

- Given a directed weighted graph G (all weights non-negative) and two vertices x and y, find the least-cost path from x to y in G.
  - Undirected graph is a special case of a directed graph, with symmetric edges
- Least-cost path may not be the path containing the fewest edges
  - “shortest path” == “least cost path”
  - “path containing fewest edges” = “path containing fewest edges”

FR-106: Single Source Shortest Path

- If all edges have unit weight,
- We can use Breadth First Search to compute the shortest path
- BFS Spanning Tree contains shortest path to each node in the graph
    - Need to do some more work to create & save BFS spanning tree
- When edges have differing weights, this obviously will not work

FR-107: Single Source Shortest Path

- Divide the vertices into two sets:
  - Vertices whose shortest path from the initial vertex is known
  - Vertices whose shortest path from the initial vertex is not known
- Initially, only the initial vertex is known
- Move vertices one at a time from the unknown set to the known set, until all vertices are known

FR-108: Dijkstra's Algorithm

- Keep a table that contains, for each vertex
  - Is the distance to that vertex known?
  - What is the best distance we've found so far?
- Repeat:
  - Pick the smallest unknown distance
  - mark it as known
  - update the distance of all unknown neighbors of that node
- Until all vertices are known

Floyd's Algorithm

- Vertices numbered from 1..n
- k-path from vertex v to vertex u is a path whose intermediate vertices (other than v and u) contain only vertices numbered k or less
- 0-path is a direct link

FR-110: Floyd's Algorithm

- Shortest n-path = Shortest path
- Shortest 0-path:
  - ∞ if there is no direct link
  - Cost of the direct link, otherwise
- If we could use the shortest k-path to find the shortest $(k + 1)$ path, we would be set

FR-111: Floyd's Algorithm

- Shortest k-path from v to u either goes through vertex k, or it does not
- If not:
  - Shortest k-path = shortest $(k-1)$-path
- If so:
  - Shortest k-path = shortest $k-1$ path from v to k, followed by the shortest $k-1$ path from k to w

Floyd's Algorithm

- If we had the shortest k-path for all pairs (v,w), we could obtain the shortest $k+1$-path for all pairs
  - For each pair v, w, compare:
    - length of the k-path from v to w
    - length of the k-path from v to k appended to the k-path from k to w
  - Set the $k+1$ path from v to w to be the minimum of the two paths above

FR-113: Floyd's Algorithm

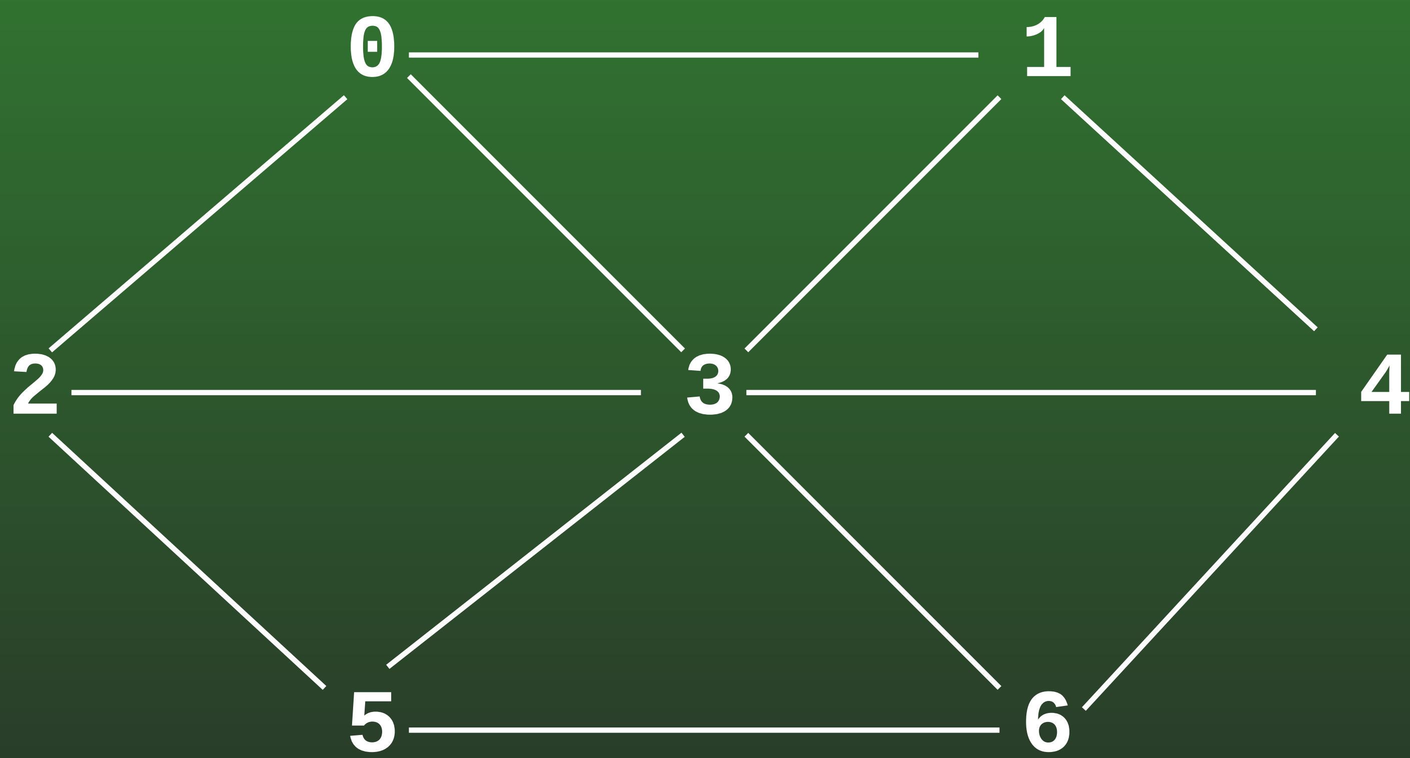
- Let $D_k[v,w]$ be the length of the shortest k-path from v to w.
- $D_0[v,w]$ = cost of arc from v to w (∞ if no direct link)
- $D_k[v,w]$ = MIN($D_{k-1}[v,w]$, $D_{k-1}[v,k] + D_{k-1}[k,w]$)
- Create D_0, use D_0 to create D_1, use D_1 to create D_2, and so on – until we have D_n

Spanning Trees

- Given a connected, undirected graph G
  - A *subgraph* of G contains a subset of the vertices and edges in G
  - A *Spanning Tree* T of G is:
    - subgraph of G
    - contains all vertices in G
    - connected
    - acyclic

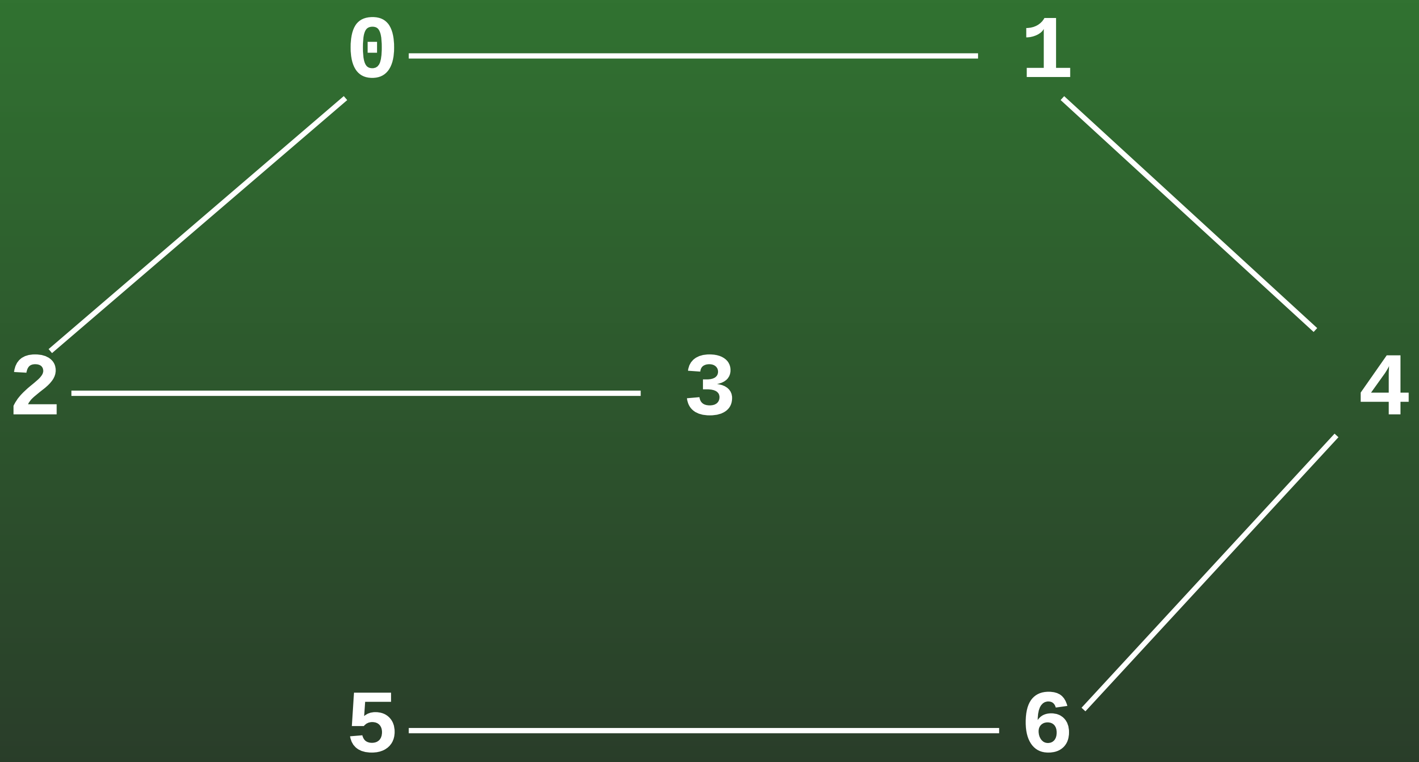
FR-115: Spanning Tree Examples

- Graph



Spanning Tree Examples

- Spanning Tree



FR-117: Minimal Cost Spanning Tree

- Minimal Cost Spanning Tree
  - Given a weighted, undirected graph G
  - Spanning tree of G which minimizes the sum of all weights on edges of spanning tree

FR-118: Kruskal's Algorithm

- Start with an empty graph (no edges)
- Sort the edges by cost
- For each edge e (in increasing order of cost)
    - Add e to G if it would not cause a cycle

Kruskal's Algorithm

- We need to:
  - Put each vertex in its own tree
  - Given any two vertices v_1 and v_2, determine if they are in the same tree
  - Given any two vertices v_1 and v_2, merge the tree containing v_1 and the tree containing v_2
- ... sound familiar?

FR-120: Kruskal's Algorithm

- Disjoint sets!
- Create a list of all edges
- Sort list of edges
- For each edge $e = (v_1, v_2)$ in the list
  - if FIND(v_1) != FIND(v_2)
    - Add e to spanning tree
    - UNION(v_1, v_2)

FR-121: Prim's Algorithm

- Grow that spanning tree out from an initial vertex
- Divide the graph into two sets of vertices
  - vertices in the spanning tree
  - vertices *not* in the spanning tree
- Initially, Start vertex is in the spanning tree, all other vertices are not in the tree
  - Pick the initial vertex arbitrarily

FR-122: Prim's Algorithm

- While there are vertices not in the spanning tree
  - Add the cheapest vertex to the spanning tree

FR-123: Indexing

- Operations:
  - Add an element
  - Remove an element
  - Find an element, using a key
  - Find all elements in a range of key values

2-3 Trees

- Generalized Binary Search Tree
  - Each node has 1 or 2 keys
  - Each (non-leaf) node has 2-3 children
    - hence the name, 2-3 Trees
  - All leaves are at the same depth

FR-125: Finding in 2-3 Trees

- How can we find an element in a 2-3 tree?
  - If the tree is empty, return false
  - If the element is stored at the root, return true
  - Otherwise, recursively find in the appropriate subtree

FR-126: Inserting into 2-3 Trees

- Always insert at the leaves
- To insert an element:
  - Find the leaf where the element would live, if it was in the tree
  - Add the element to that leaf
    - What if the leaf already has 2 elements?
    - Split!

FR-127: Splitting nodes

- To split a node in a 2-3 tree that has 3 elements:
  - Split nodes into two nodes
    - One node contains the smallest element
    - Other node contains the largest element
  - Add median element to parent
    - Parent can then handle the extra pointer

FR-128: 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree

FR-129: 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree

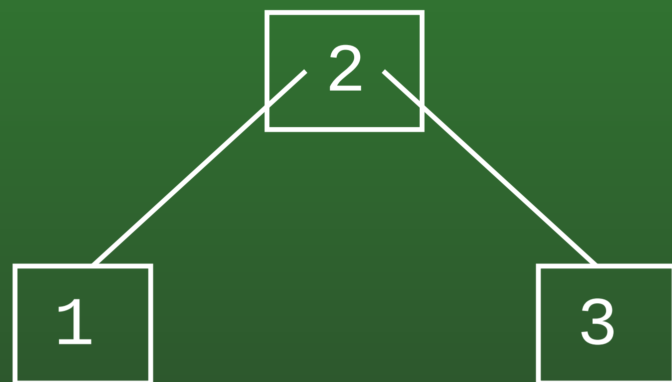
FR-130: 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree


```
Too many keys,
need to split
```

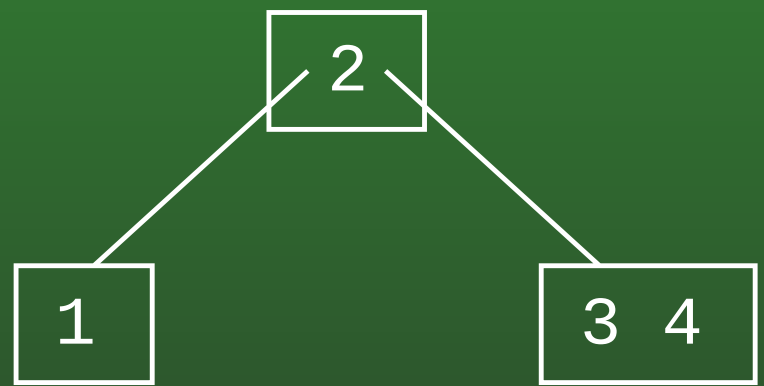
FR-131: 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree



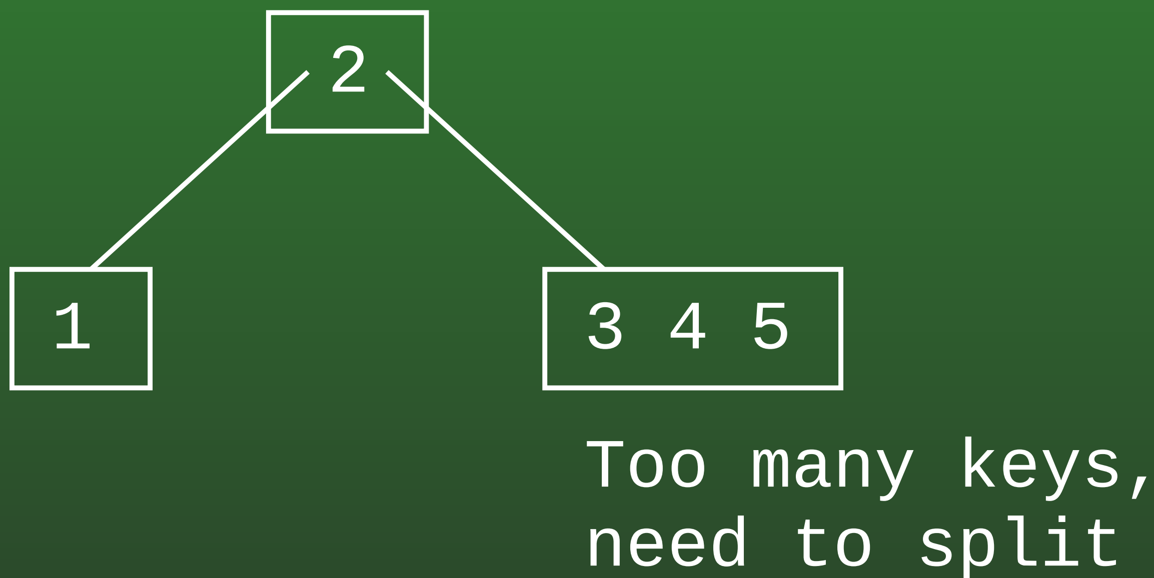
2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree



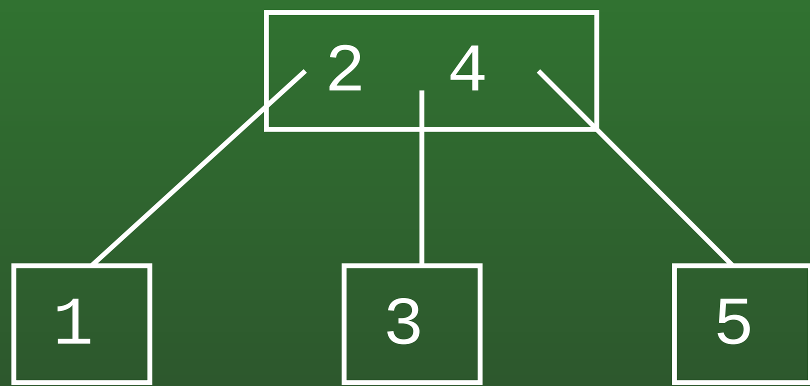
FR-133: 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree



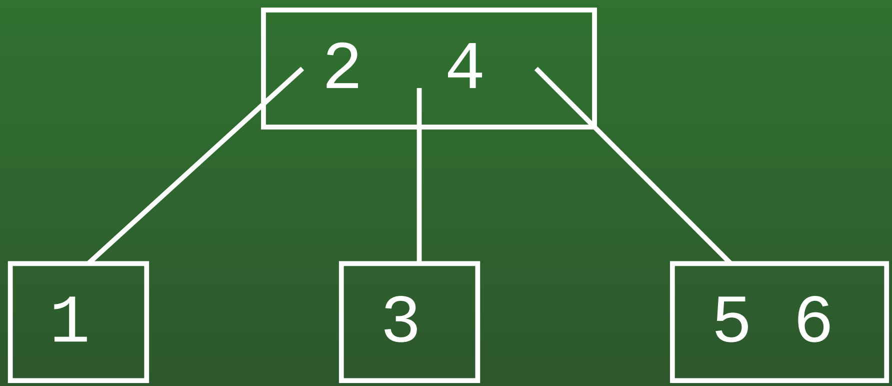
FR-134: 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree



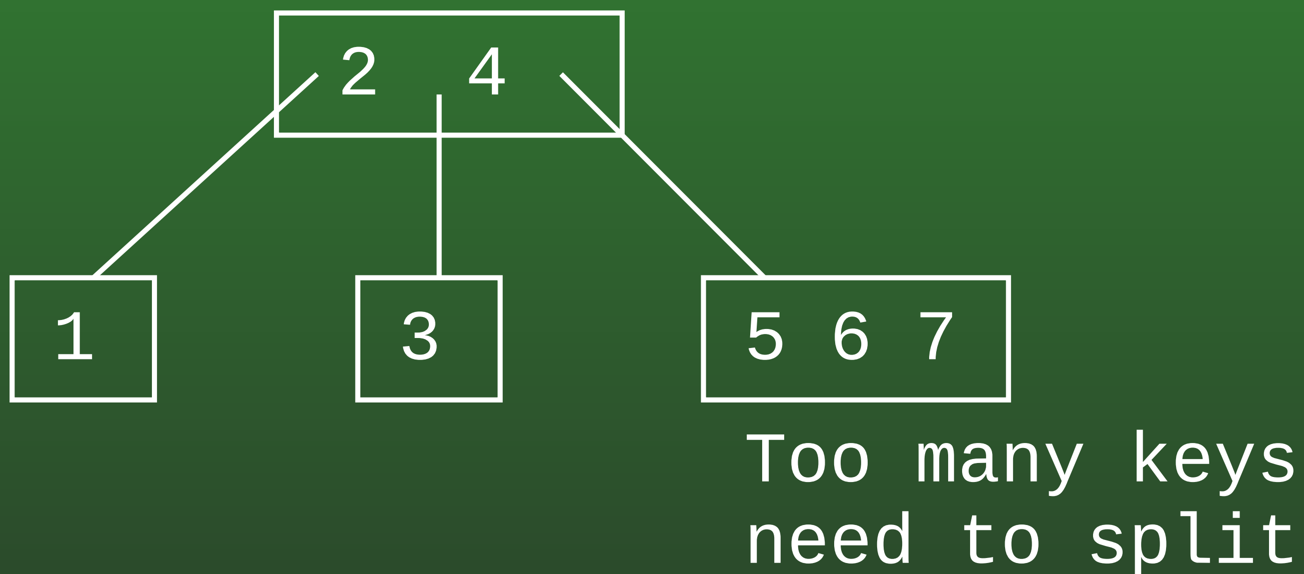
2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree



FR-136: 2-3 Tree Example

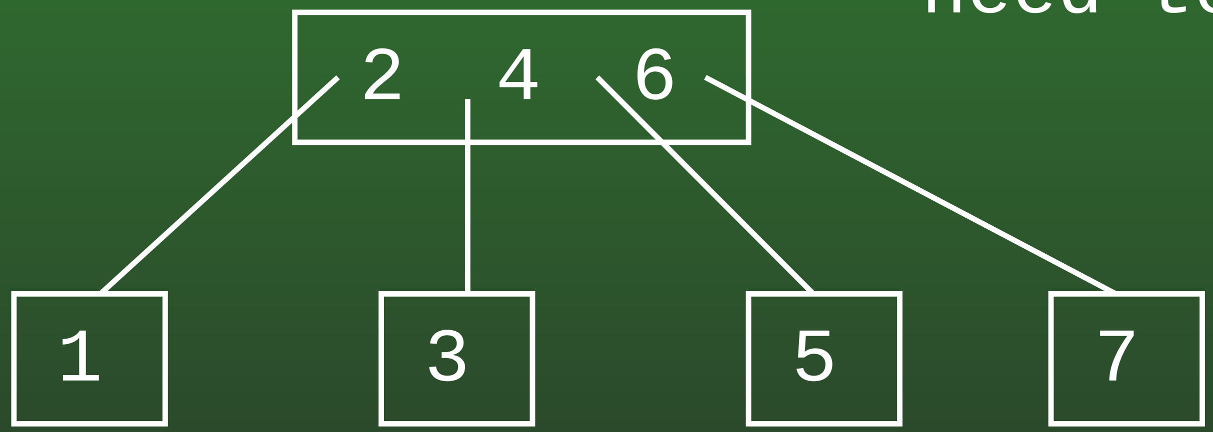
- Inserting elements 1-9 (in order) into a 2-3 tree



FR-137: 2-3 Tree Example

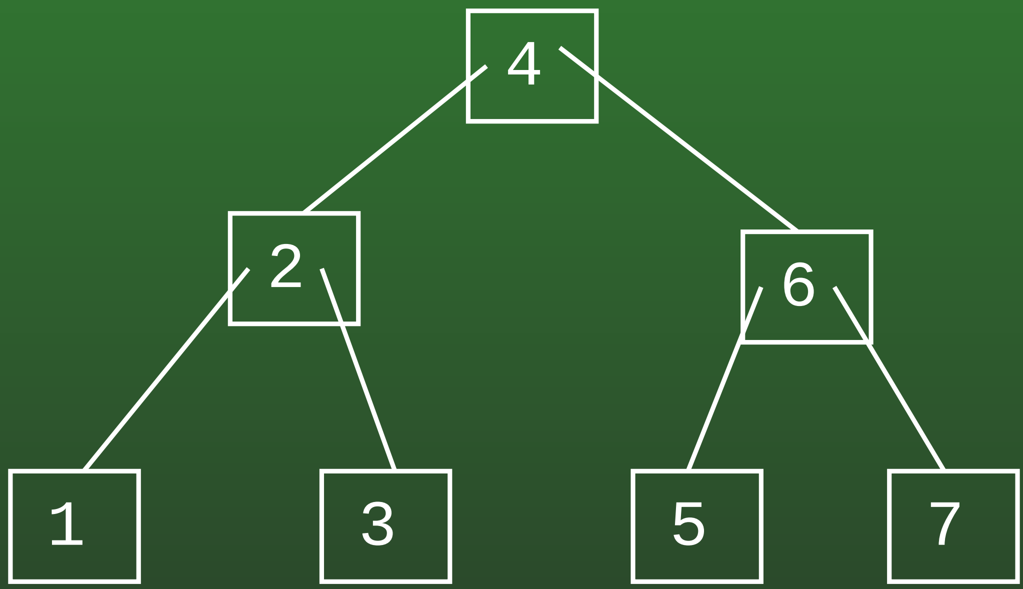
- Inserting elements 1-9 (in order) into a 2-3 tree

Too many keys
need to split



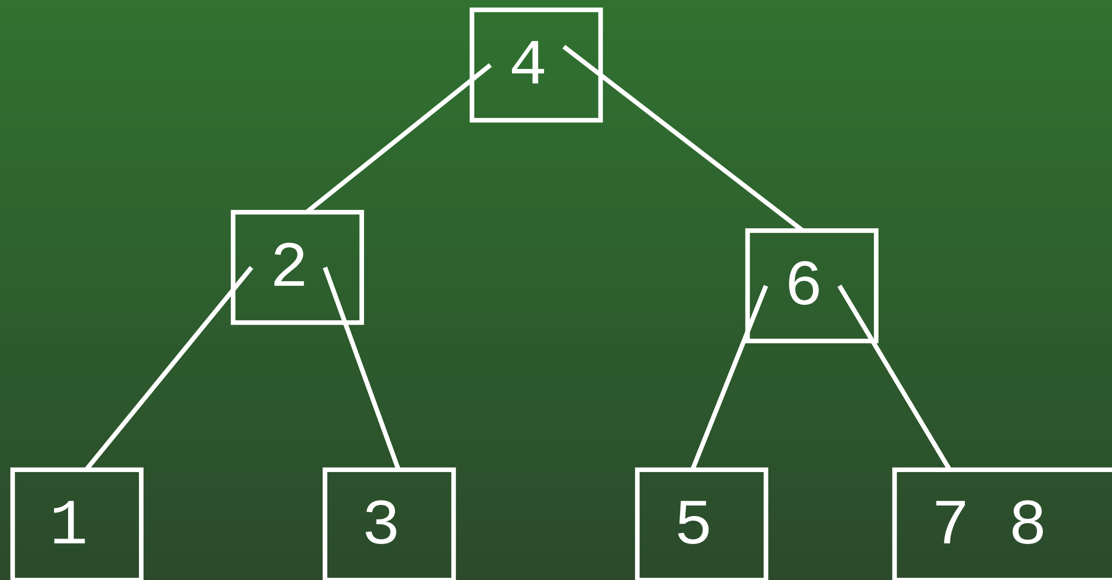
2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree



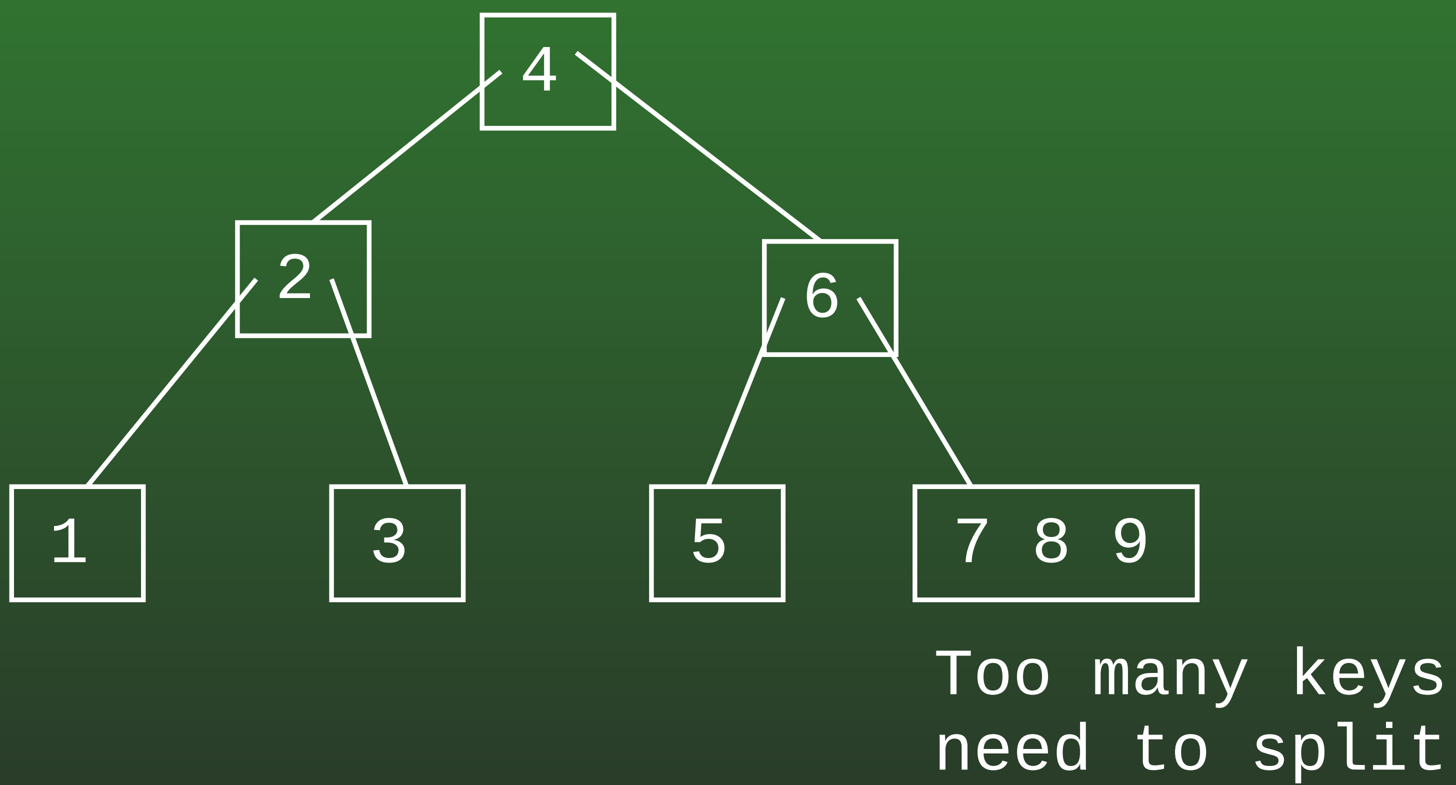
 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree



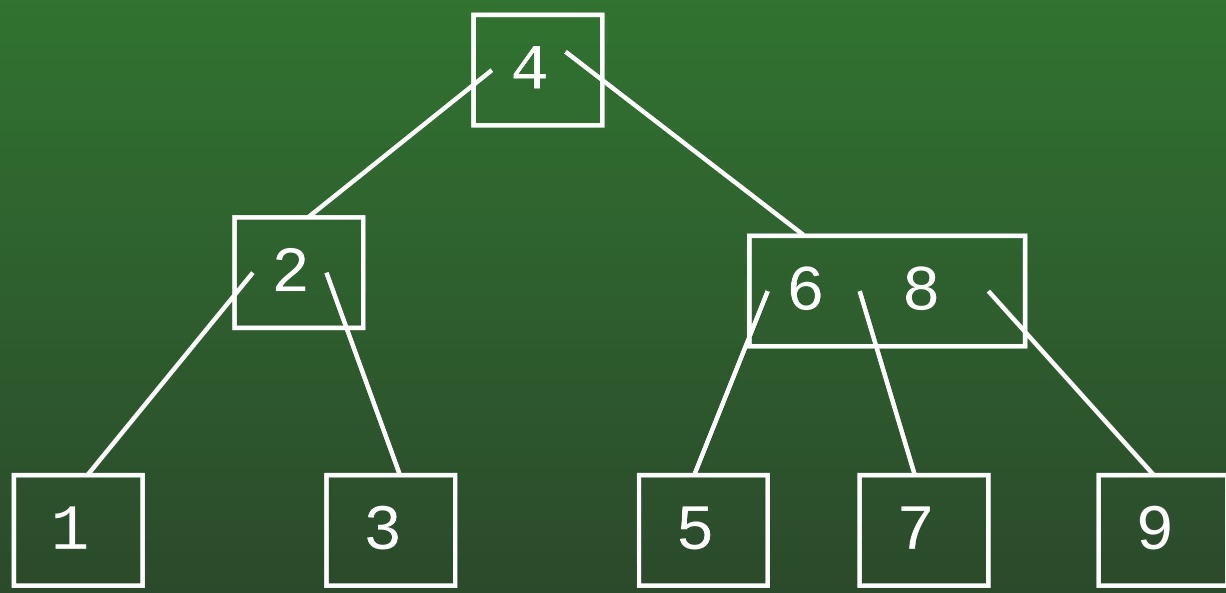
FR-140: 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree



FR-141: 2-3 Tree Example

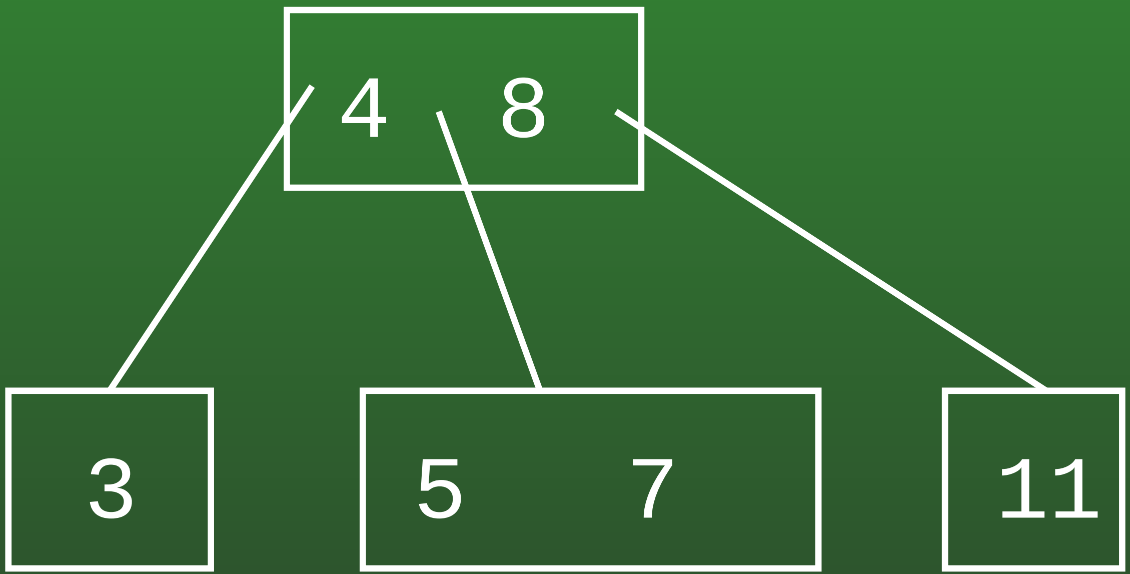
- Inserting elements 1-9 (in order) into a 2-3 tree



FR-142: Deleting Leaves

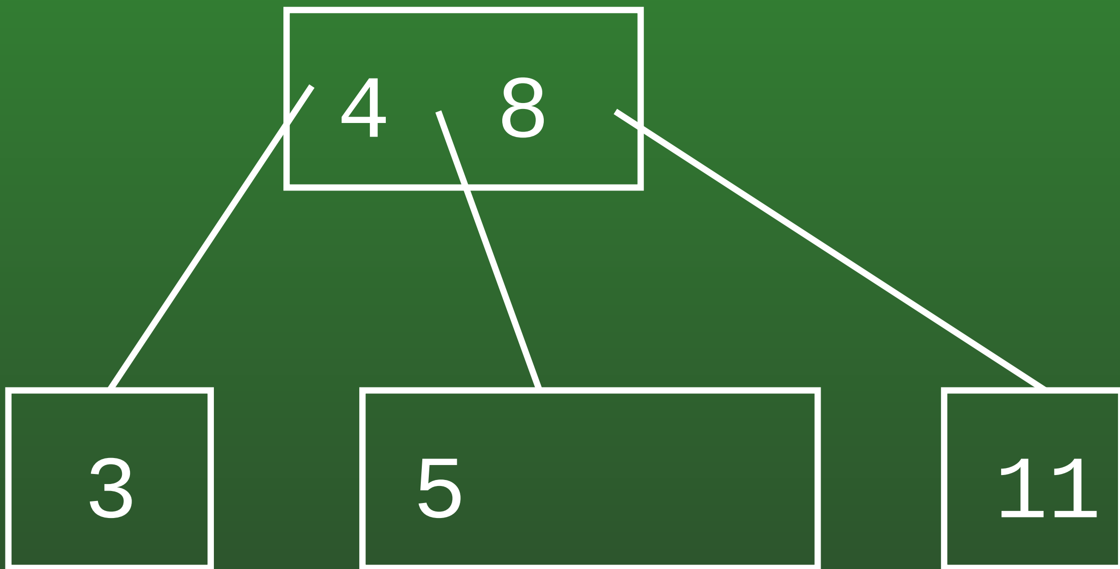
- If leaf contains 2 keys
  - Can safely remove a key

FR-143: Deleting Leaves



- Deleting 7

FR-144: Deleting Leaves

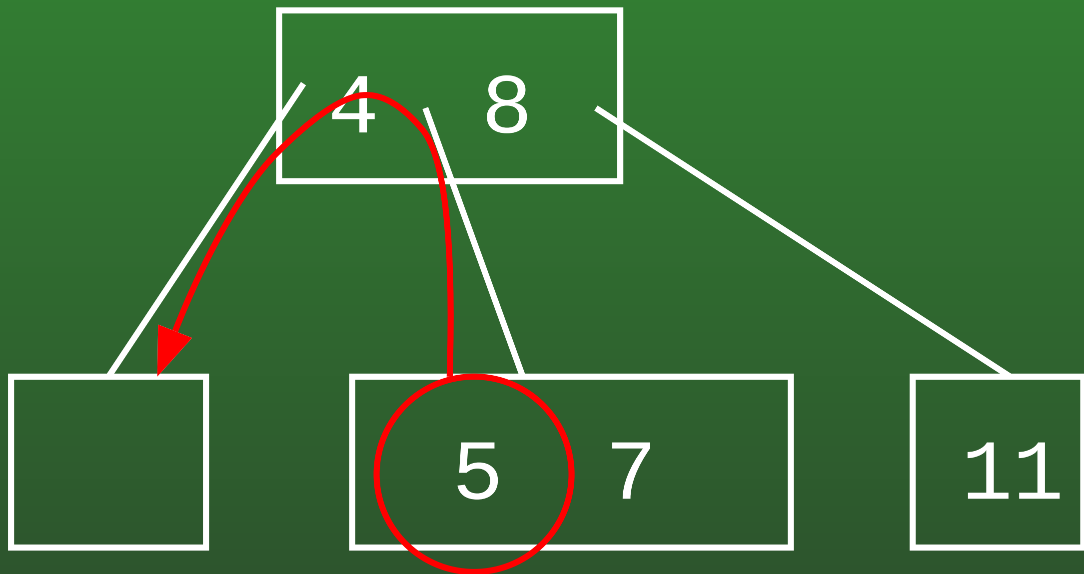


- Deleting 7

FR-145: Deleting Leaves

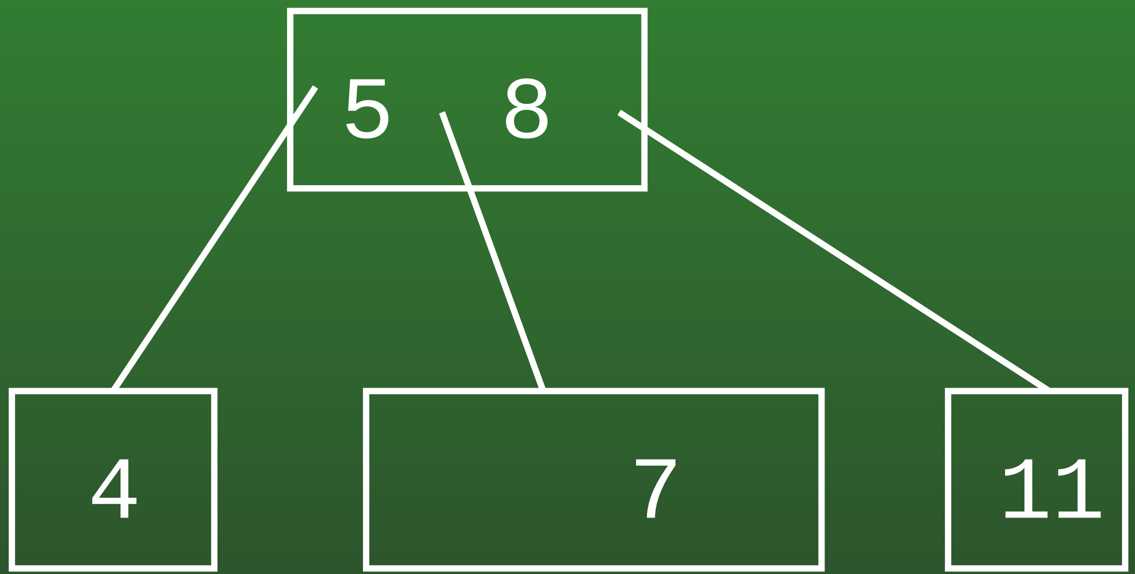
- If leaf contains 1 key
  - Cannot remove key without making leaf empty
  - Try to steal extra key from sibling

FR-146: Deleting Leaves



- Steal key from sibling *through parent*

FR-147: Deleting Leaves

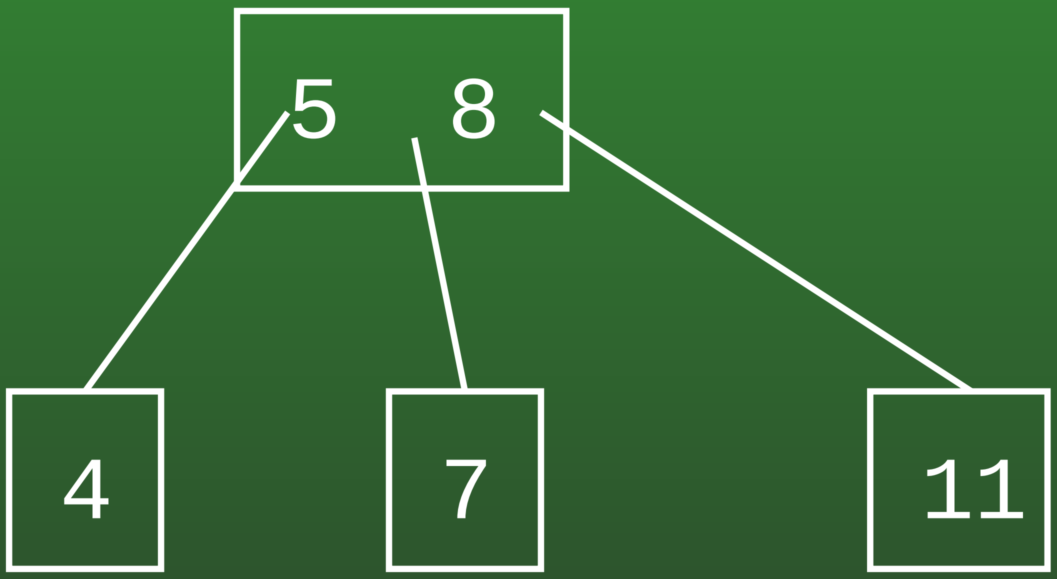


- Steal key from sibling *through parent*

FR-148: Deleting Leaves

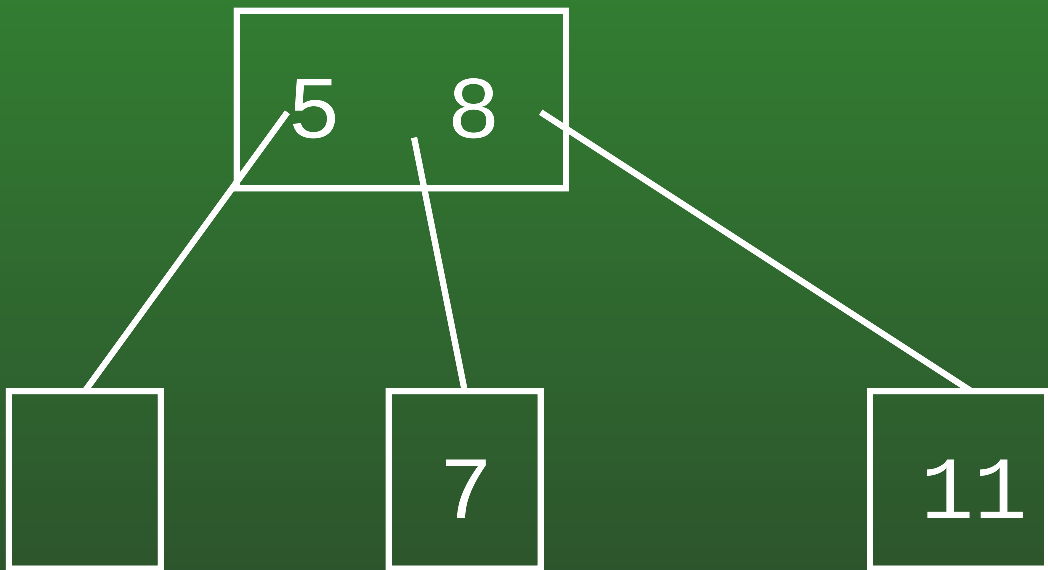
- If leaf contains 1 key, and no sibling contains extra keys
  - Cannot remove key without making leaf empty
  - Cannot steal a key from a sibling
  - Merge with sibling
    - split in reverse

FR-149: Merging Nodes



- Removing the 4

Merging Nodes

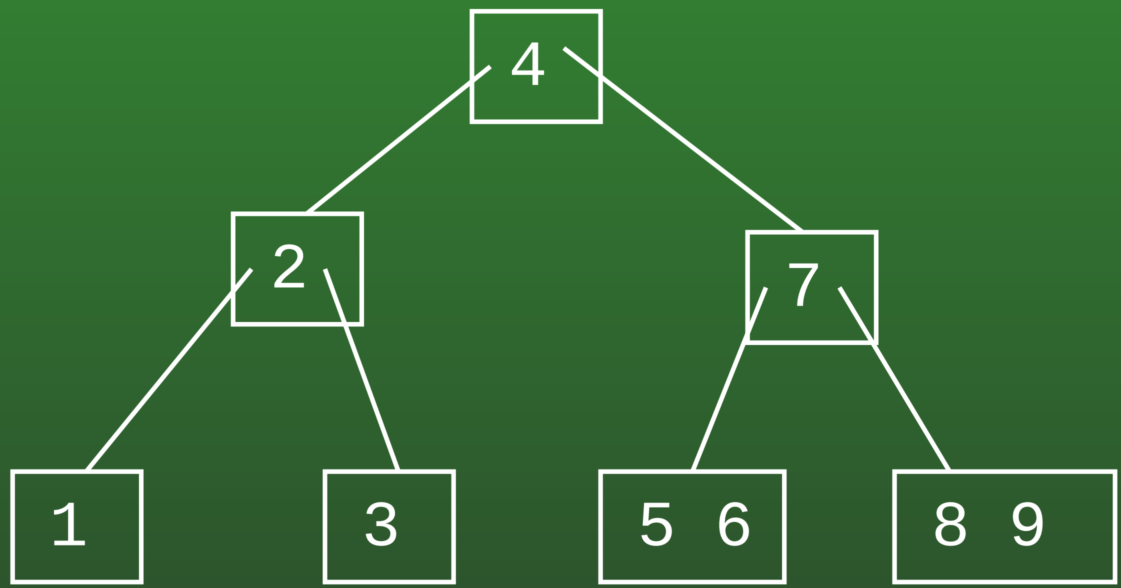


- Removing the 4
- Combine 5, 7 into one node

FR-151: Deleting Interior Keys

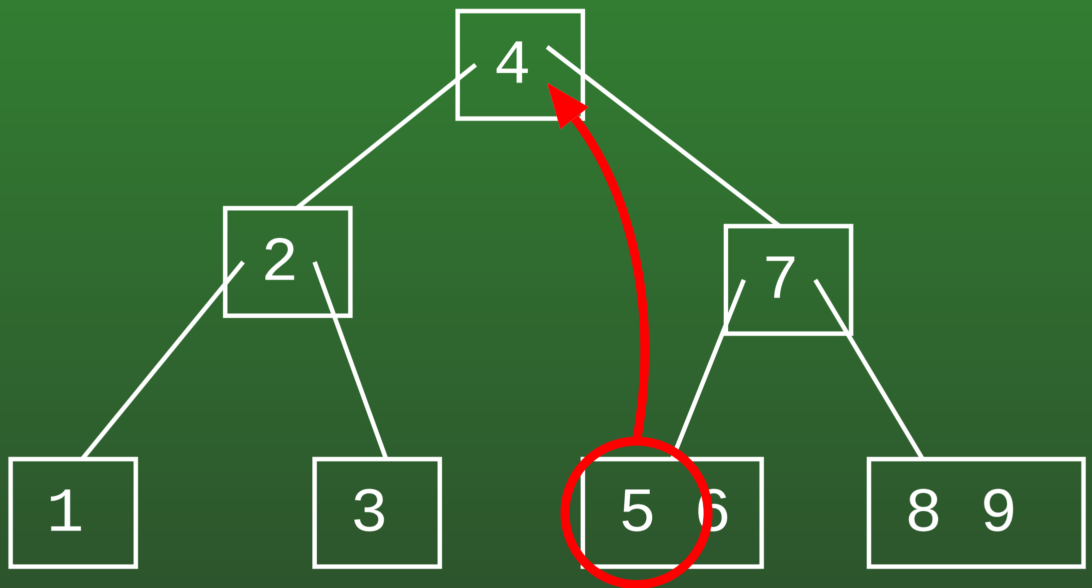
- How can we delete keys from non-leaf nodes?
  - Replace key with smallest element subtree to right of key
  - Recursivly delete smallest element from subtree to right of key
- (can also use largest element in subtree to left of key)

FR-152: Deleting Interior Keys



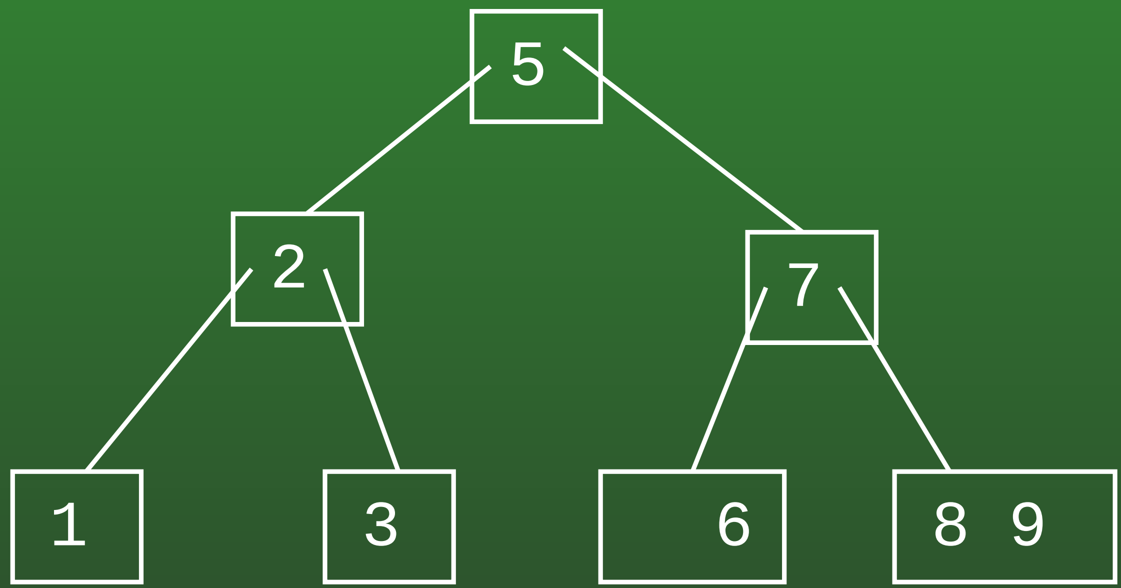
- Deleting the 4

Deleting Interior Keys

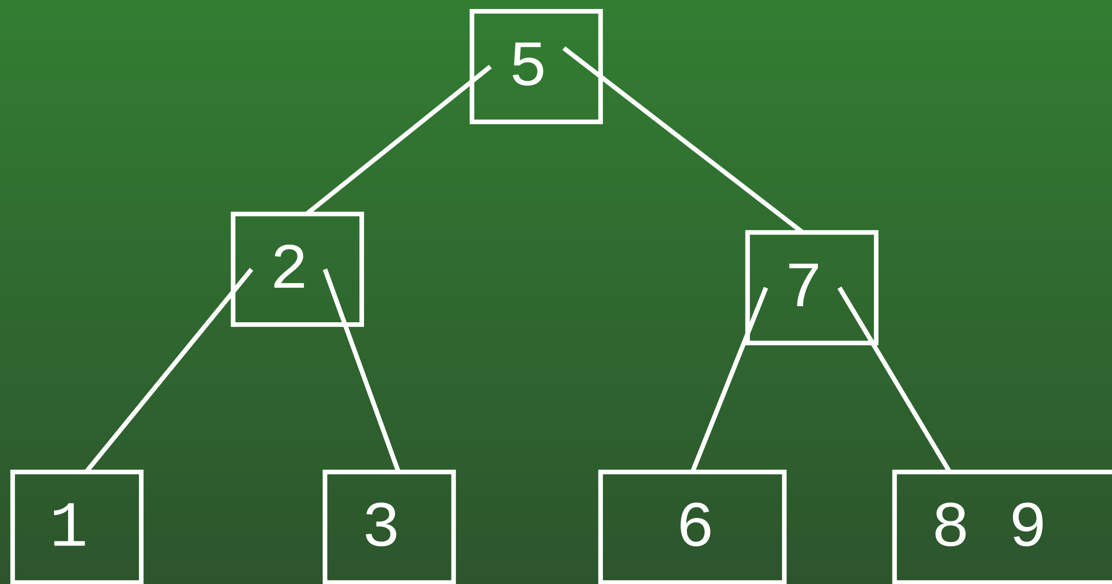


- Deleting the 4
- Replace 4 with smallest element in tree to right of 4

FR-154: Deleting Interior Keys

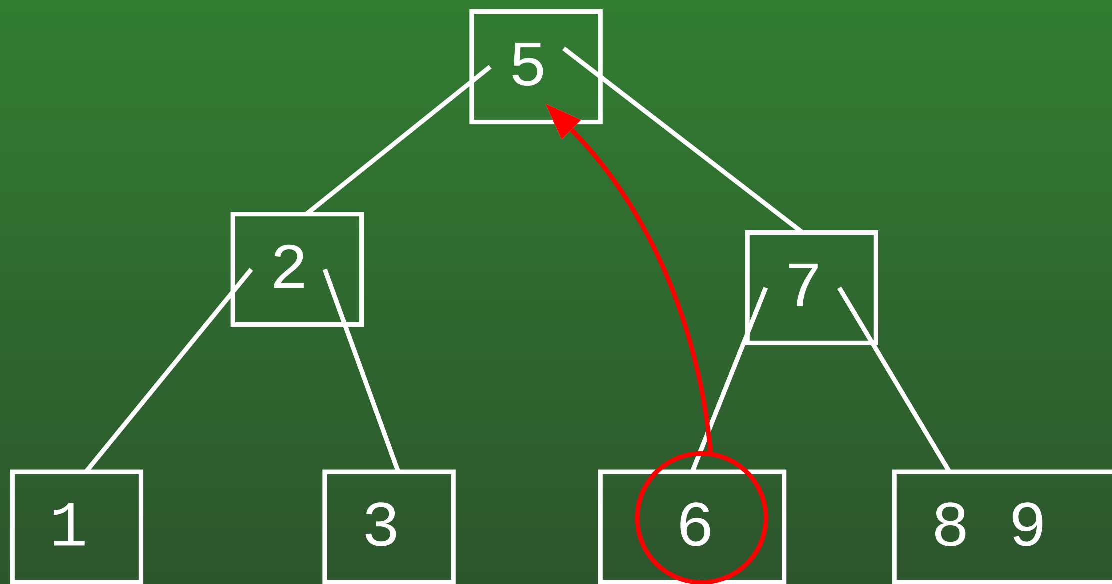


FR-155: Deleting Interior Keys



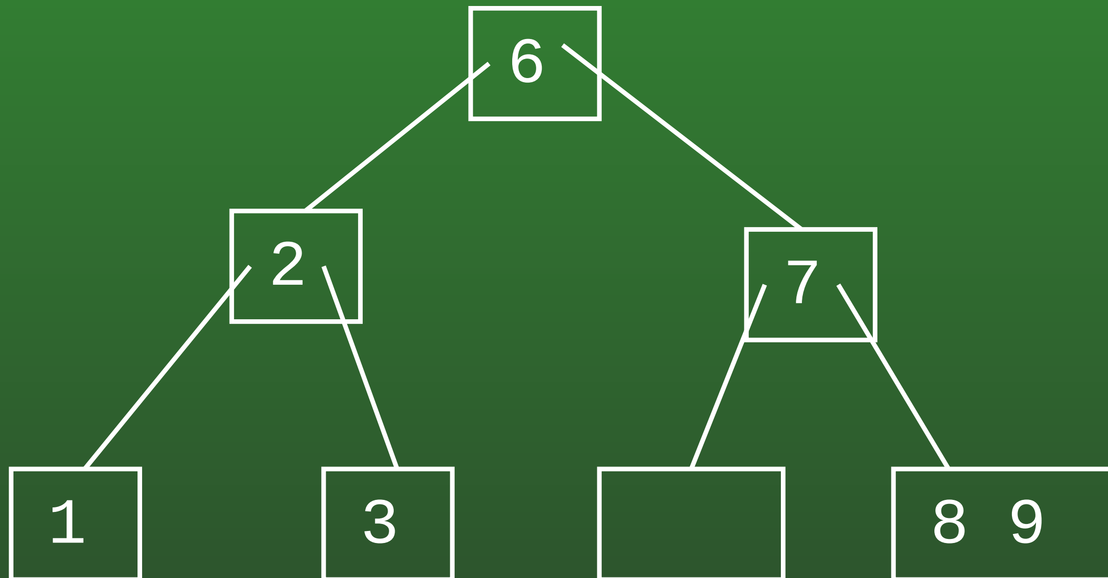
- Deleting the 5

FR-156: Deleting Interior Keys



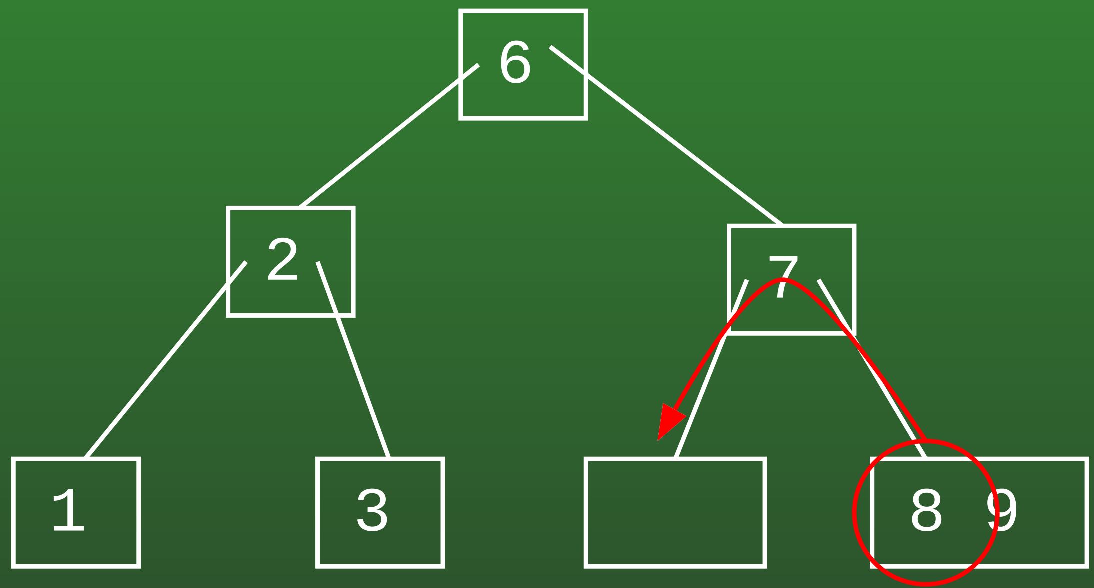
- Deleting the 5
- Replace the 5 with the smallest element in tree to right of 5

FR-157: Deleting Interior Keys



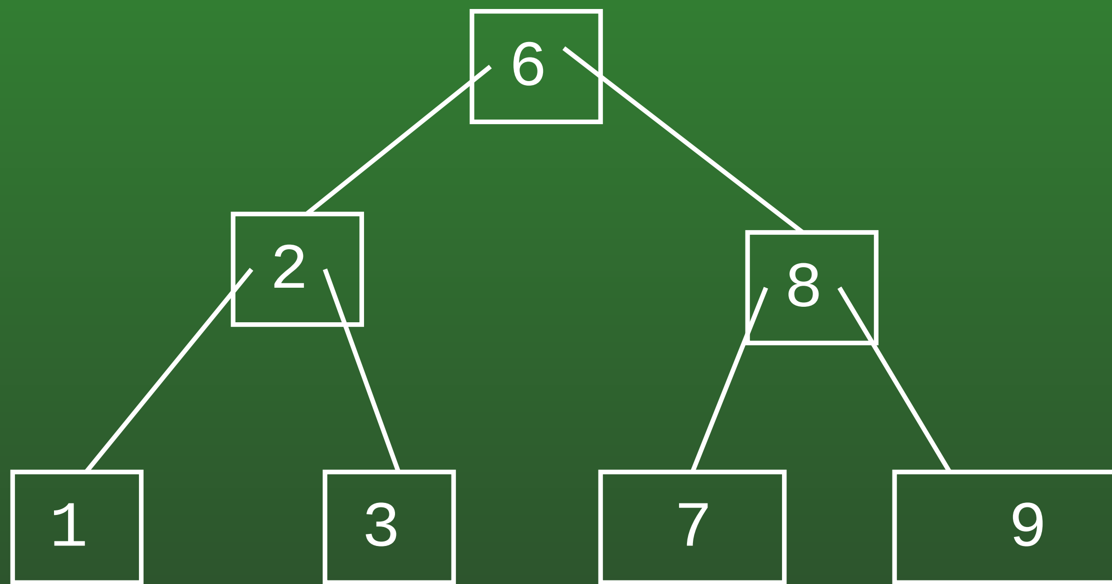
- Deleting the 5
- Replace the 5 with the smallest element in tree to right of 5
- Node with two few keys

Deleting Interior Keys

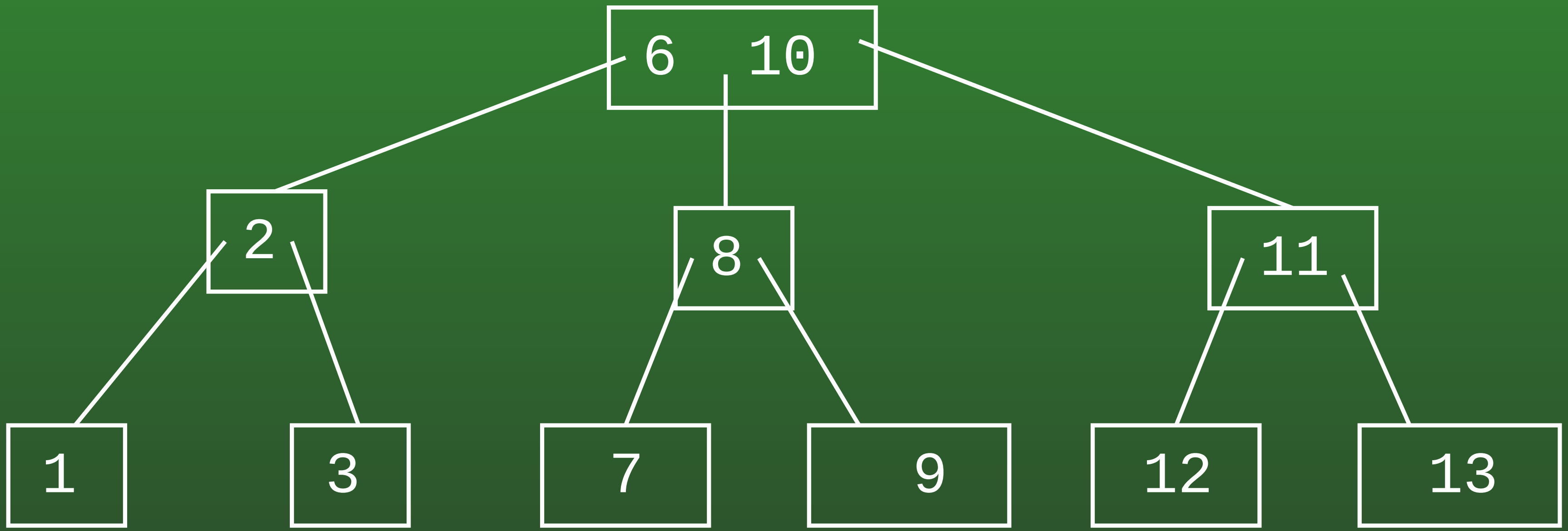


- Node with two few keys
- Steal a key from a sibling

Deleting Interior Keys

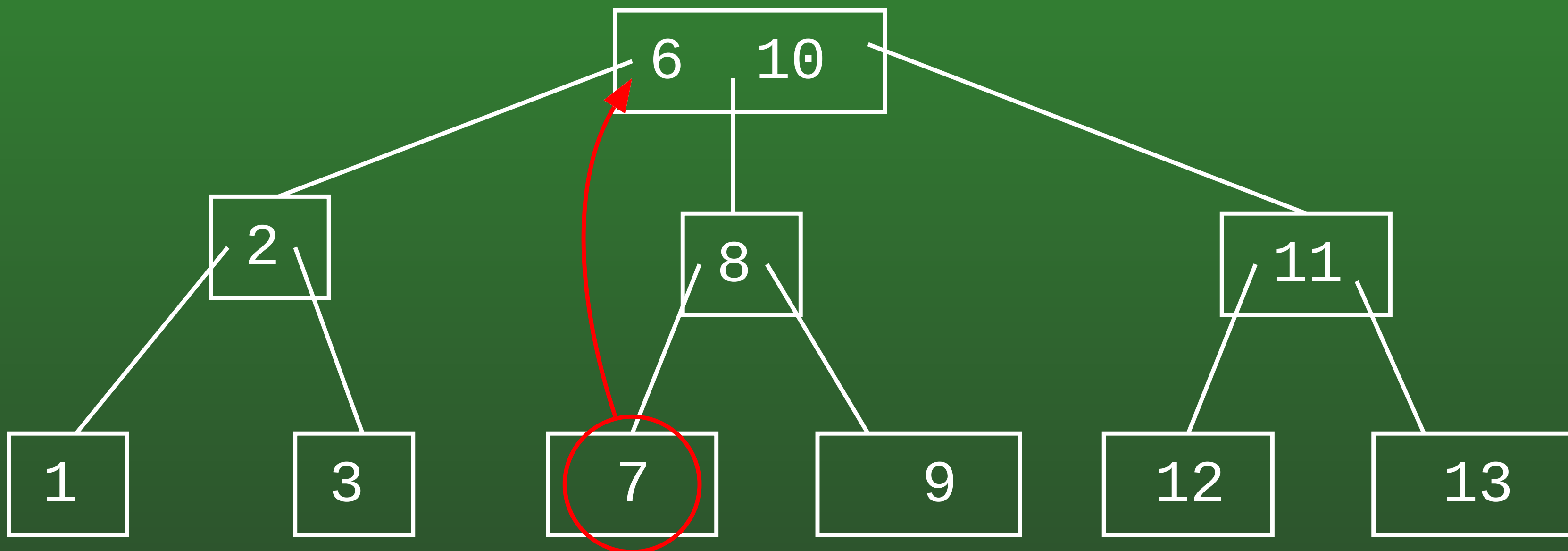


FR-160: Deleting Interior Keys



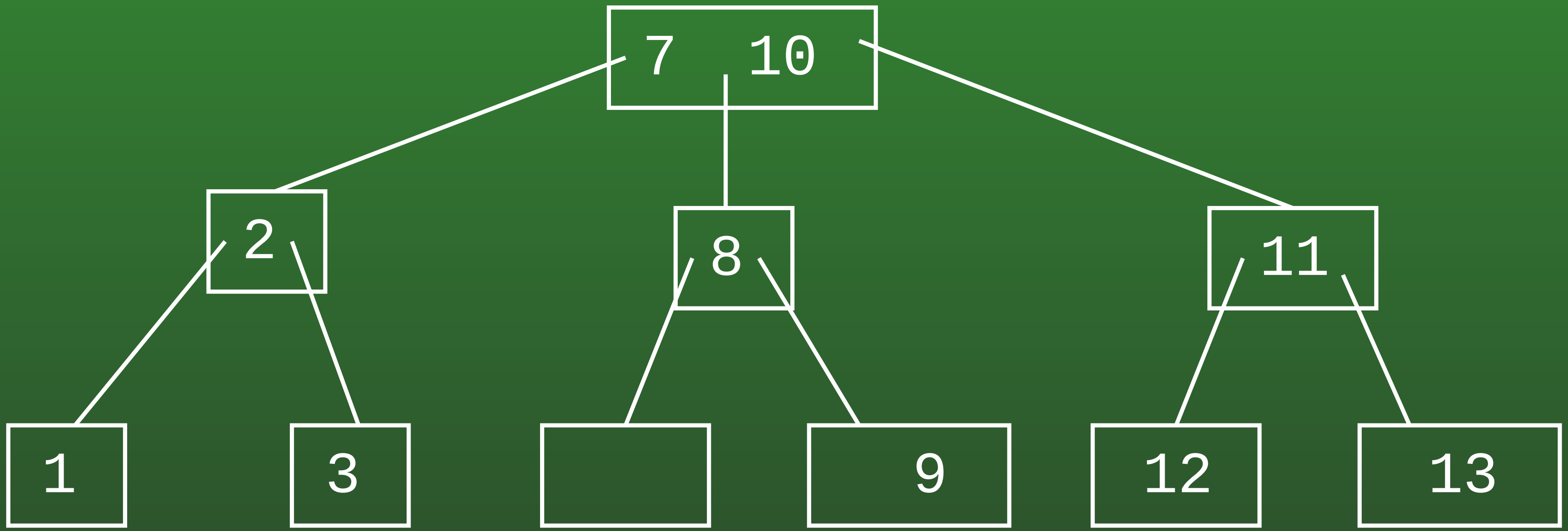
- Removing the 6

FR-161: Deleting Interior Keys



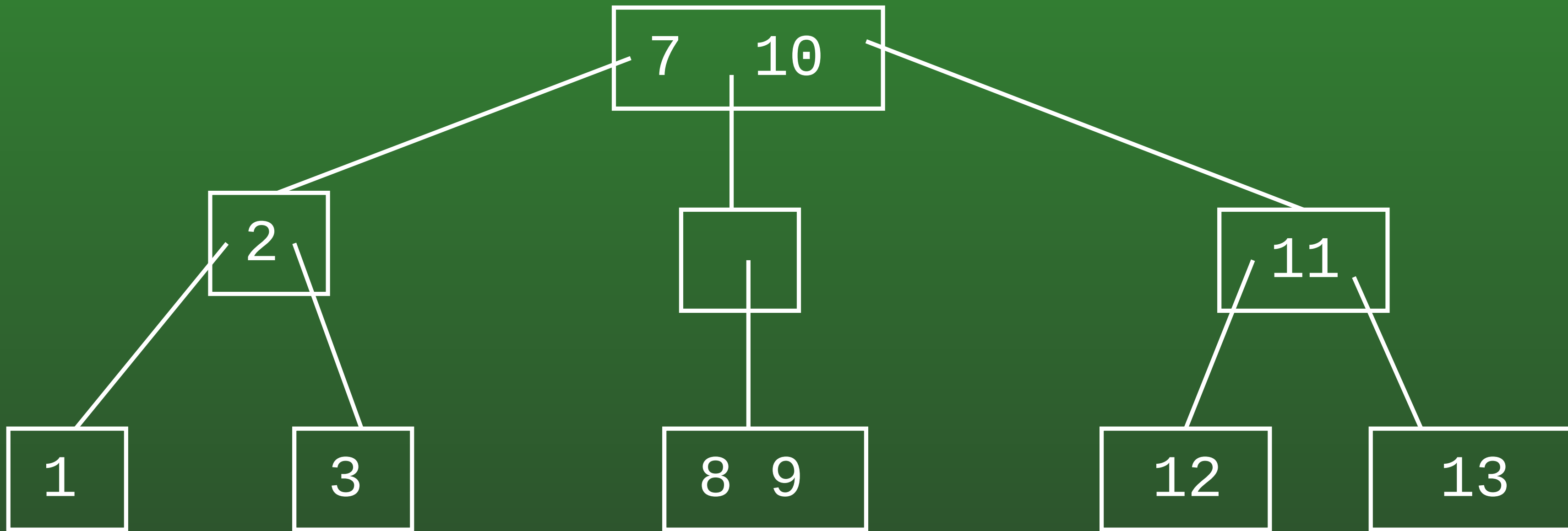
- Removing the 6
- Replace the 6 with the smallest element in the tree to the right of the 6

Deleting Interior Keys



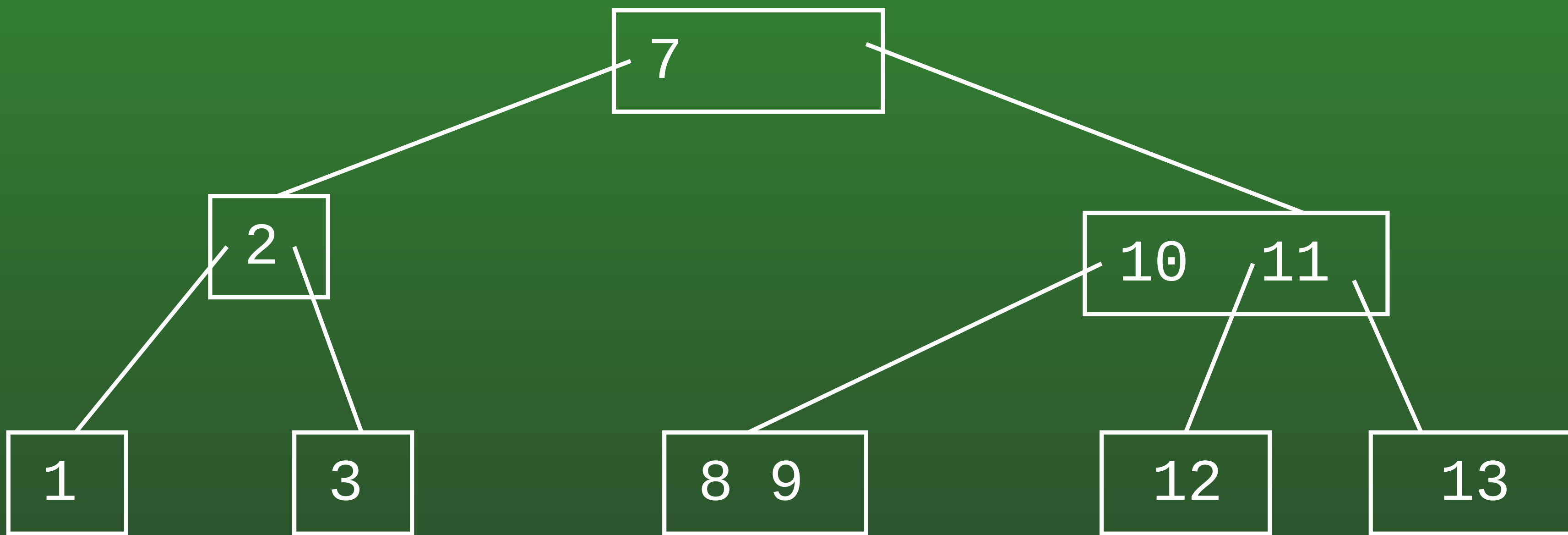
- Node with too few keys
  - Can’t steal key from sibling
  - Merge with sibling

Deleting Interior Keys



- Node with too few keys
  - Can't steal key from sibling
  - Merge with sibling
  - (arbitrarily pick right sibling to merge with)

FR-164: Deleting Interior Keys



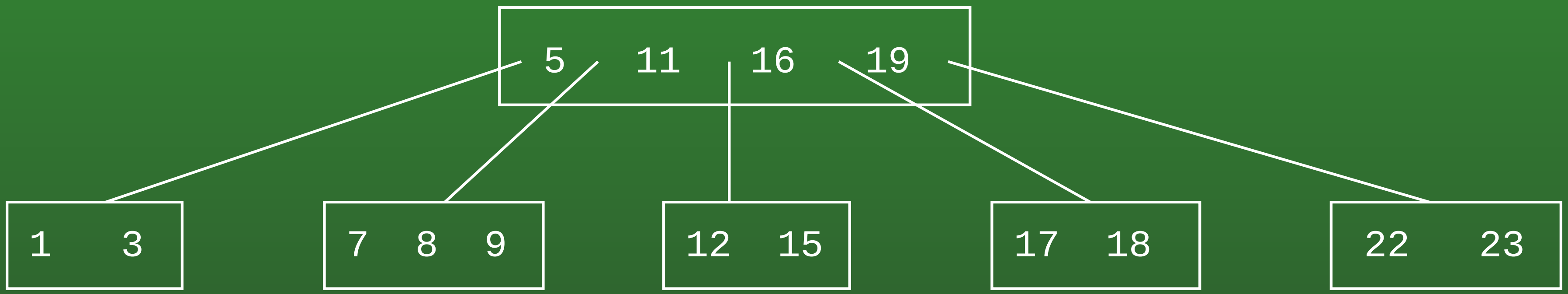
Generalizing 2-3 Trees

- In 2-3 Trees:
  - Each node has 1 or 2 keys
  - Each interior node has 2 or 3 children
- We can generalize 2-3 trees to allow more keys / node

FR-166: B-Trees

- A B-Tree of maximum degree k:
  - All interior nodes have $\lceil k/2 \rceil \ldots k$ children
  - All nodes have $\lceil k/2 \rceil - 1 \ldots k - 1$ keys
- 2-3 Tree is a B-Tree of maximum degree 3

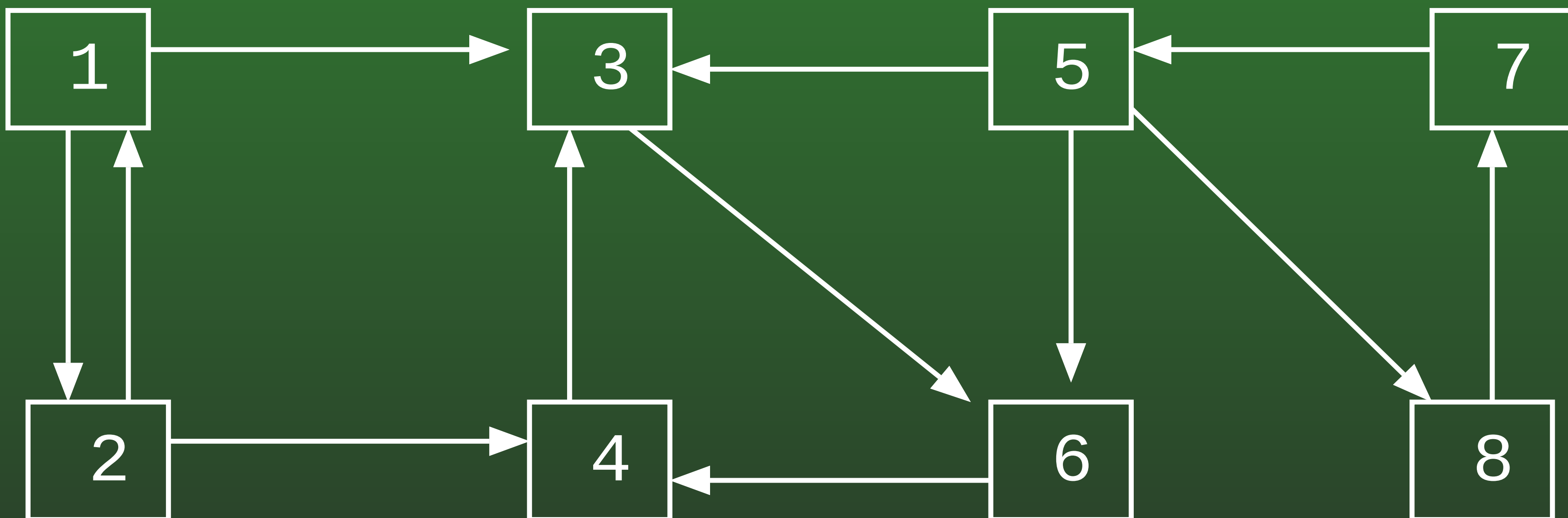
FR-167: B-Trees



- B-Tree with maximum degree 5
  - Interior nodes have 3 – 5 children
  - All nodes have 2-4 keys

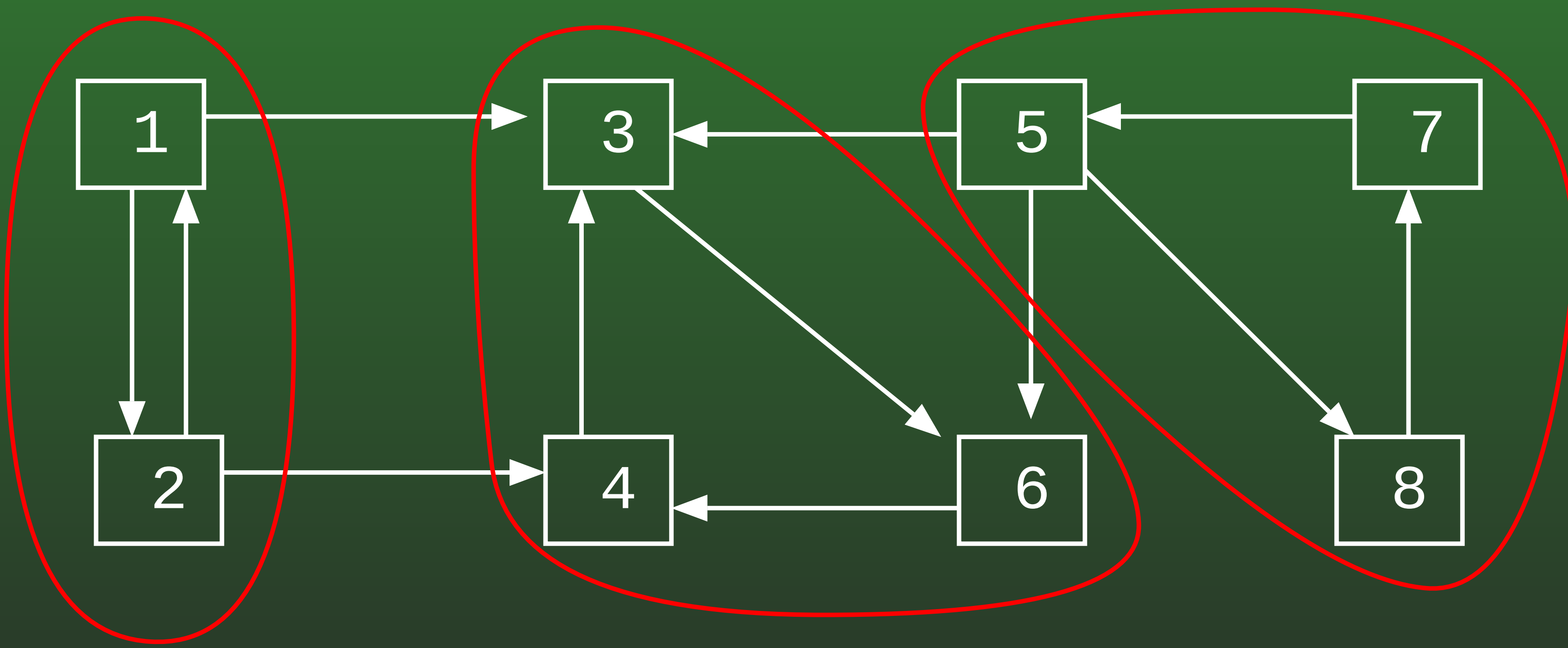
Connected Components

- Subgraph (subset of the vertices) that is strongly connected.



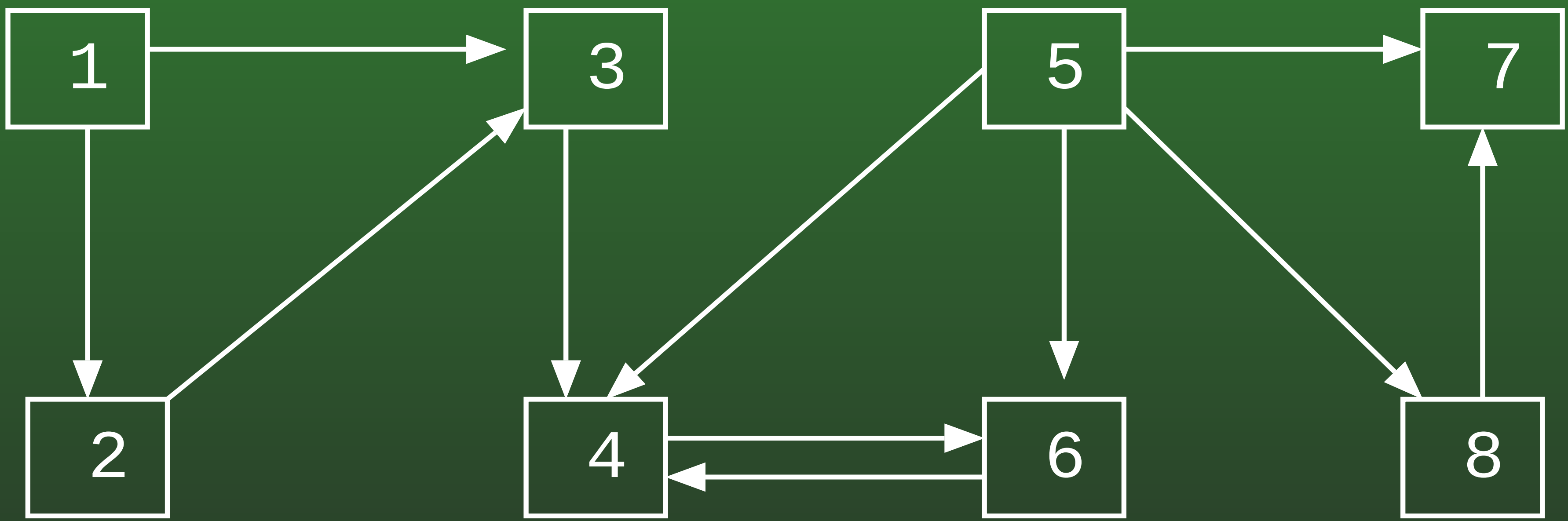
FR-169: Connected Components

- Subgraph (subset of the vertices) that is strongly connected.



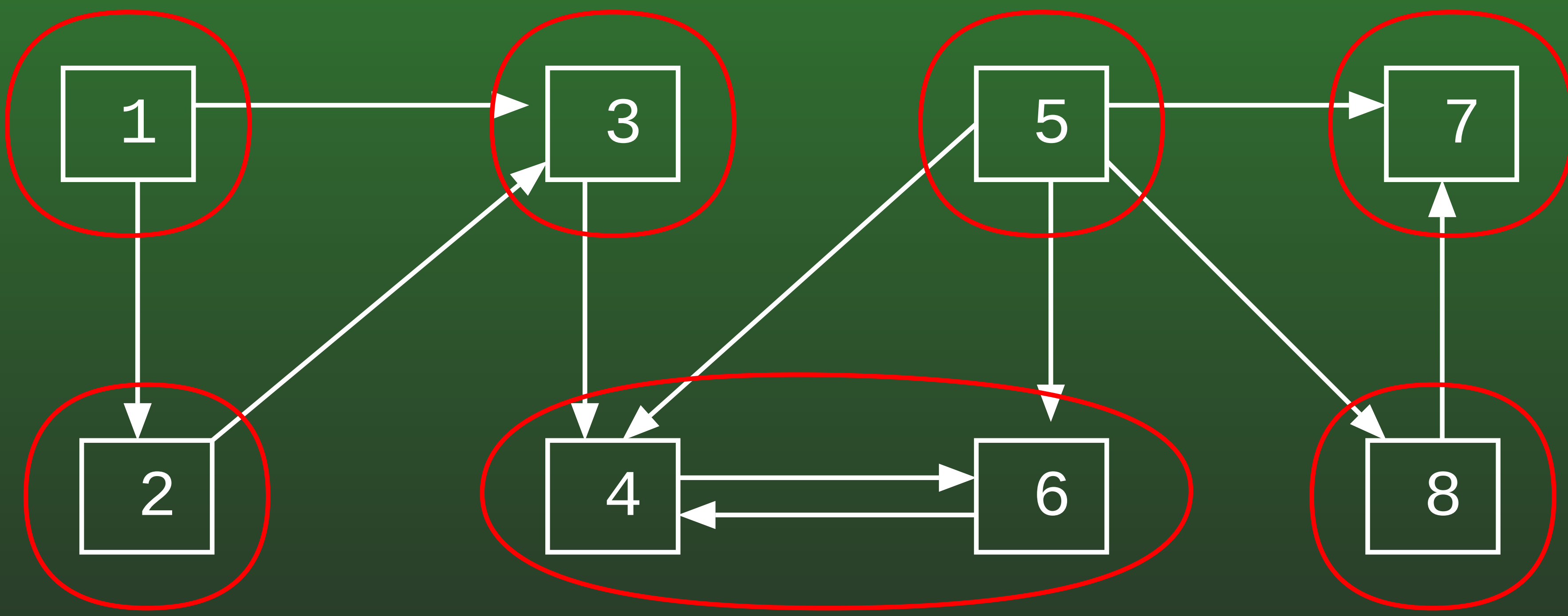
FR-170: Connected Components

- Subgraph (subset of the vertices) that is strongly connected.



Connected Components

- Subgraph (subset of the vertices) that is strongly connected.



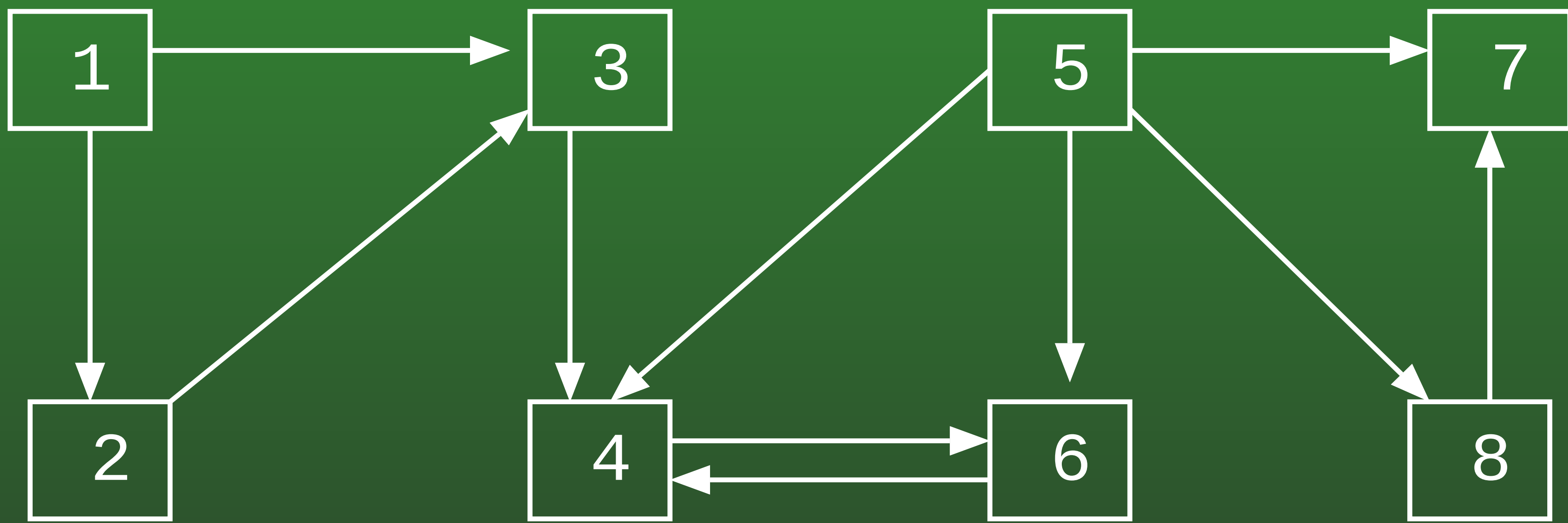
FR-172: DFS Revisited

- We can keep track of the order in which we visit the elements in a Depth-First Search
- For any vertex v in a DFS:
  - d[v] = *Discovery* time – when the vertex is first visited
  - f[v] = *Finishing* time – when we have finished with a vertex (and all of its children

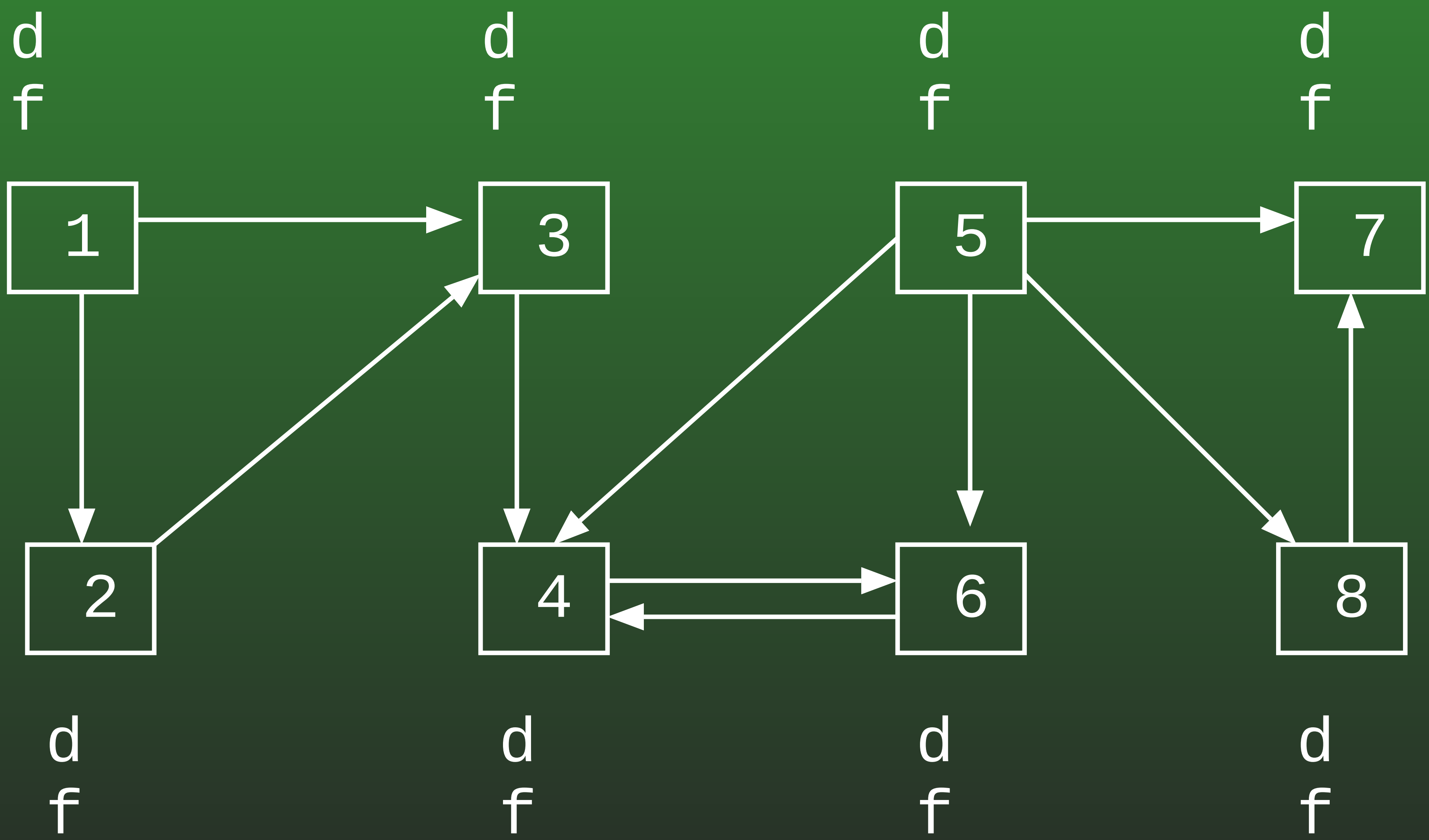
FR-173: DFS Revisited

```
class Edge {
   public int neighbor;
   public int next;
}

void DFS(Edge G[], int vertex, boolean Visited[], int d[], int f[]) {
  Edge tmp;
  Visited[vertex] = true;
  d[vertex] = time++;
  for (tmp = G[vertex]; tmp != null; tmp = tmp.next) {
    if (!Visited[tmp.neighbor])
      DFS(G, tmp.neighbor, Visited);
  }
  f[vertex] = time++;
}
```

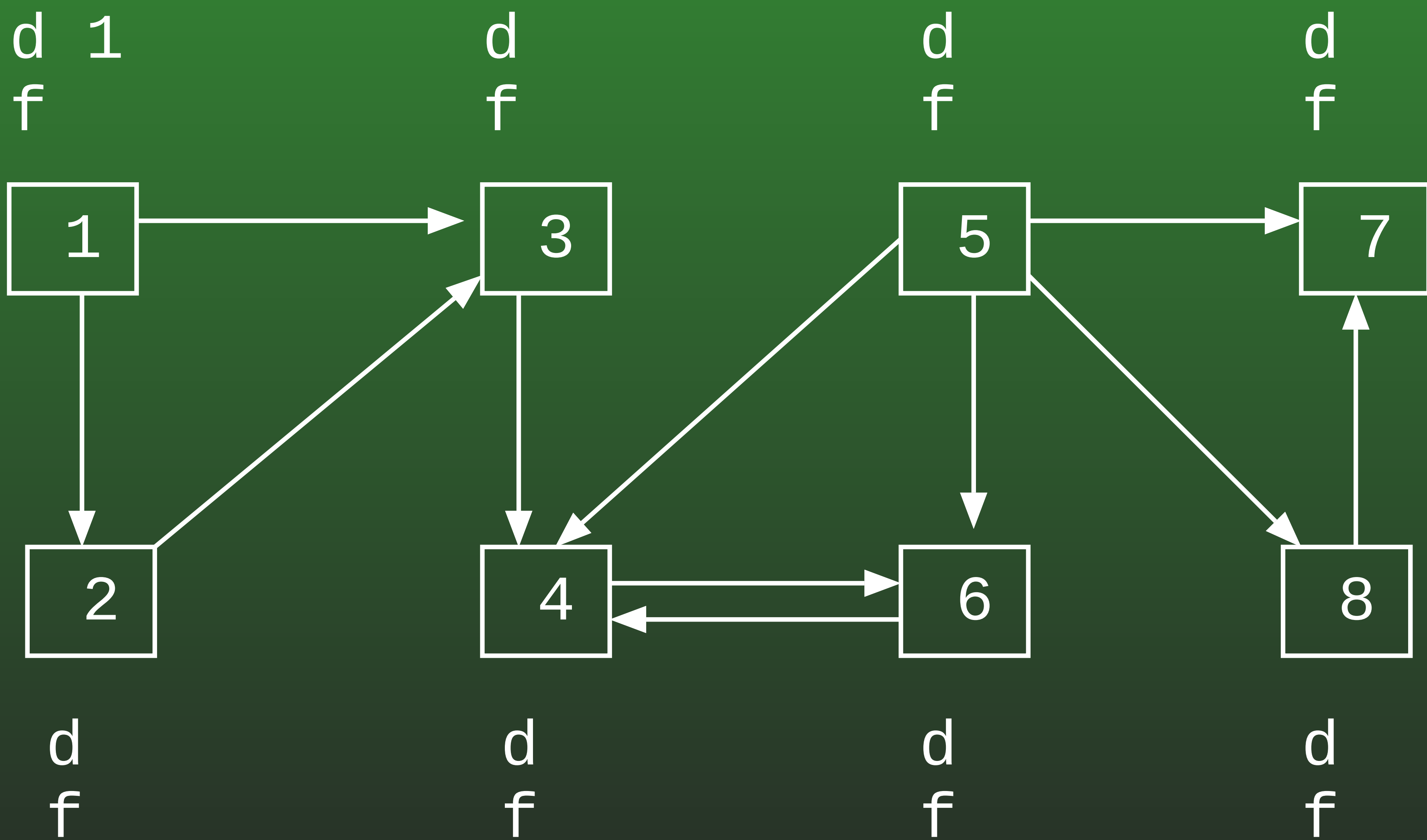
FR-174: DFS Example



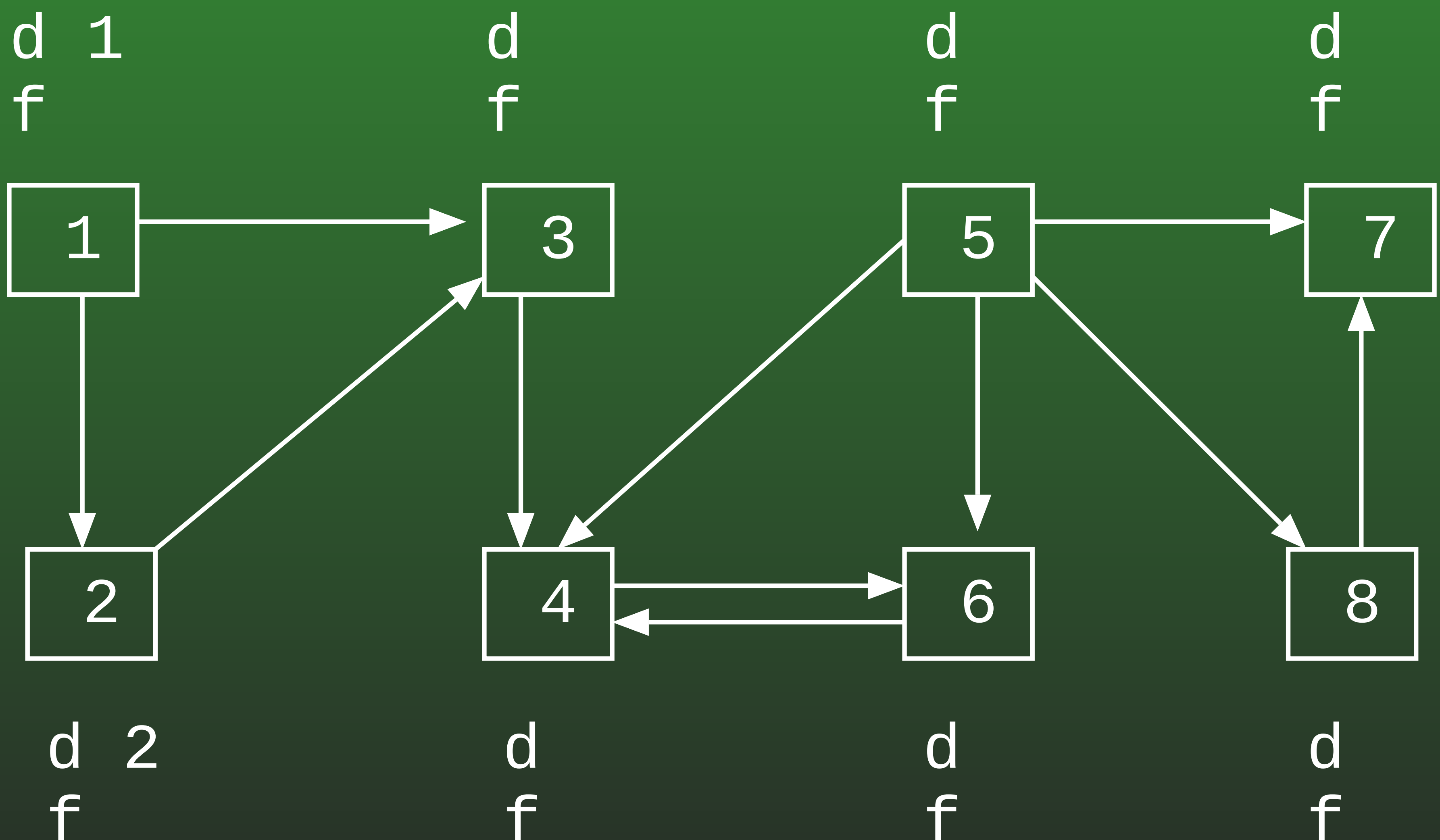
FR-175: DFS Example



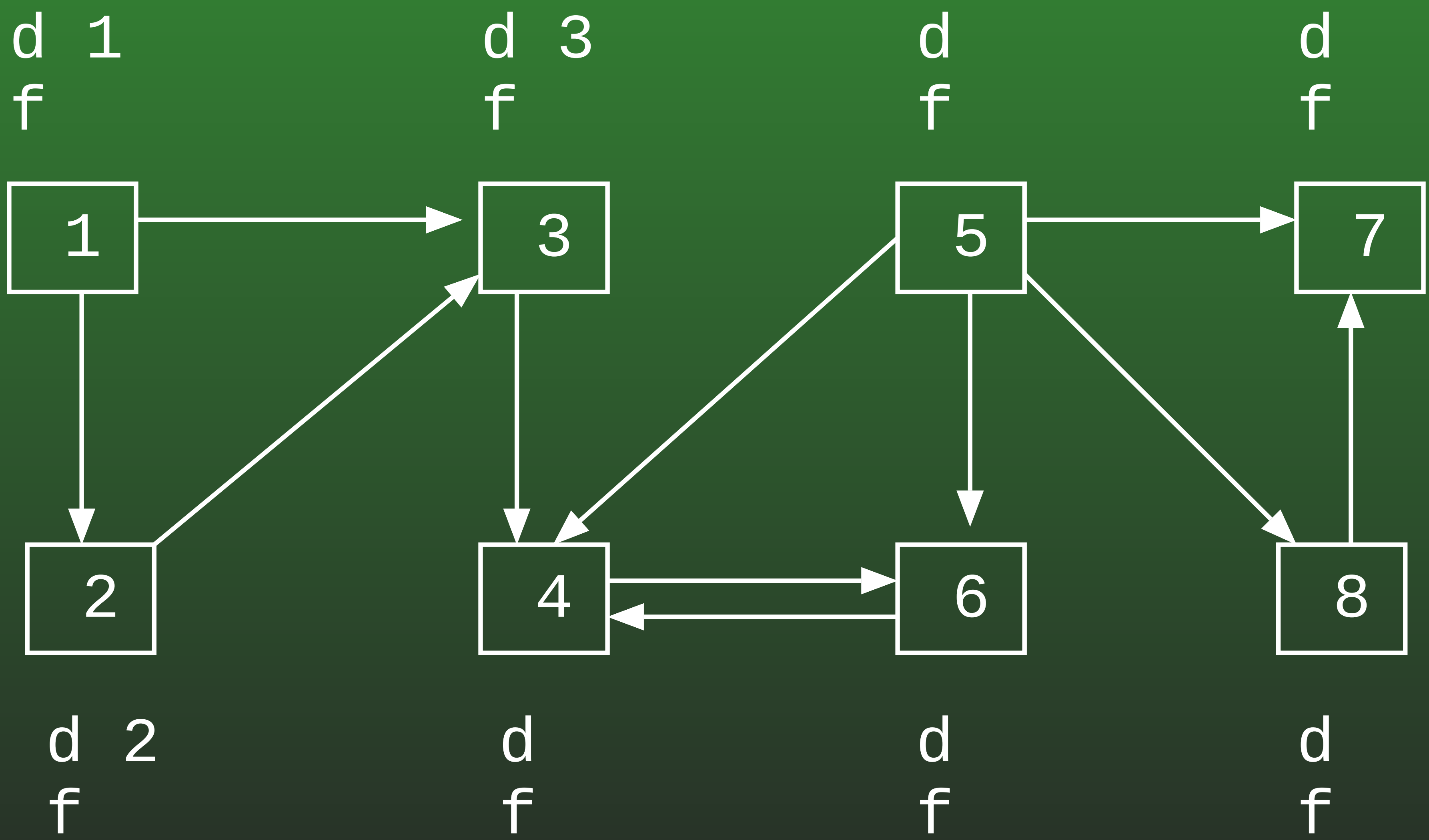
FR-176: DFS Example



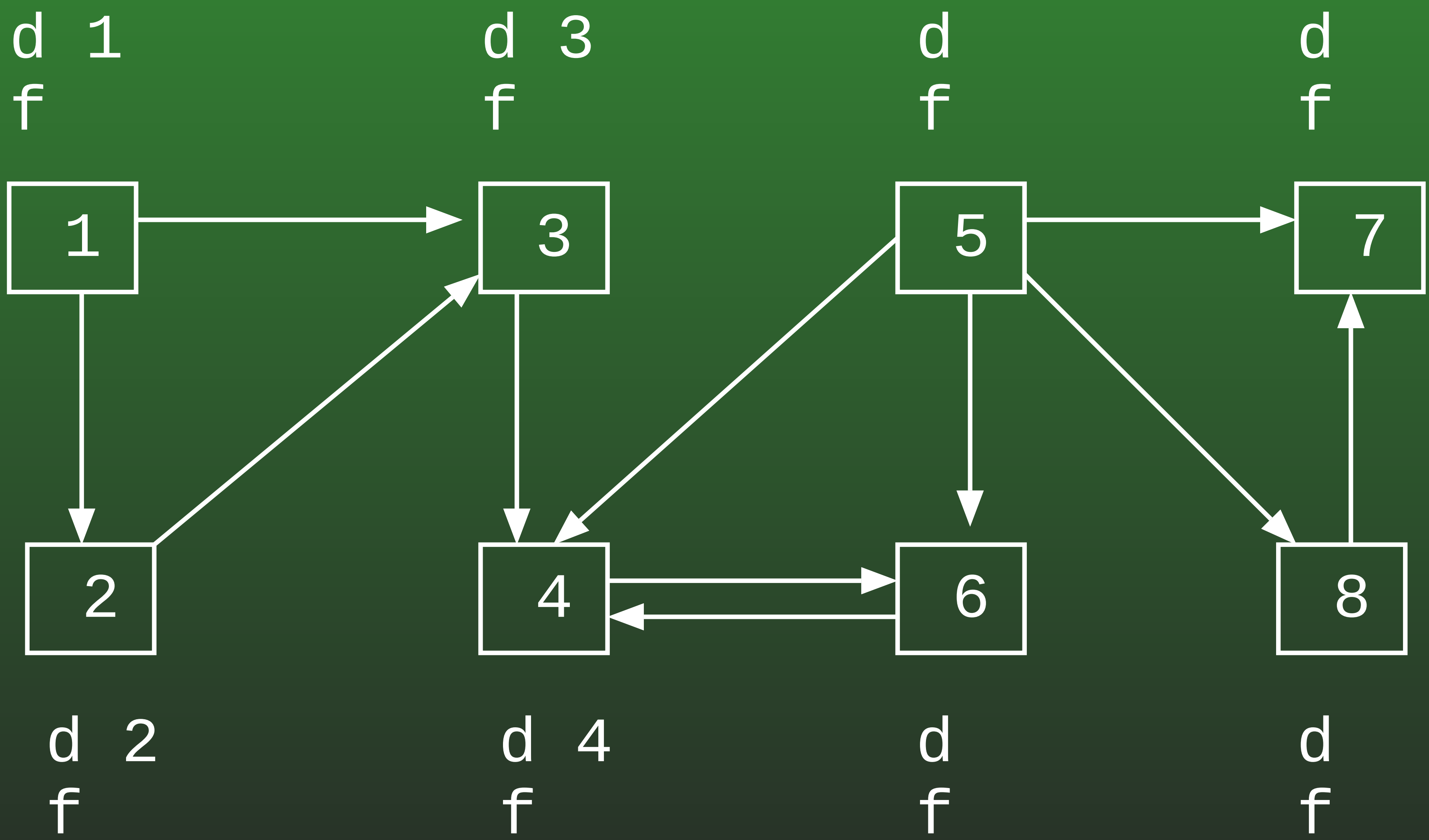
DFS Example



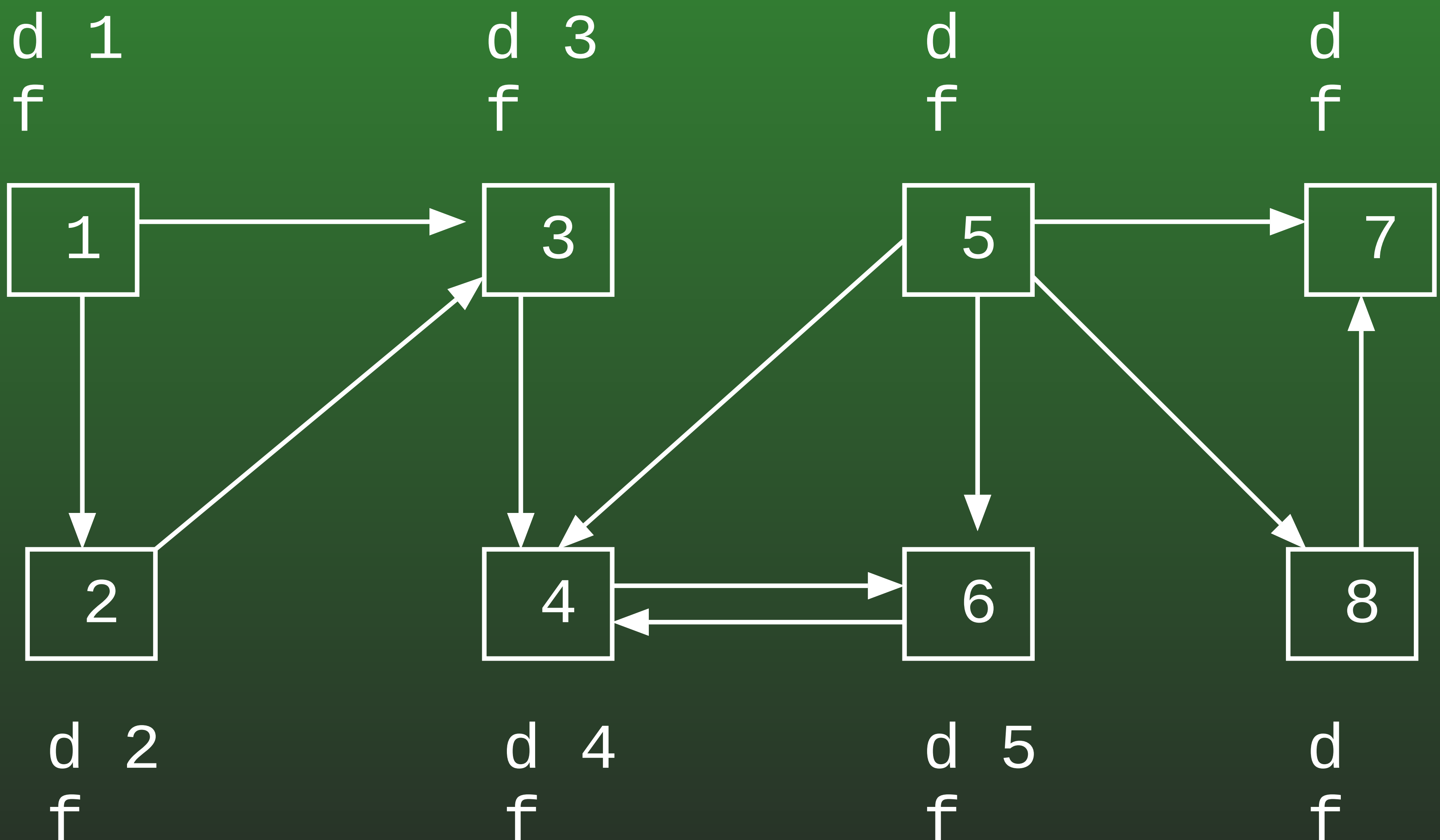
 # DFS Example



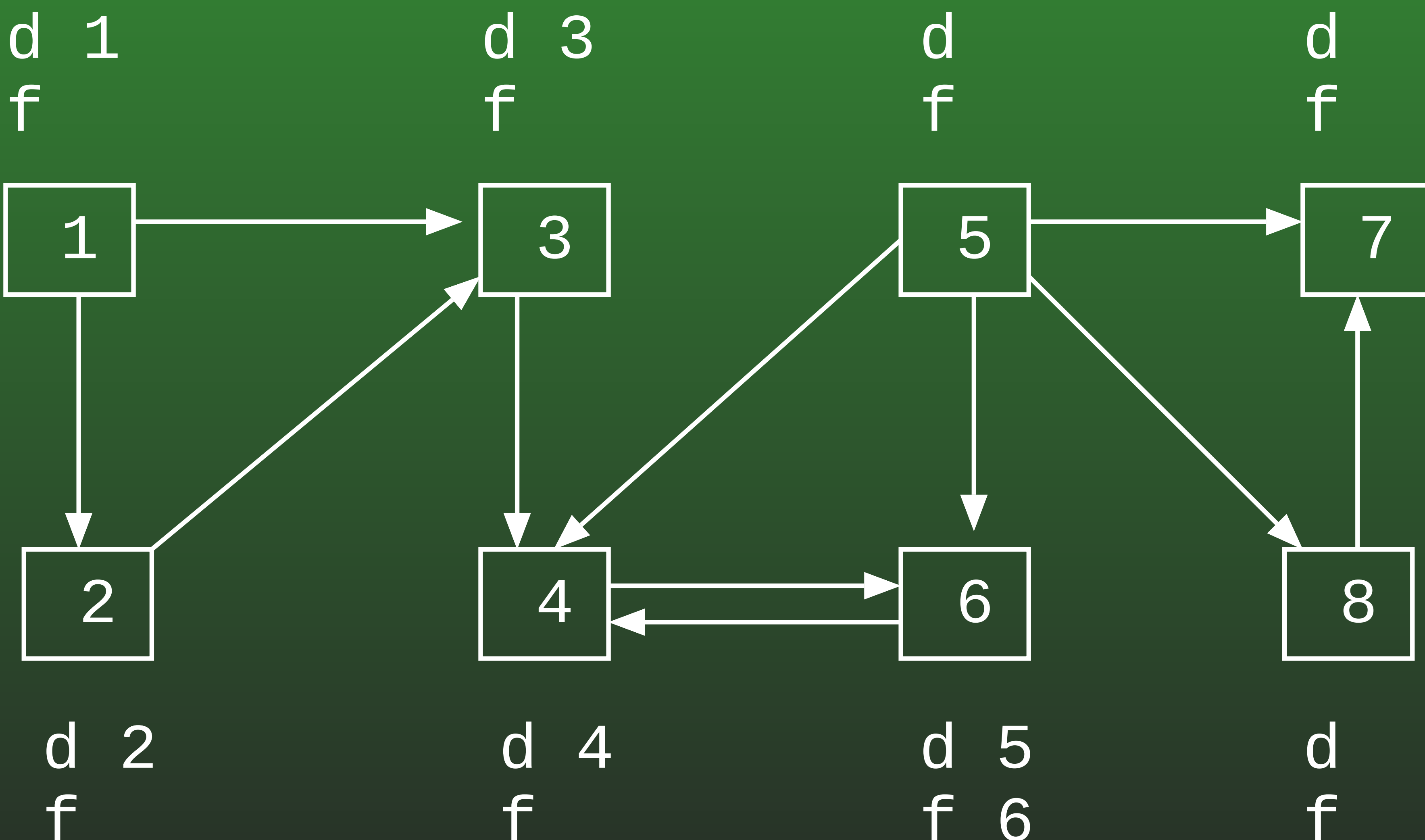
DFS Example



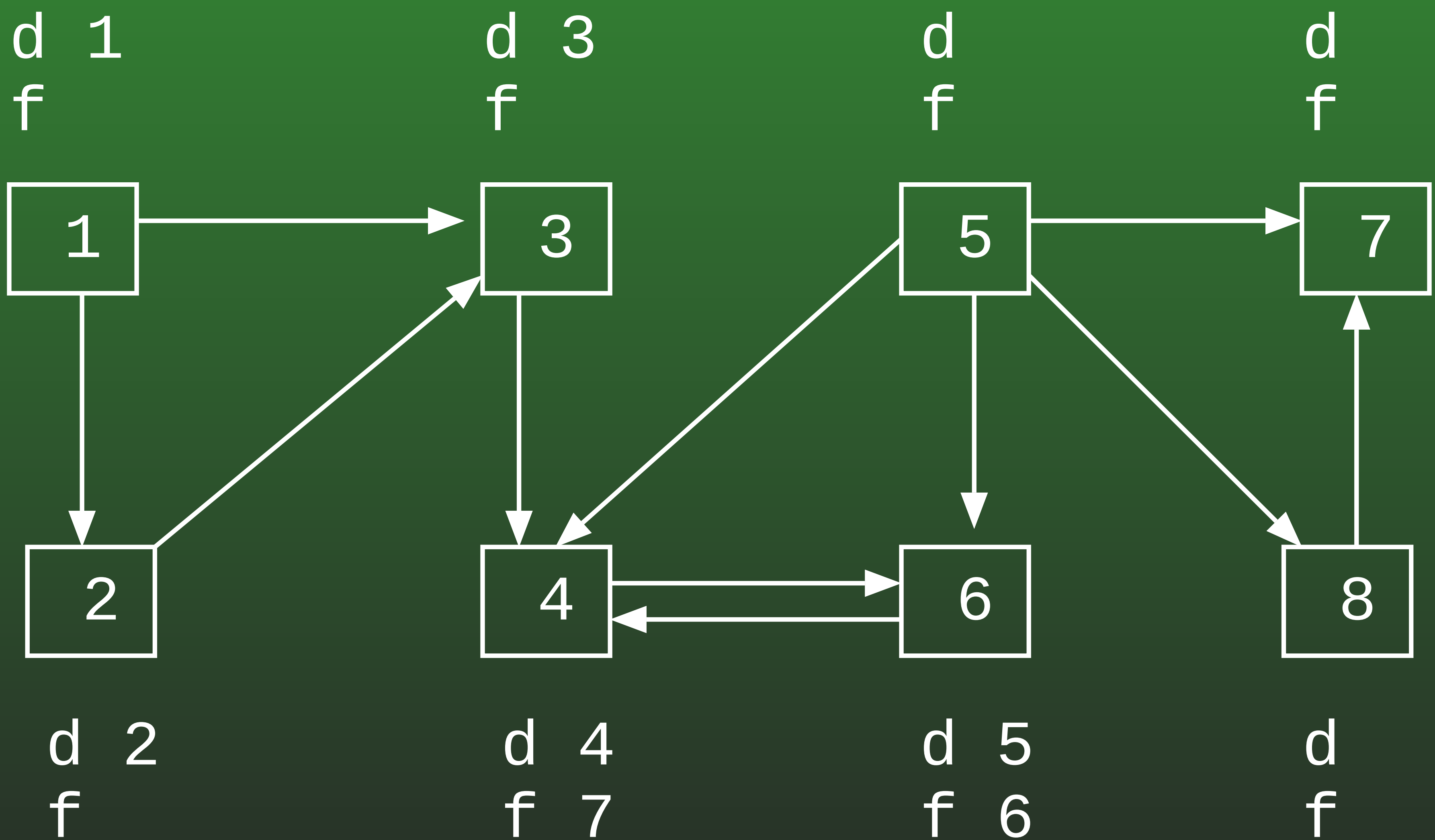
DFS Example



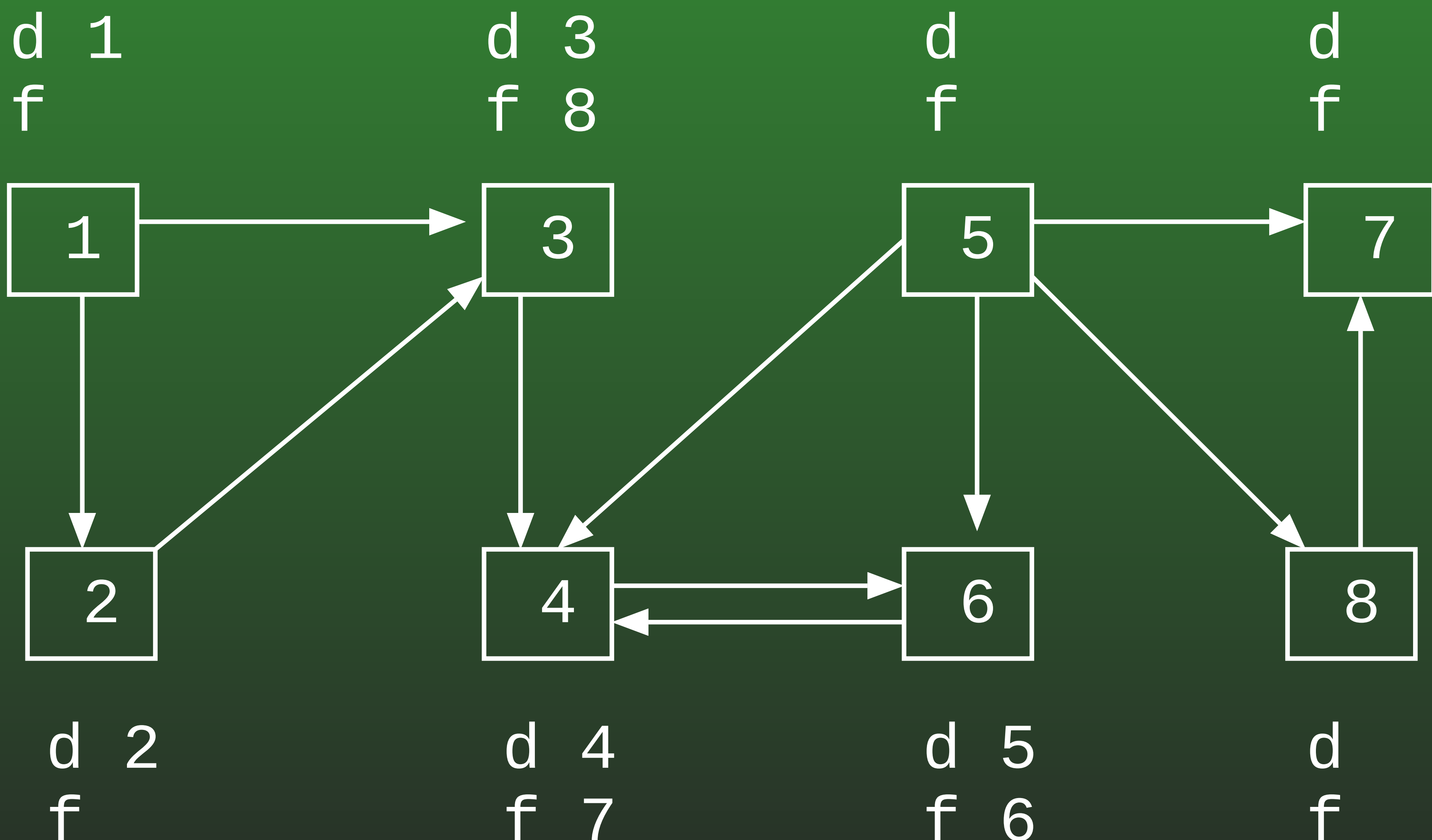
FR-181: DFS Example



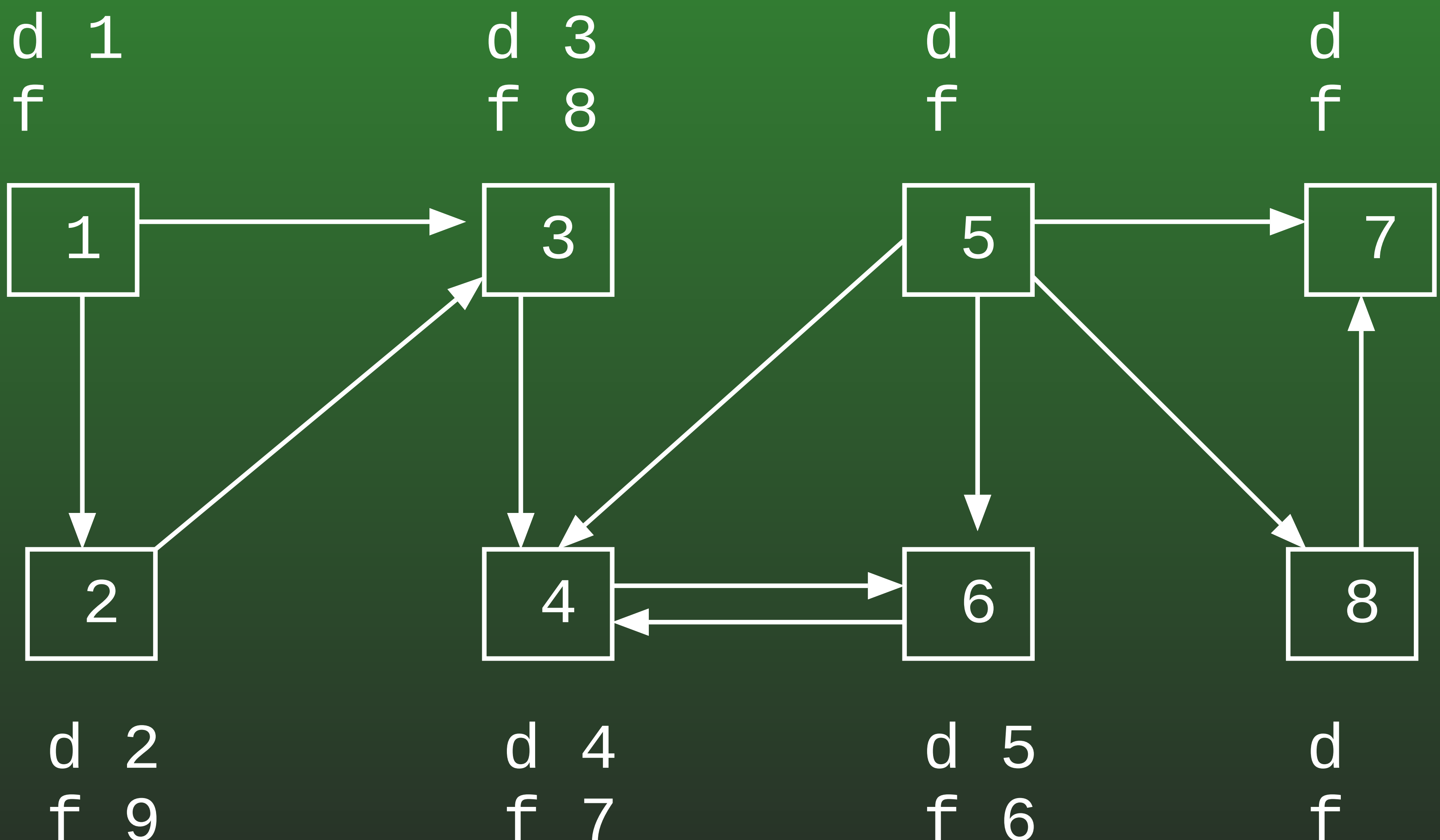
FR-182: DFS Example



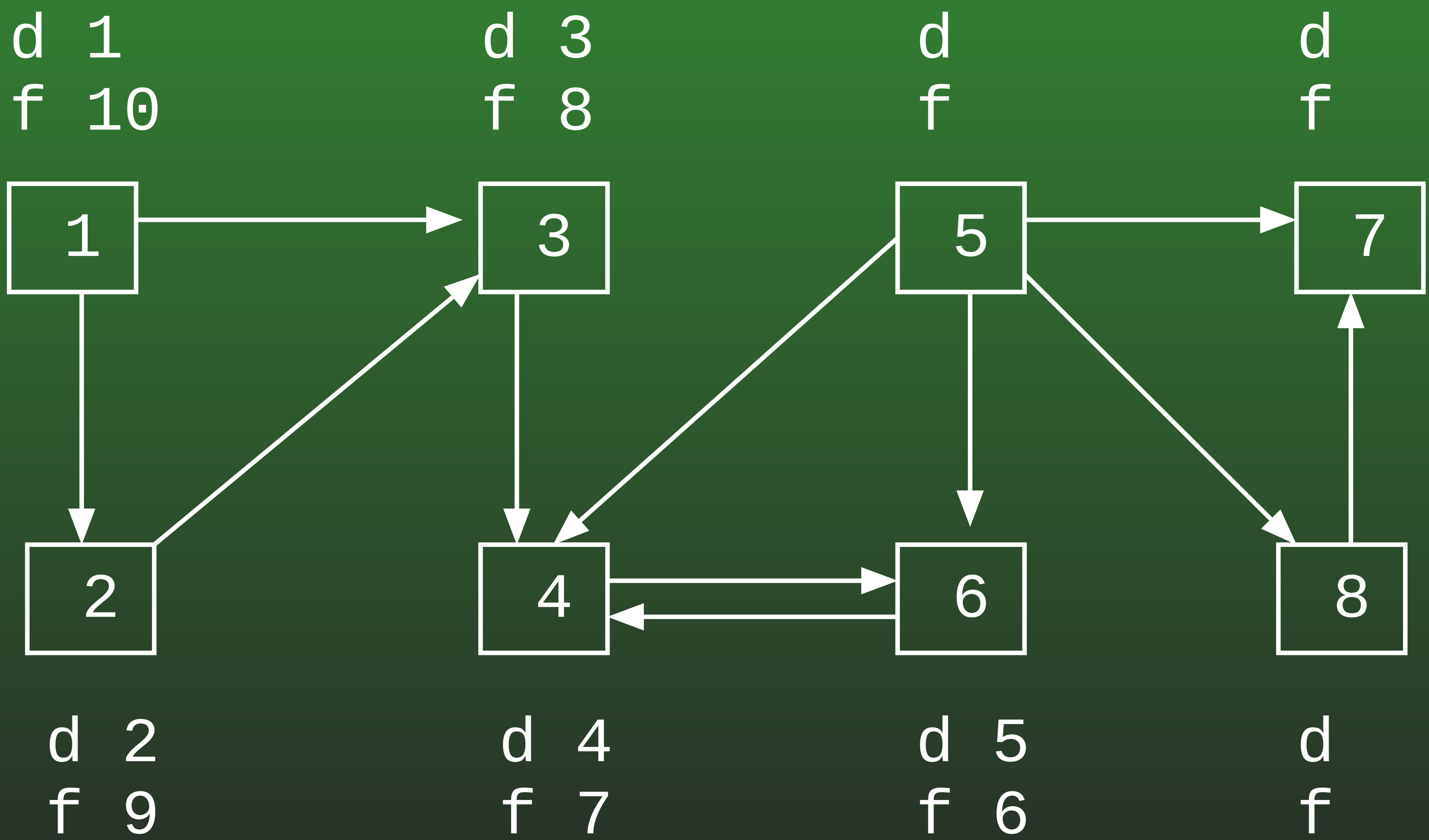
FR-183: DFS Example



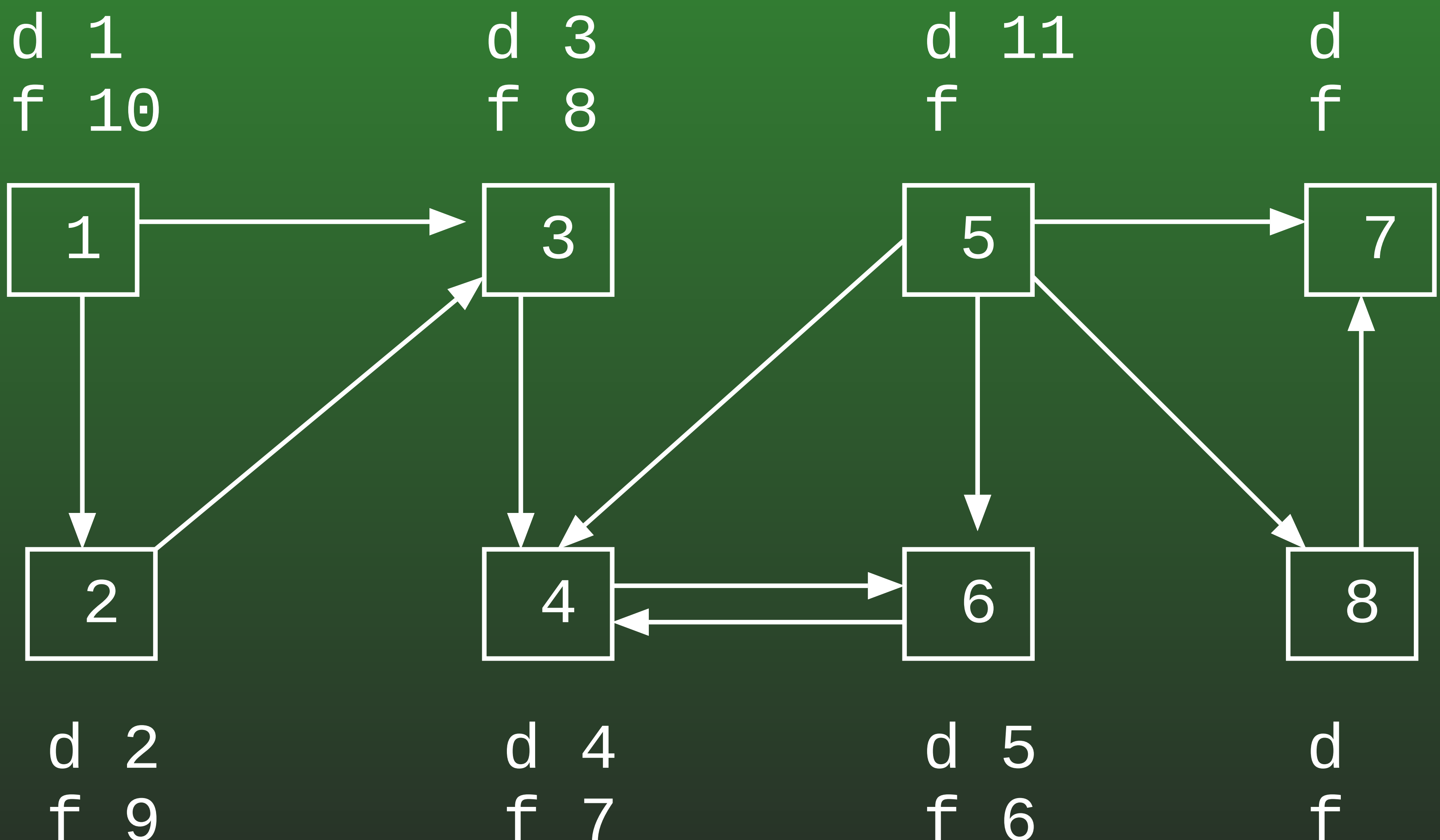
DFS Example



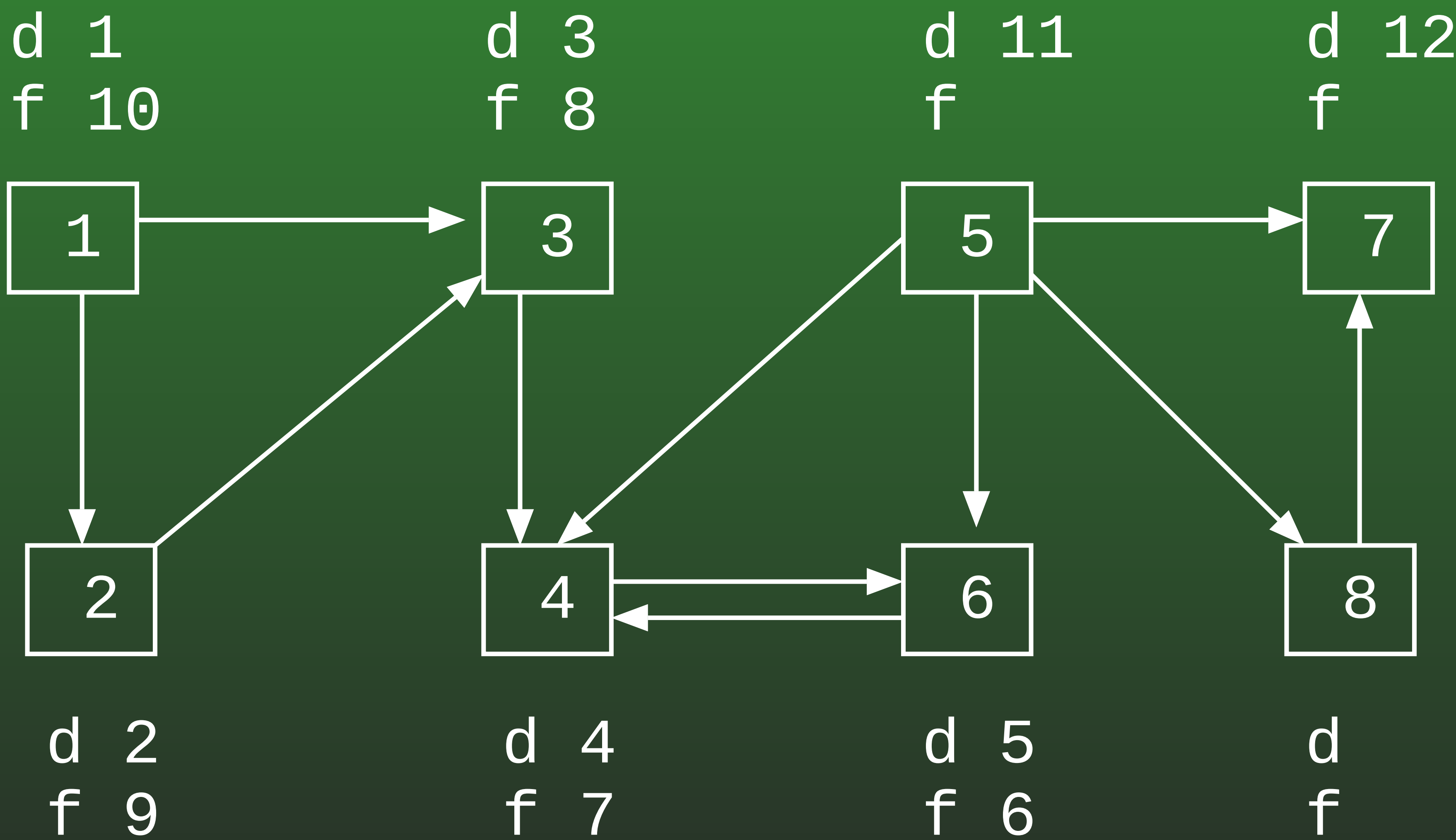
DFS Example



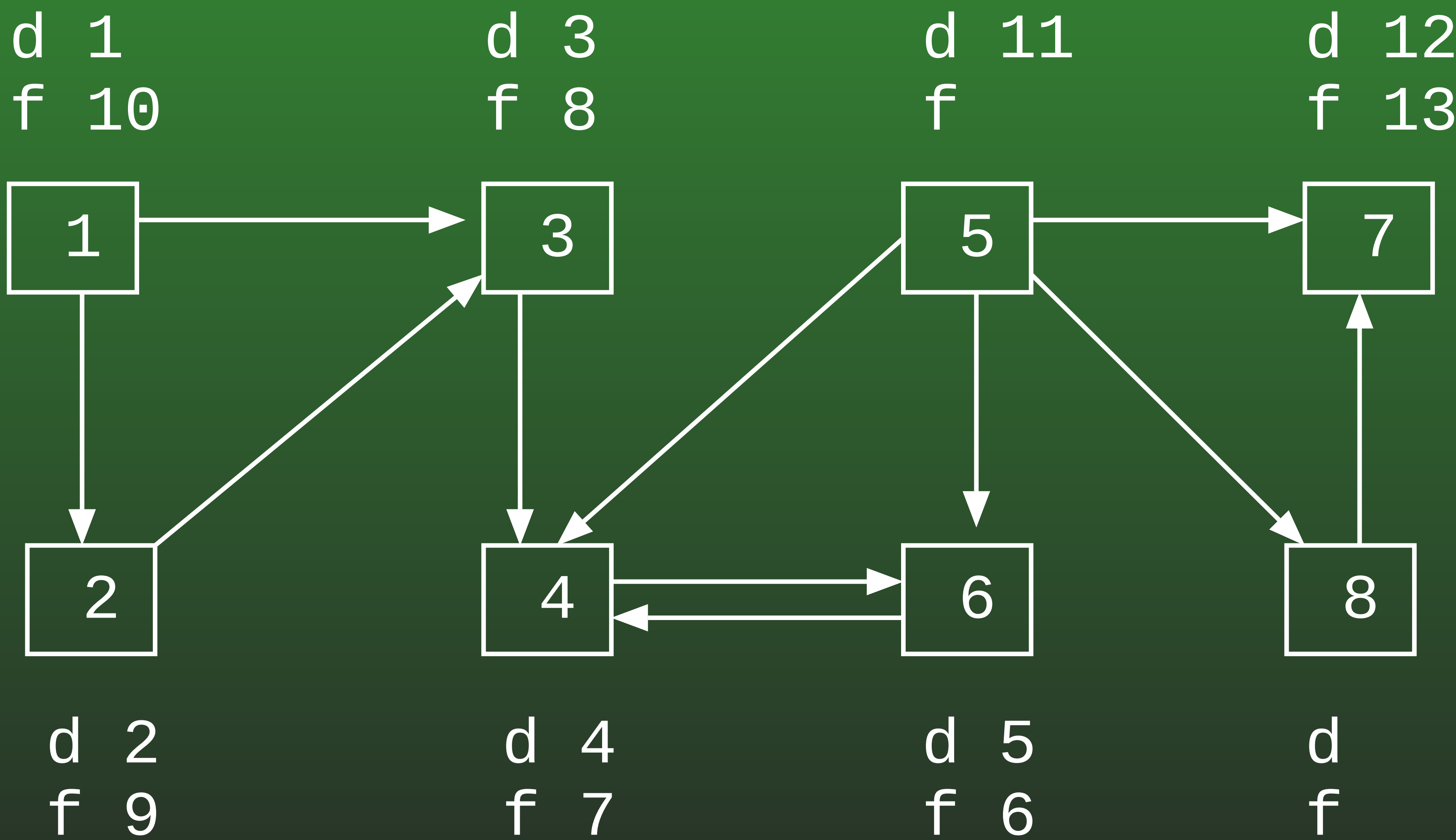
FR-186: DFS Example



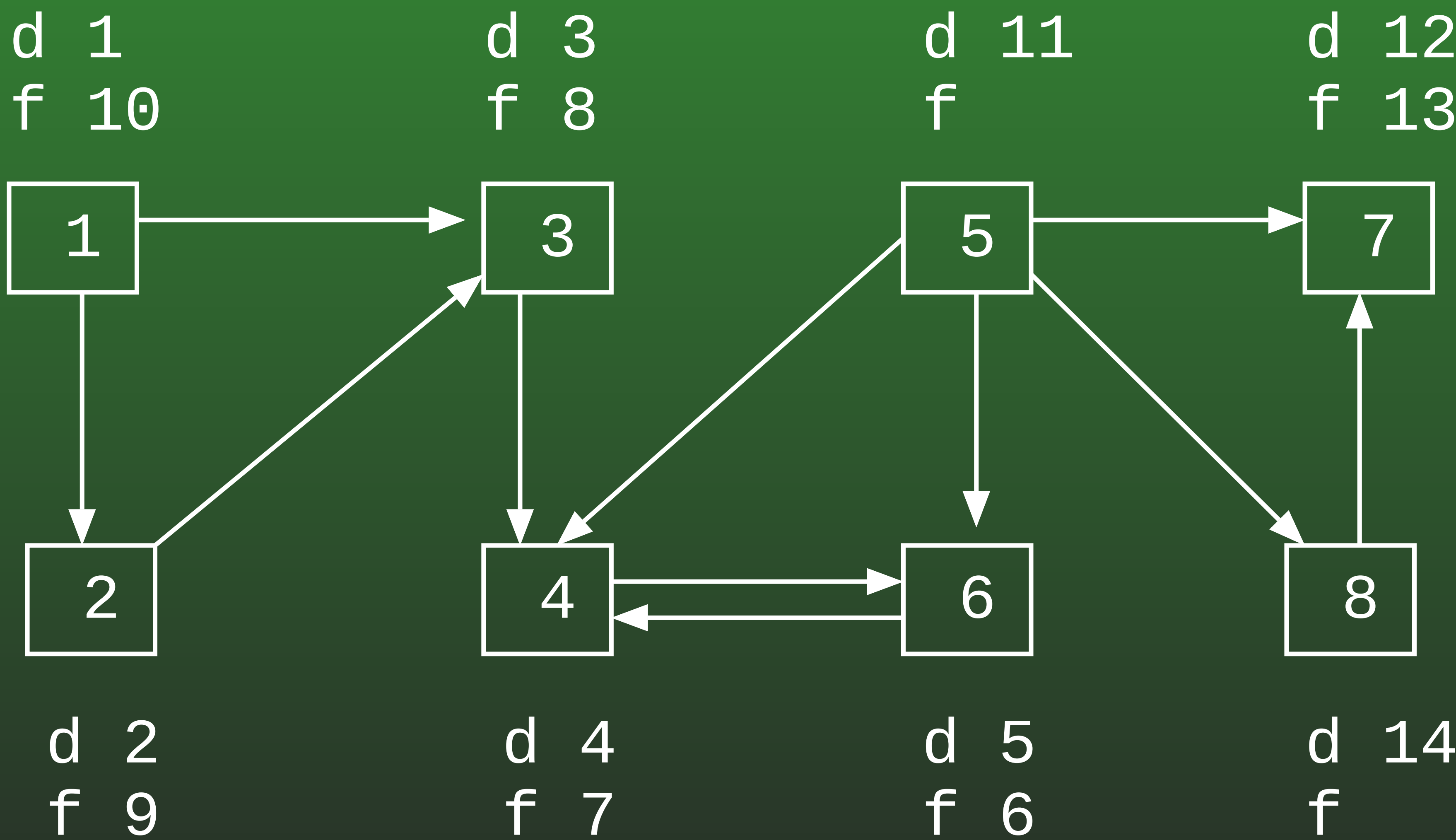
DFS Example



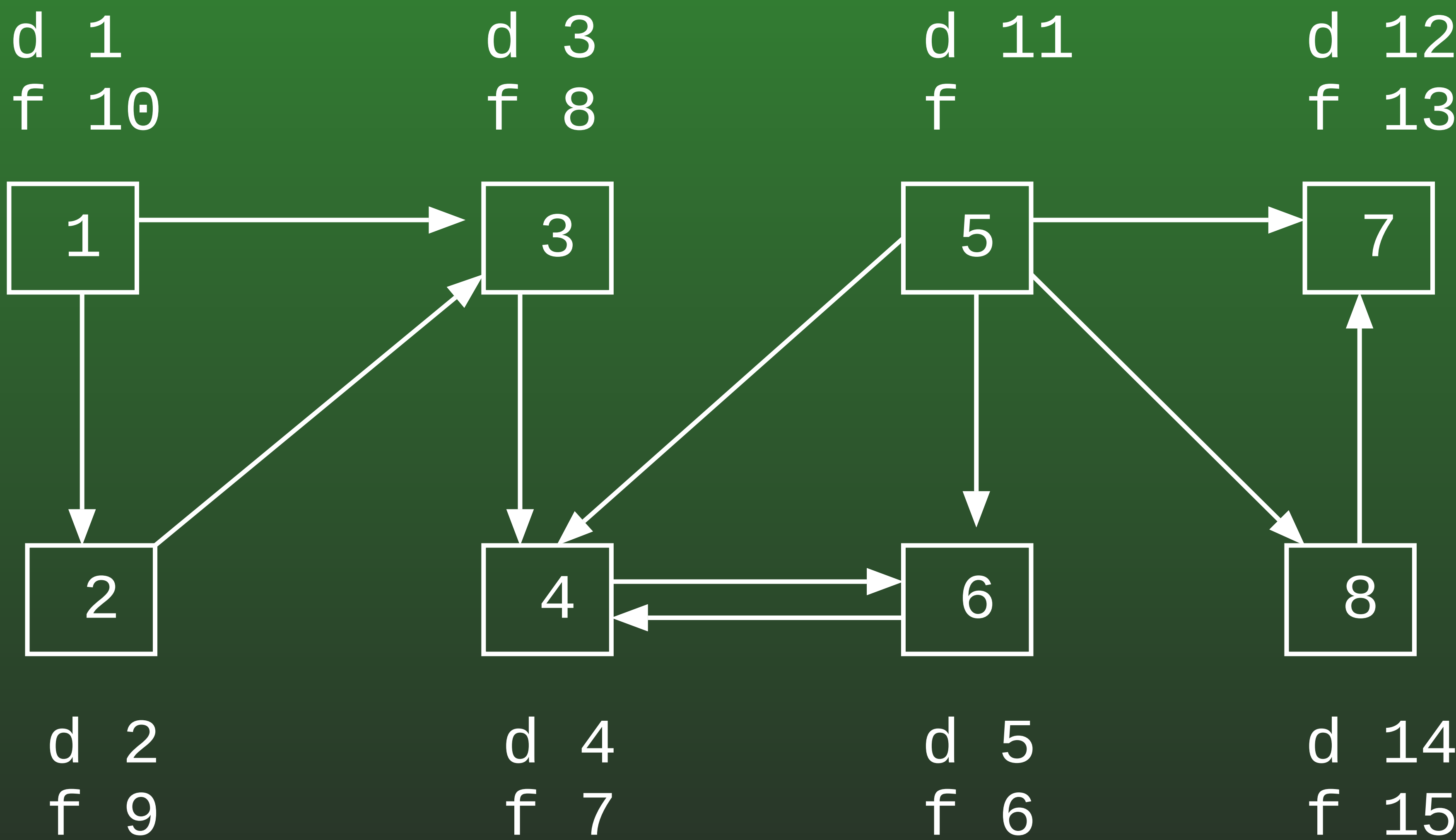
DFS Example



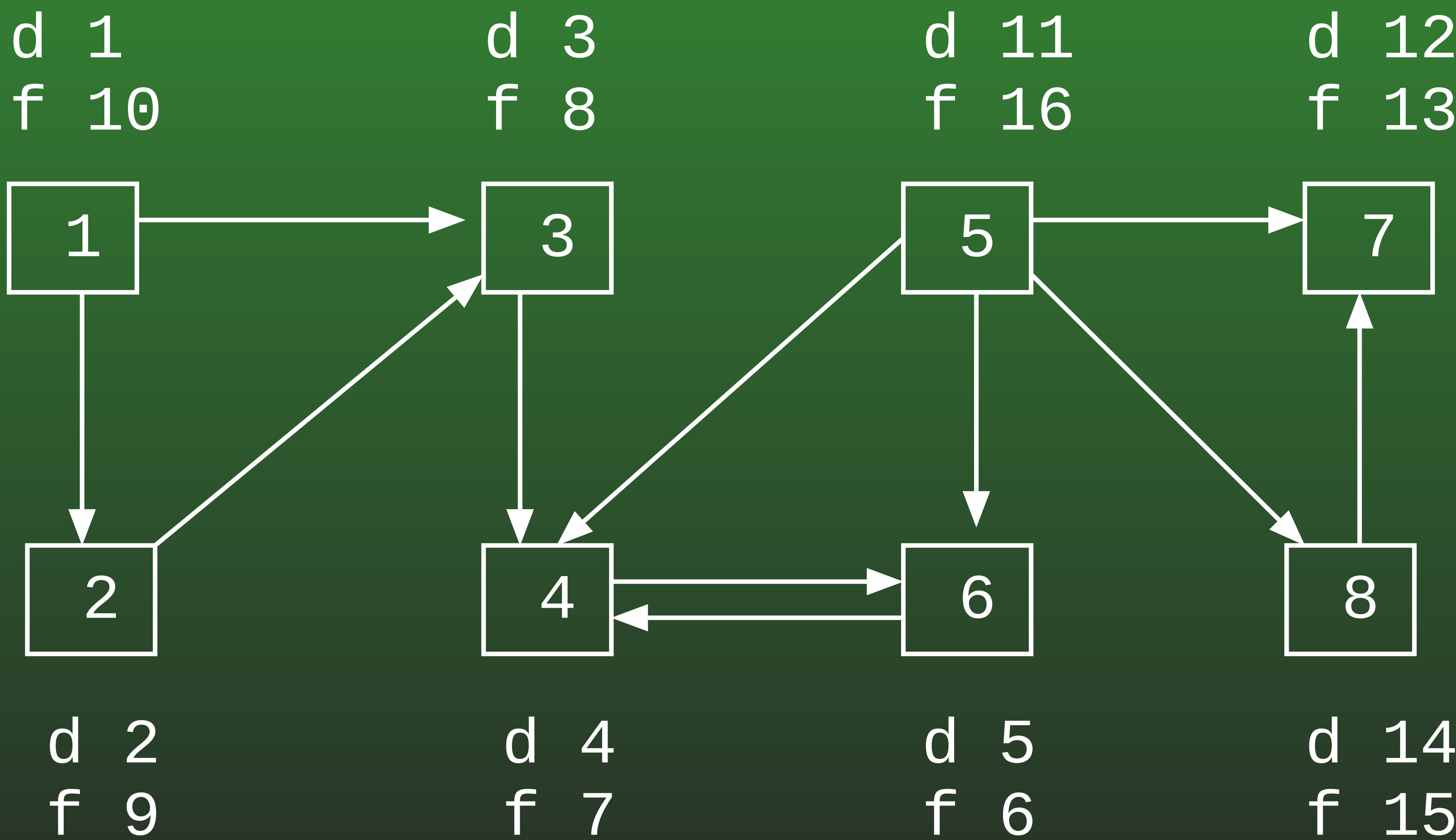
DFS Example



DFS Example



DFS Example



FR-192: Using d[] & f[]

- Given two vertices v_1 and v_2, what do we know if $f[v_2] < f[v_1]$?
  - Either:
    - Path from v_1 to v_2
      - Start from v_1
      - Eventually visit v_2
      - Finish v_2
      - Finish v_1

Using d[] & f[]

- Given two vertices v_1 and v_2, what do we know if $f[v_2] < f[v_1]$?
  - Either:
    - Path from v_1 to v_2
    - No path from v_2 to v_1
      - Start from v_2
      - Eventually finish v_2
      - Start from v_1
      - Eventually finish v_1

FR-194: Using d[] & f[]

- If $f[v_2] < f[v_1]$:
  - Either a path from v_1 to v_2, or no path from v_2 to v_1
  - If there is a path from v_2 to v_1, then there must be a path from v_1 to v_2
- $f[v_2] < f[v_1]$ and a path from v_2 to v_1 $\Rightarrow$ v_1 and v_2 are in the same connected component

Connected Components

- Run DFS on G, calculating f[] times
- Compute G^T
- Run DFS on G^T – examining nodes in *inverse order of finishing times* from first DFS
- Any nodes that are in the same DFS search tree in G^T must be in the same connected component

FR-196: Dynamic Programming

- Simple, recursive solution to a problem
- Naive solution recalculates same value many times
- Leads to exponential running time

FR-197: Dynamic Programming

- Recalculating values can lead to unacceptable run times
  - Even if the total number of values that needs to be calculated is small
- Solution: Don’t recalculate values
  - Calculate each value once
  - Store results in a table
  - Use the table to calculate larger results

FR-198: Faster Fibonacci

```
int Fibonacci(int n) {

 int[] FIB = new int[n+1];

  FIB[0] = 1;
  FIB[1] = 1;

  for (i=2; i<=n; i++)
    FIB[i] = FIB[i-1] + FIB[i-2];

  return FIB[n];
}
```

FR-199: Dynamic Programming

- To create a dynamic programming solution to a problem:
  - Create a simple recursive solution (that may require a large number of repeat calculations
  - Design a table to hold partial results
  - Fill the table such that whenever a partial result is needed, it is already in the table

FR-200: Memoization

- Can be difficult to determine order to fill the table
- We can use a table together with recursive solution
  - Initialize table with sentinel value
  - In recursive function:
    - Check table – if entry is there, use it
    - Otherwise, call function recursively

      Set appropriate table value

      return table value

Fibonacci Memoized

```
int Fibonacci(int n) {

  if (n == 0)
      return 1;

  if (n == 1)
      return 1;

  if (T[n] == -1)
    T[n] = Fibonacci(n-1) + Fibonacci(n-2);

  return T[n];
}
```

FR-202: Hard Problems

- Some algorithms take exponential time
  - Simple version of Fibonacci
  - Faster versions of Fibonacci that take linear time
- Some *Problems* take exponential time
  - *All* algorithms that solve the problem take exponential time
  - Towers of Hanoi

FR-203: Reductions

- A reduction from Problem 1 to Problem 2 allows us to solve Problem 1 in terms of Problem 2
  - Given an instance of Problem 1, create an instance of Problem 2
  - Solve the instance of Problem 2
  - Use the solution of Problem 2 to create a solution to Problem 1

FR-204: Reductions

- We can use a Reduction to compare problems
- If there is a reduction from problem A to problem B that can be done quickly
- Problem B is known to be hard (cannot be solved quickly)
- Problem A cannot be solved quickly, either

NP Problems

- A problem is NP if a solution can be verified easily
  - Traveling Salesman Problem (TSP)
    - Given a graph with weighted vertices, and a cost bound k
    - Is there a cycle that contains all vertices in the graph, that has a total cost less than k?
  - Given any potential solution to the TSP, we can easily verify that the solution is correct

FR-206: Non-Deterministic Machine

- Two Definitions of Non-Deterministic Machines:
  - "Oracle" – allows machine to magically make a correct guess
  - Massively parallel – simultaneously try to verify all possible solutions
    - Try all permutations of vertices in a graph, see if any form a cycle with cost < k
    - Try all colorings of a graph with up to k colors, see if any are legal
    - Try all permutations of a list, see if any are sorted

FR-207: NP vs. P

- A problem is NP if a non-deterministic machine can solve it in polynomial time
  - Of course, we have no real non-deterministic machines
- A problem is in P (Polynomial), if a deterministic machine can solve it in polynomial time
  - Sorting is in P – can sort a list in polynomial time
  - All problems in P are also in NP
    - Ignore the oracle

FR-208: NP-Complete

- An NP problem is “NP-Complete” if there is a reduction from *any* NP problem to that problem
- For example, Traveling Salesman (TSP) is NP-Complete
  - We can reduce *any* NP problem to TSP
  - If we could solve TSP in polynomial time, we could solve *all* NP problems in polynomial time
- TSP is not unique – many NP-Complete problems

FR-209: NP =? P

- If we could solve any NP-Complete problem quickly (polynomial time), we could solve all NP problems quickly
- If that is the case, then NP=P
  - P is set of problems that can be solved by a standard machine in polynomial time
- Most everyone believes that NP $\neq$ P, and all NP-Complete problems require exponential time on standard computers – not yet been proven

NP-Completeness

- What can we do, if we need to solve a problem that is NP-Complete?
  - If the problem we need to solve is very small (< 20), an exponential solution might be OK
  - We can solve an *approximation* of the problem
    - Color a graph using an non-optimal number of colors
    - Find a Traveling Salesman tour that is not optimal

FR-211: Impossible Problems

- Some problems are “easy” – require a fairly small amount of time to solve
  - Sorting
- Some problems are “probably hard” – believed to require exponential time to solve
  - TSP, Graph Coloring, etc
- Some problems are “hard” – known to require an exponential amount of time to solve
  - Towers of Hanoi
- Some problems are impossible – *cannot* be solved

FR-212: Halting Problem

- Program is running – seems to be taking a long time
- We'd like to know if the program will eventually finish, or if it is in an infinite loop
- Great debugging tool:
  - Takes as input the source code to a program p, and an input i
  - Determines if p will run forever when run on i
- No such tool can exist!

FR-213: Halting Problem

```
boolean halt(char [] program, char [] input) {

     /* code to determine if the program
        halts when run on the input */

     if (program halts on input)
        return true;
     else
        return false;
}
```

FR-214: Halting Problem

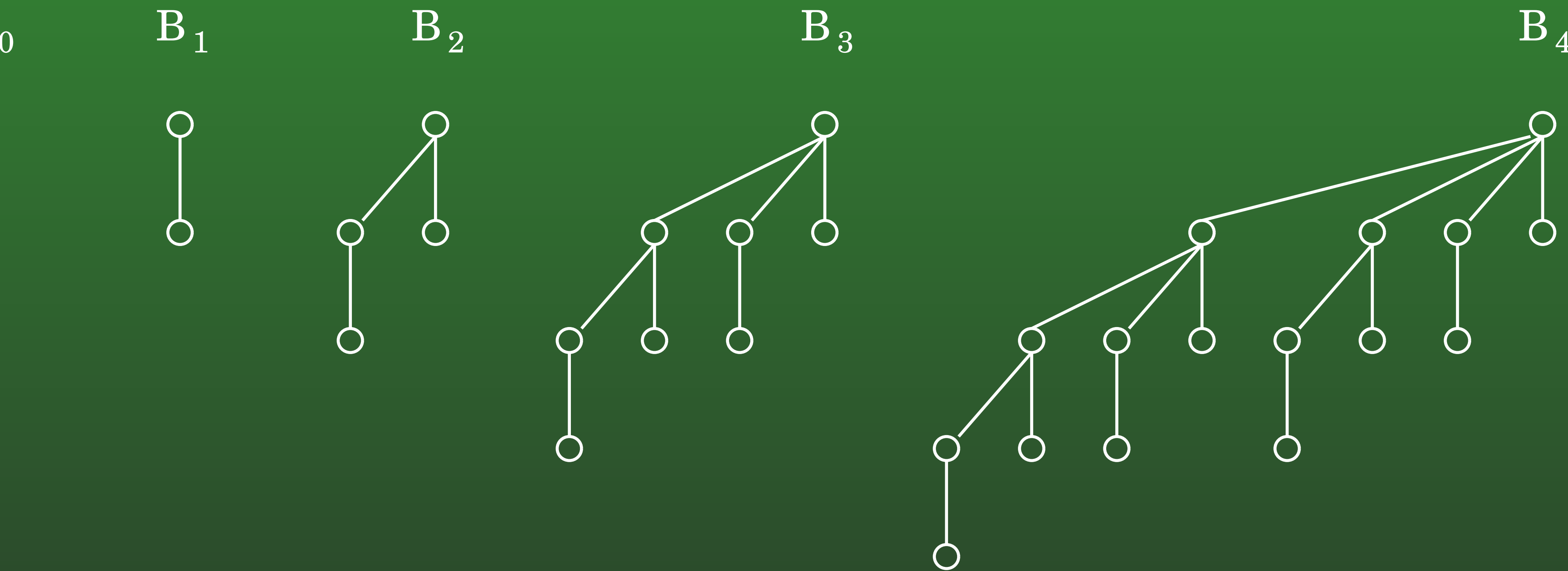
```
boolean selfhalt(char [] program) {
    if (halt(program, program))
        return true;
    else
        return false;
}

void contrary(char [] program) {
    if (selfhalt(program)
      while(true);   /* infinite loop */
}
```

- what happens when we call contrary, passing in its own source code as input?

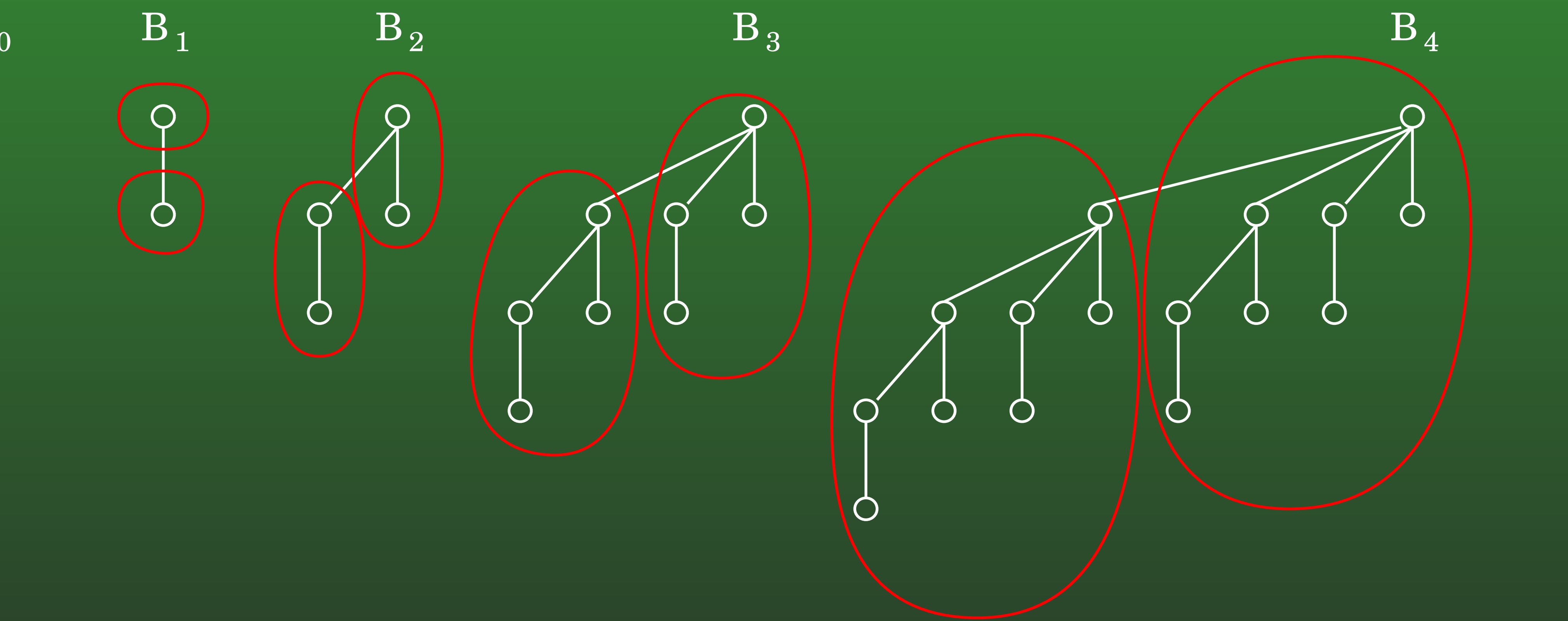
Binomial Trees

- B_0 is a tree containing a single node
- To build B_k:
  - Start with B_{k-1}
  - Add B_{k-1} as left subtree

Binomial Trees



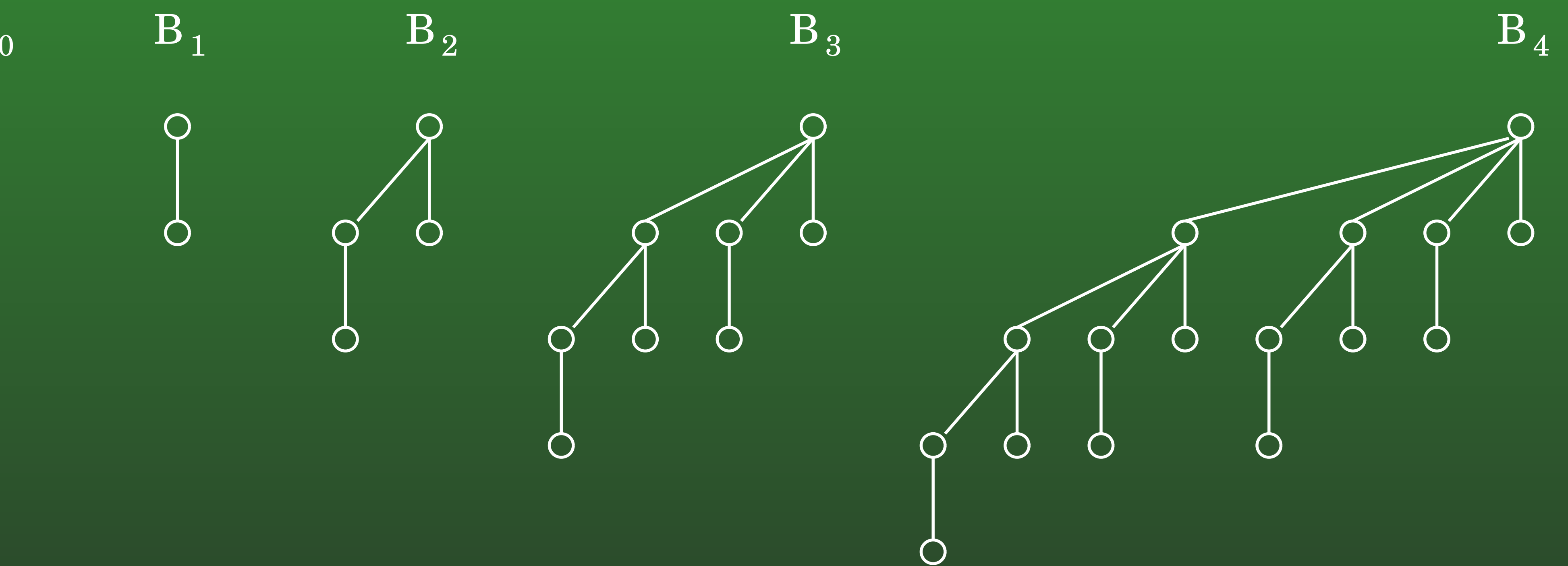
FR-217: Binomial Trees



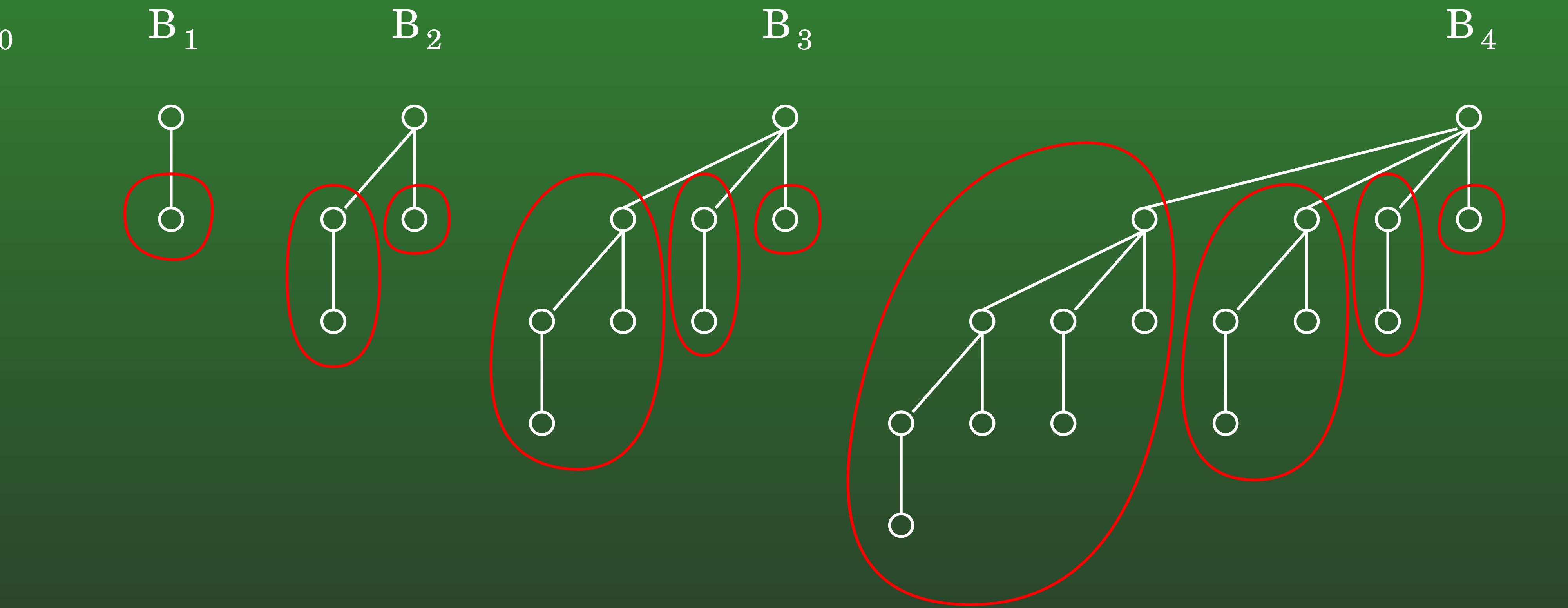
FR-218: Binomial Trees

- Equivalent defintion
  - B_0 is a binomial heap with a single node
  - B_k is a binomial heap with k children:
    - $B_0 \ldots B_{k-1}$

FR-219: Binomial Trees



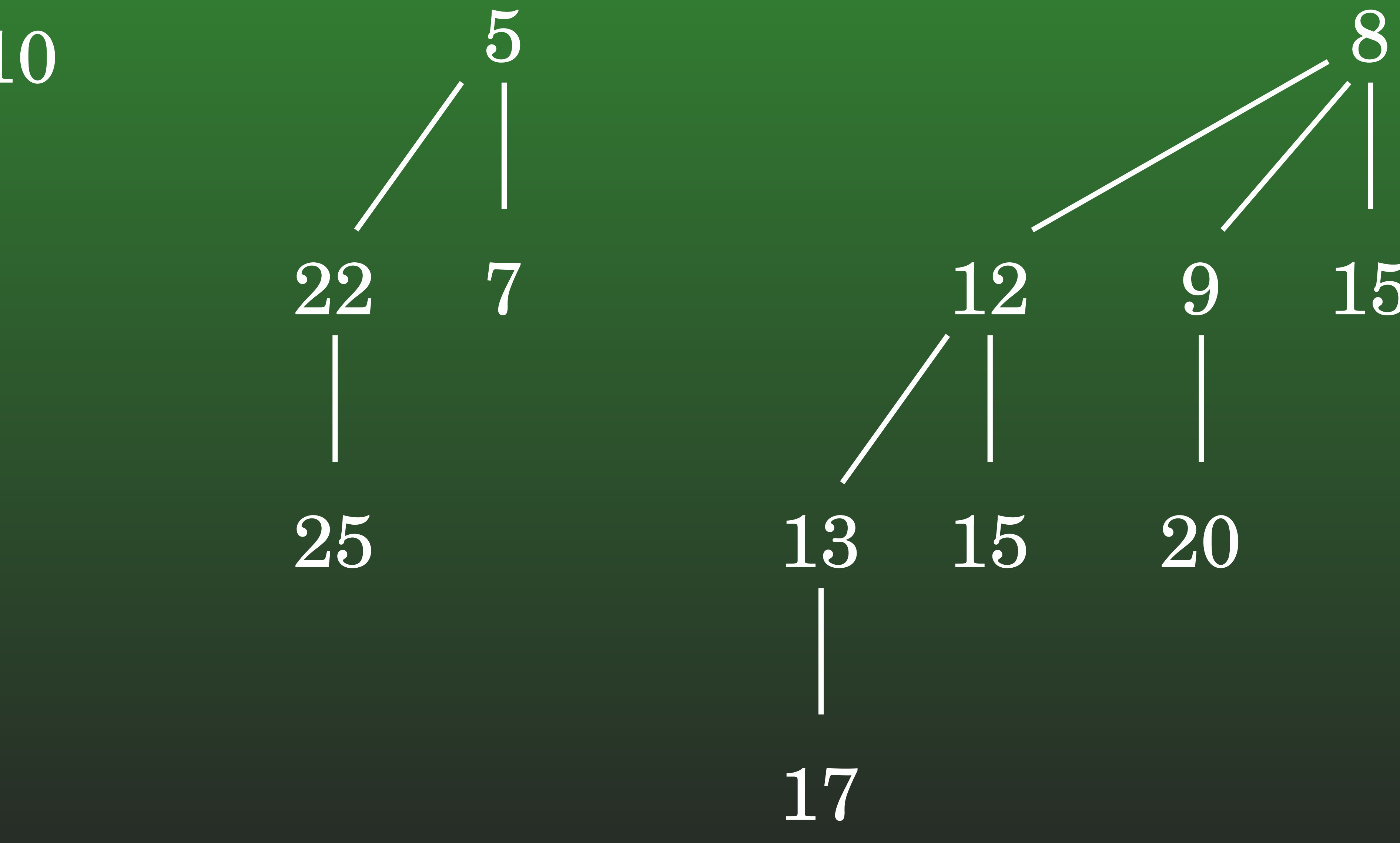
FR-220: Binomial Trees



Binomial Heaps

- A Binomial Heap is:
  - Set of binomial trees, each of which has the heap property
    - Each node in every tree is $<=$ all of its children
  - All trees in the set have a different root degree
    - Can't have two B_3's, for instance

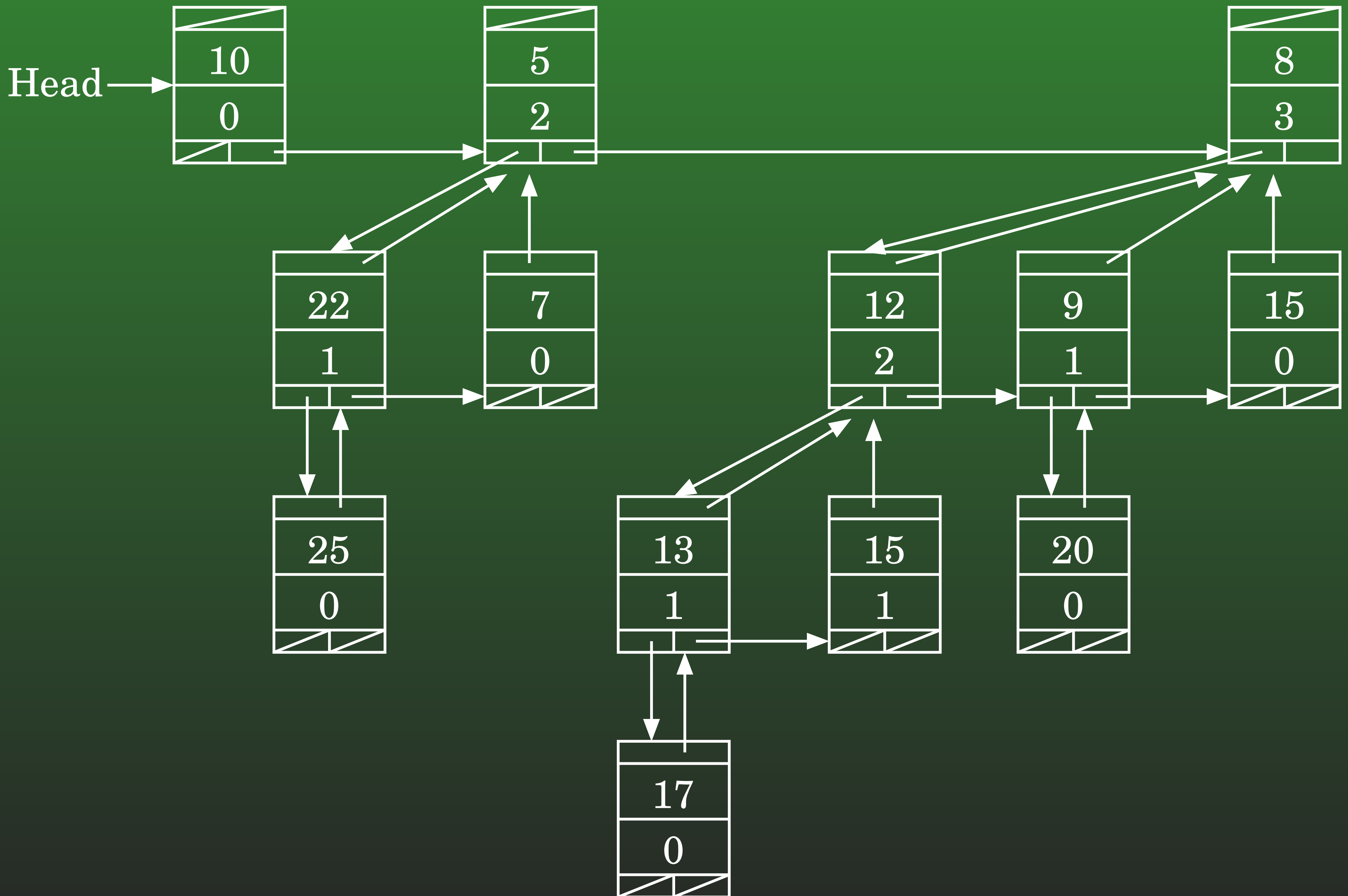
Binomial Heaps



FR-223: Binomial Heaps

- Representing Binomial Heaps
  - Each node contains:
    - left child, right sibling, parent pointers
    - degreee (is the tree rooted at this node B_0, B_1, etc.)
    - data
  - Each list of children sorted by degree

FR-224: Binomial Heaps



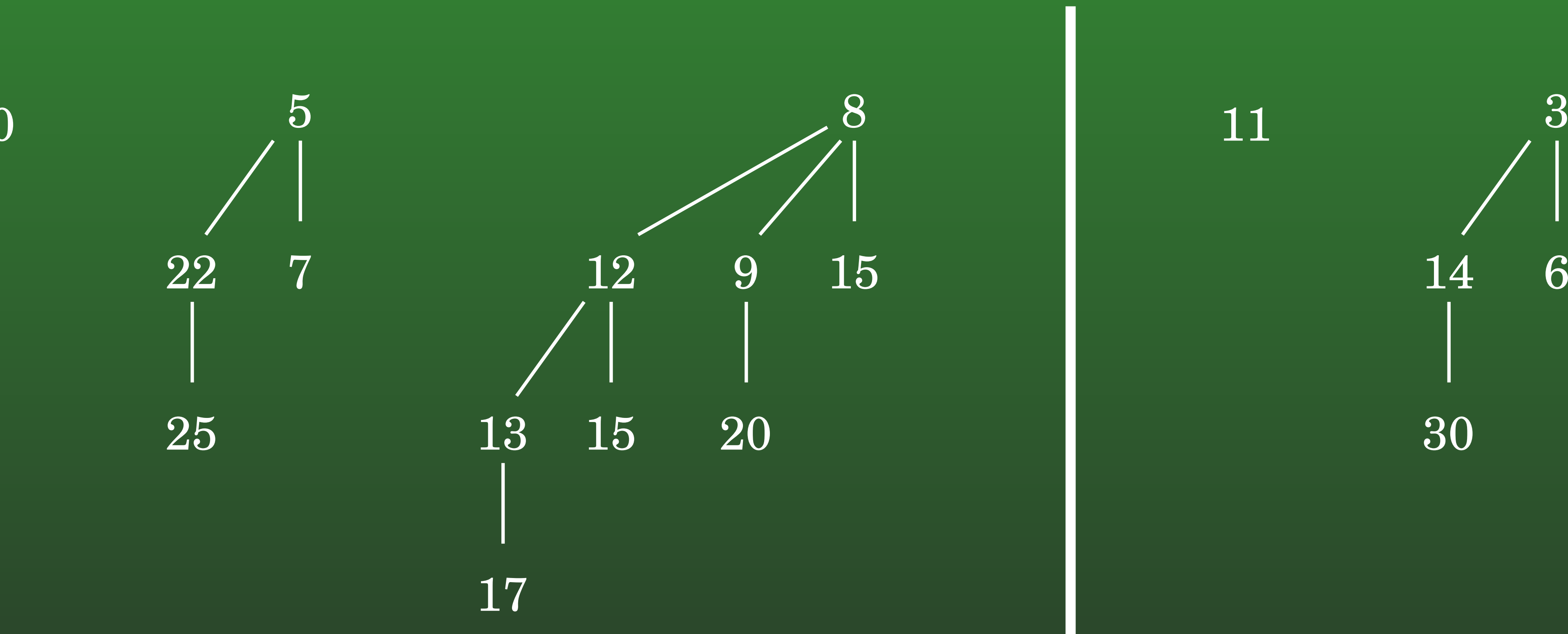
FR-225: Binomial Heaps

- How can we find the minimum element in a binomial heap?
  - Look at the root of each tree in the list, find smallest value
- How long does it take?
  - Heap has n elements
  - Represent n as a binary number
  - B_k is in heap iff kth binary digit of n is 1
  - Number of trees in heap $\in O(\lg n)$

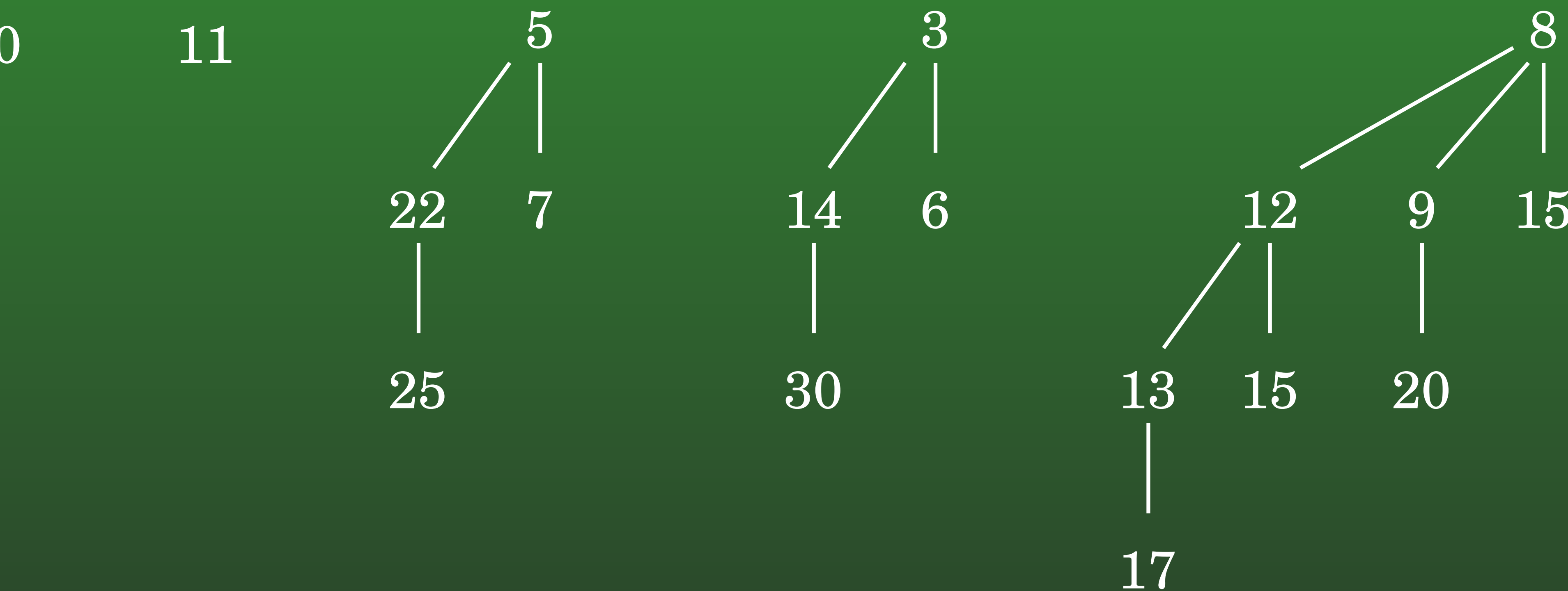
Binomial Heaps

- Merging Heaps H_1 and H_2
  - Merge root lists of H_1 and H_2
    - Could now have two trees with same degree
  - Go through list from smallest degree to largest degree
    - If two trees have same degree, combine them into one tree of larger degree
    - If three trees have same degree (how can this happen?) leave one, combine other two into tree of larger degree

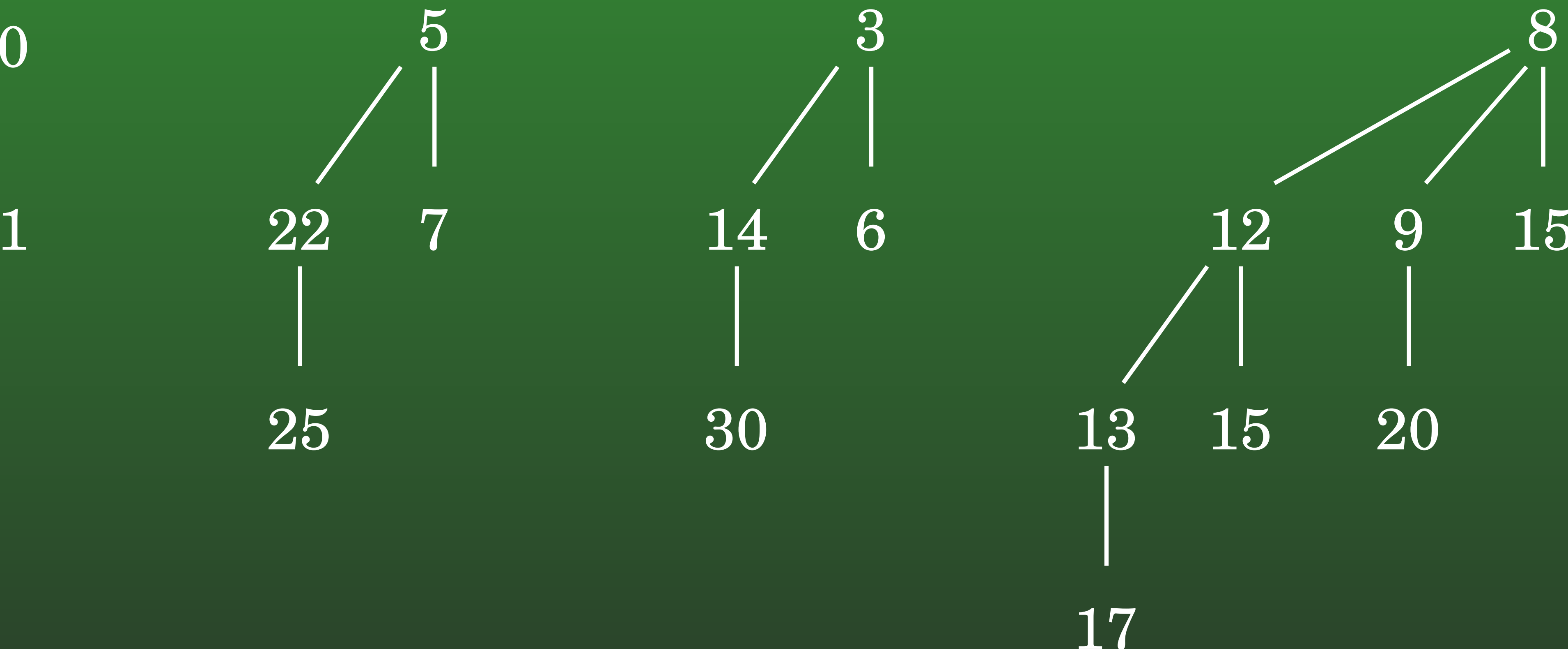
FR-227: Binomial Heaps



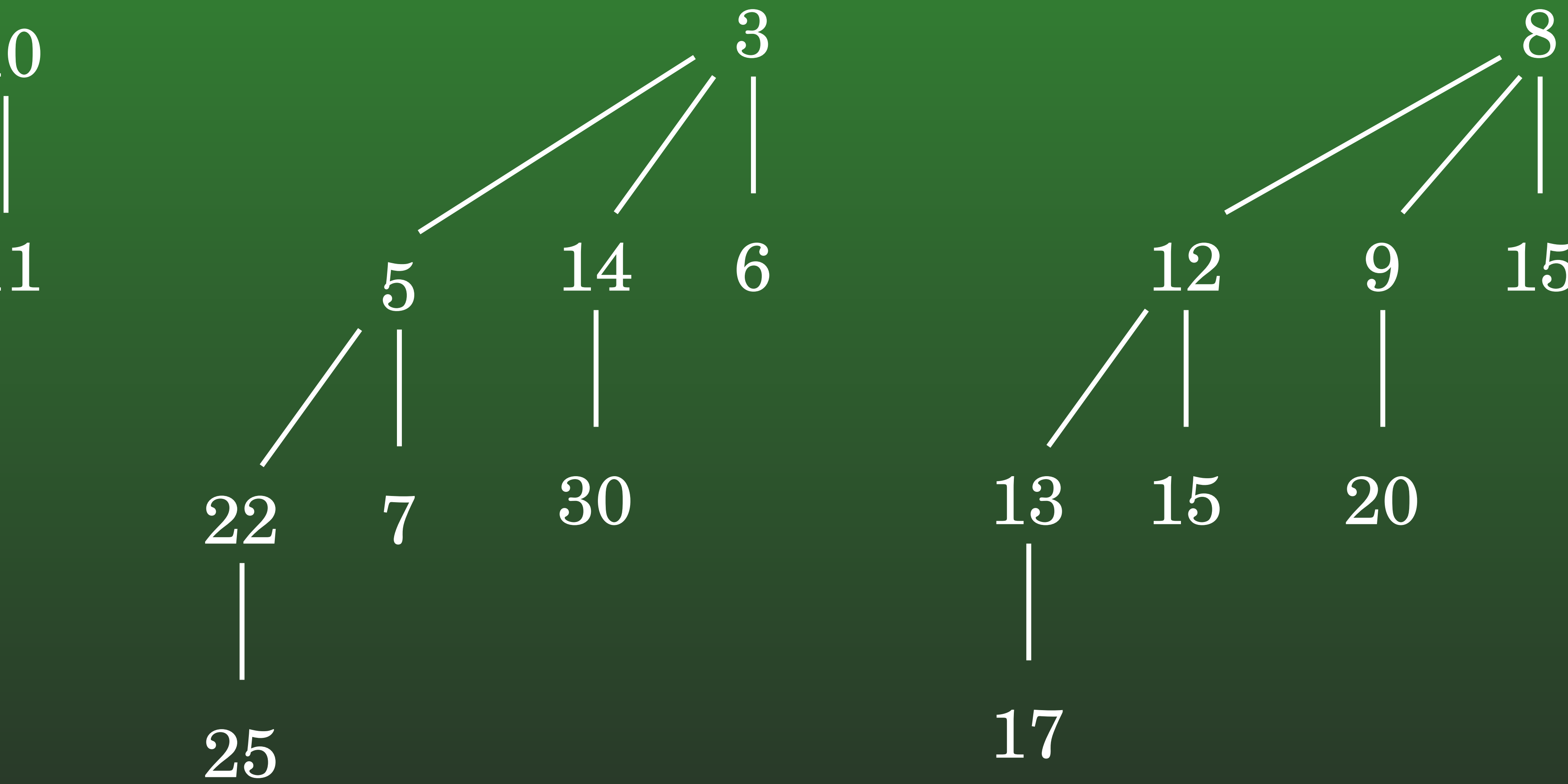
FR-228: Binomial Heaps



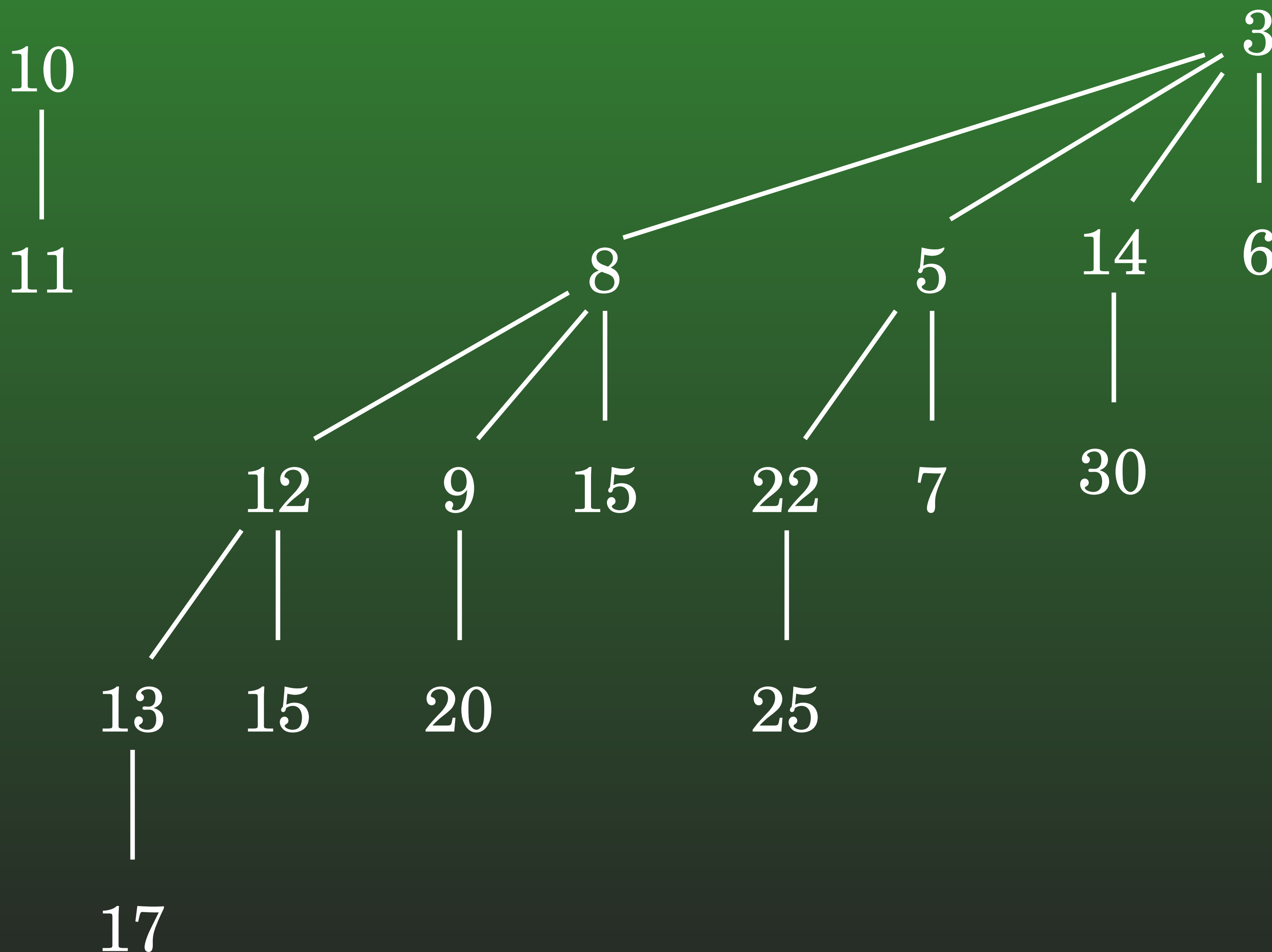
FR-229: Binomial Heaps



Binomial Heaps



FR-231: Binomial Heaps

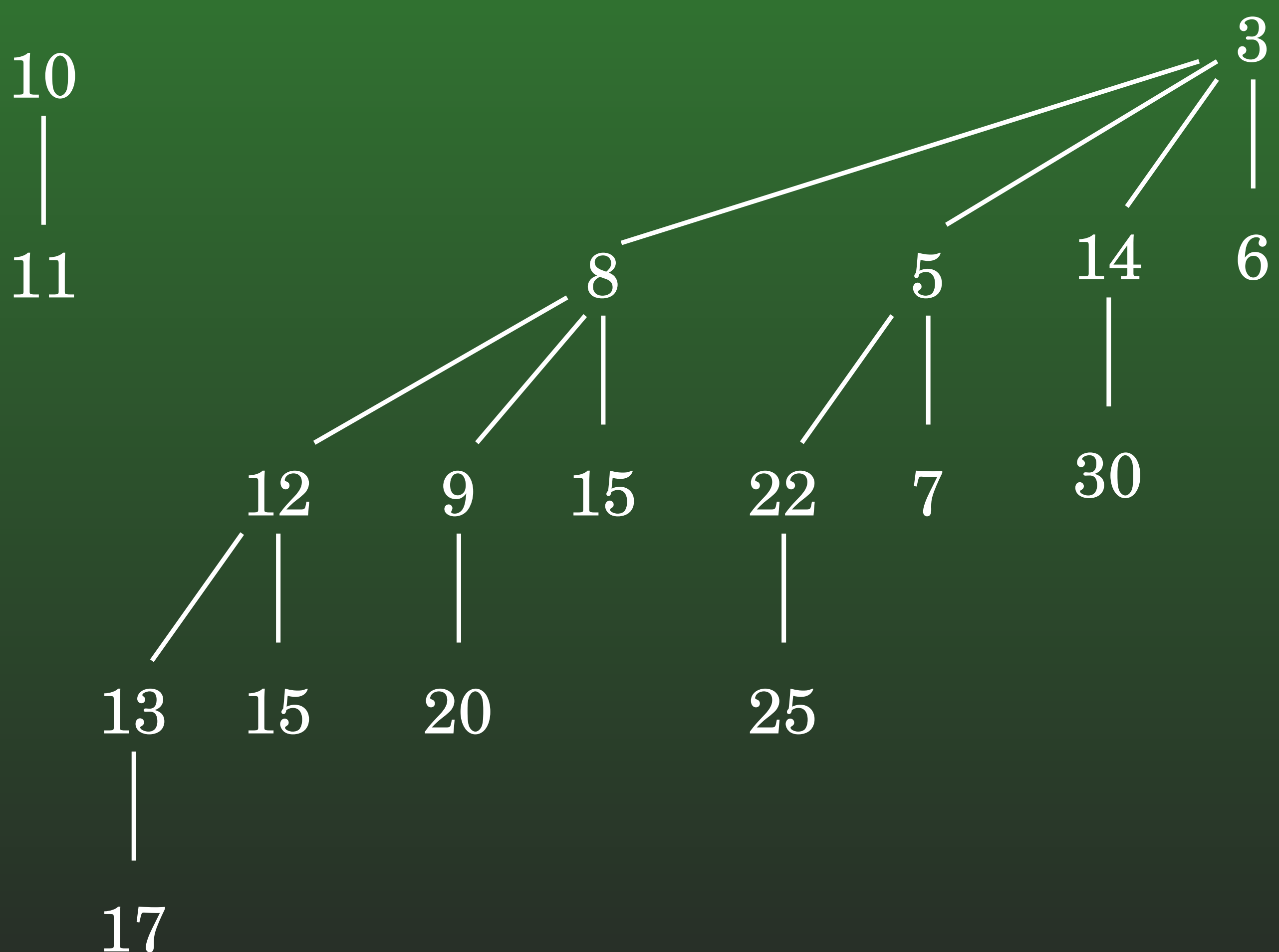


FR-232: Binomial Heaps

- Removing minimum element
  - Find tree T that has minimum value at root, remove T from the list
  - Remove the root of T
    - Leaving a list of smaller trees
  - Reverse list of smaller trees
  - Merge two lists of trees together

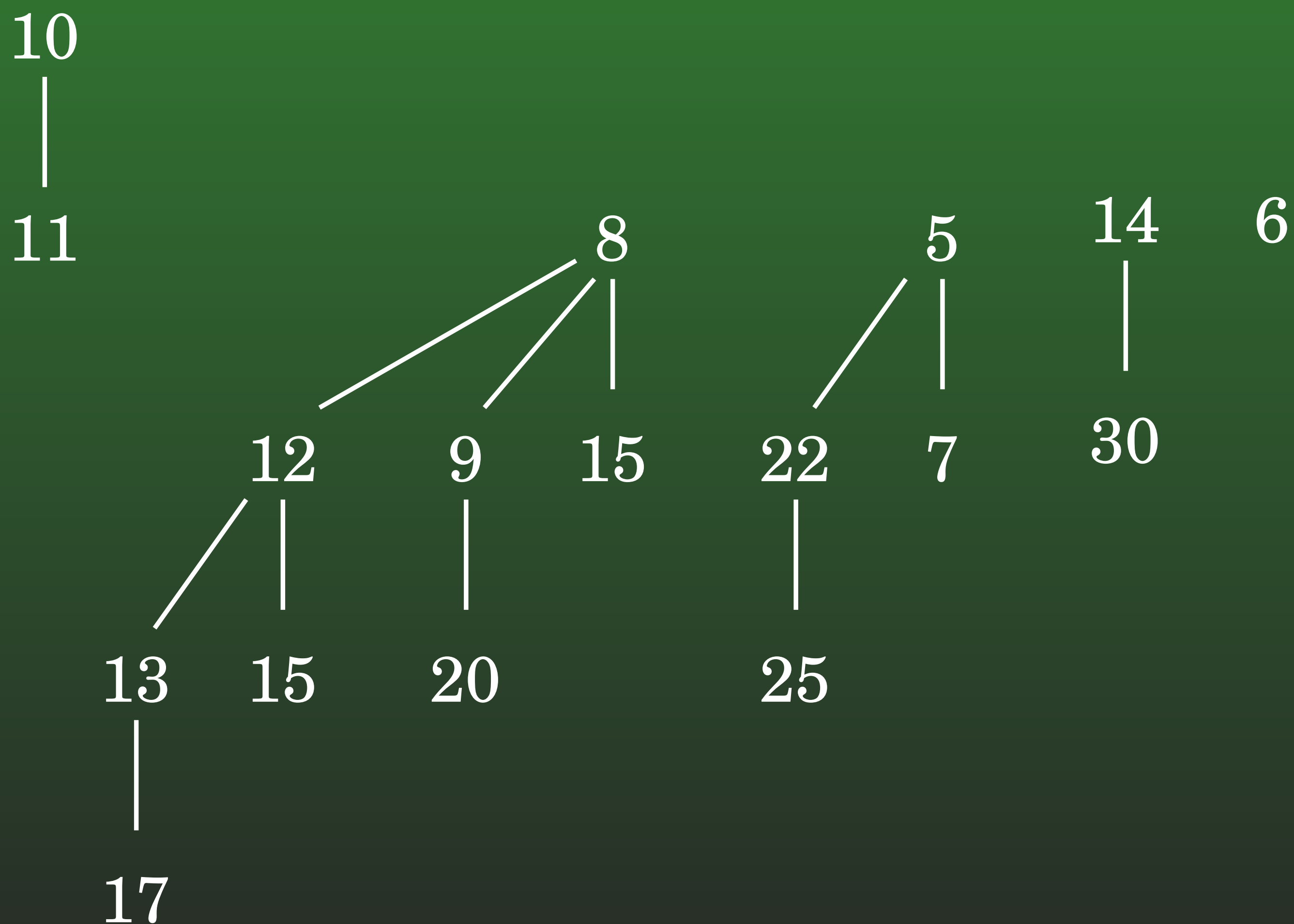
Binomial Heaps

- Removing minimum element



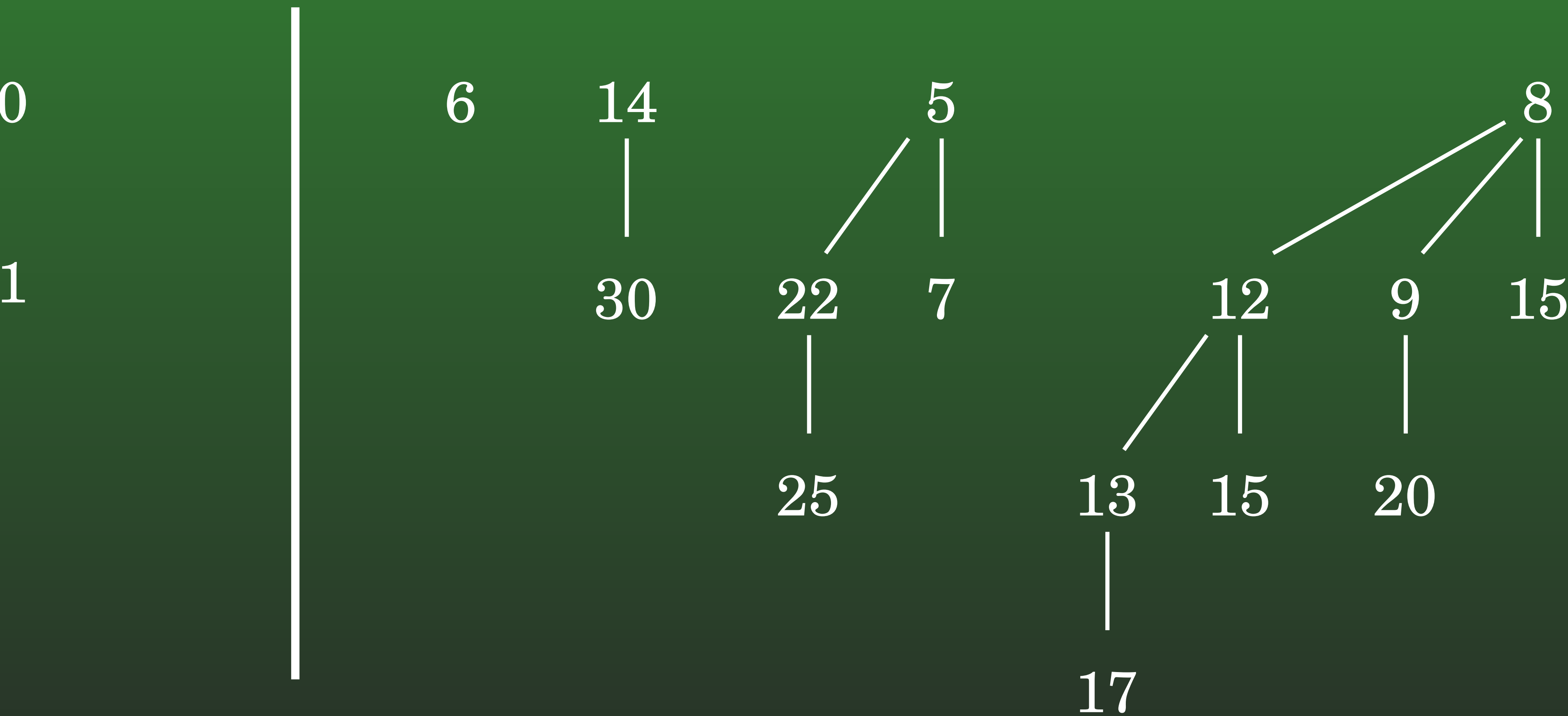
FR-234: Binomial Heaps

- Removing minimum element



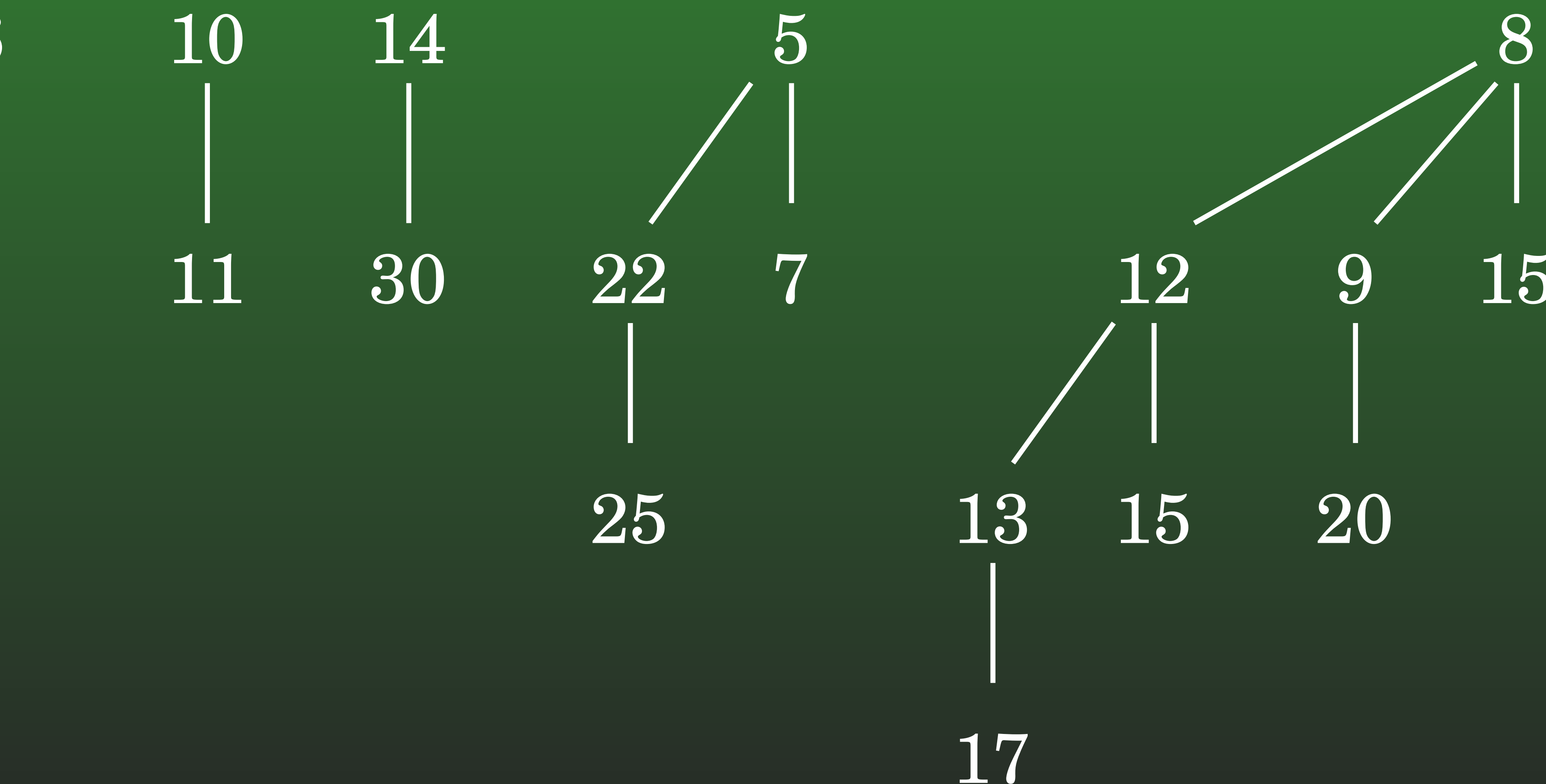
FR-235: Binomial Heaps

- Removing minimum element



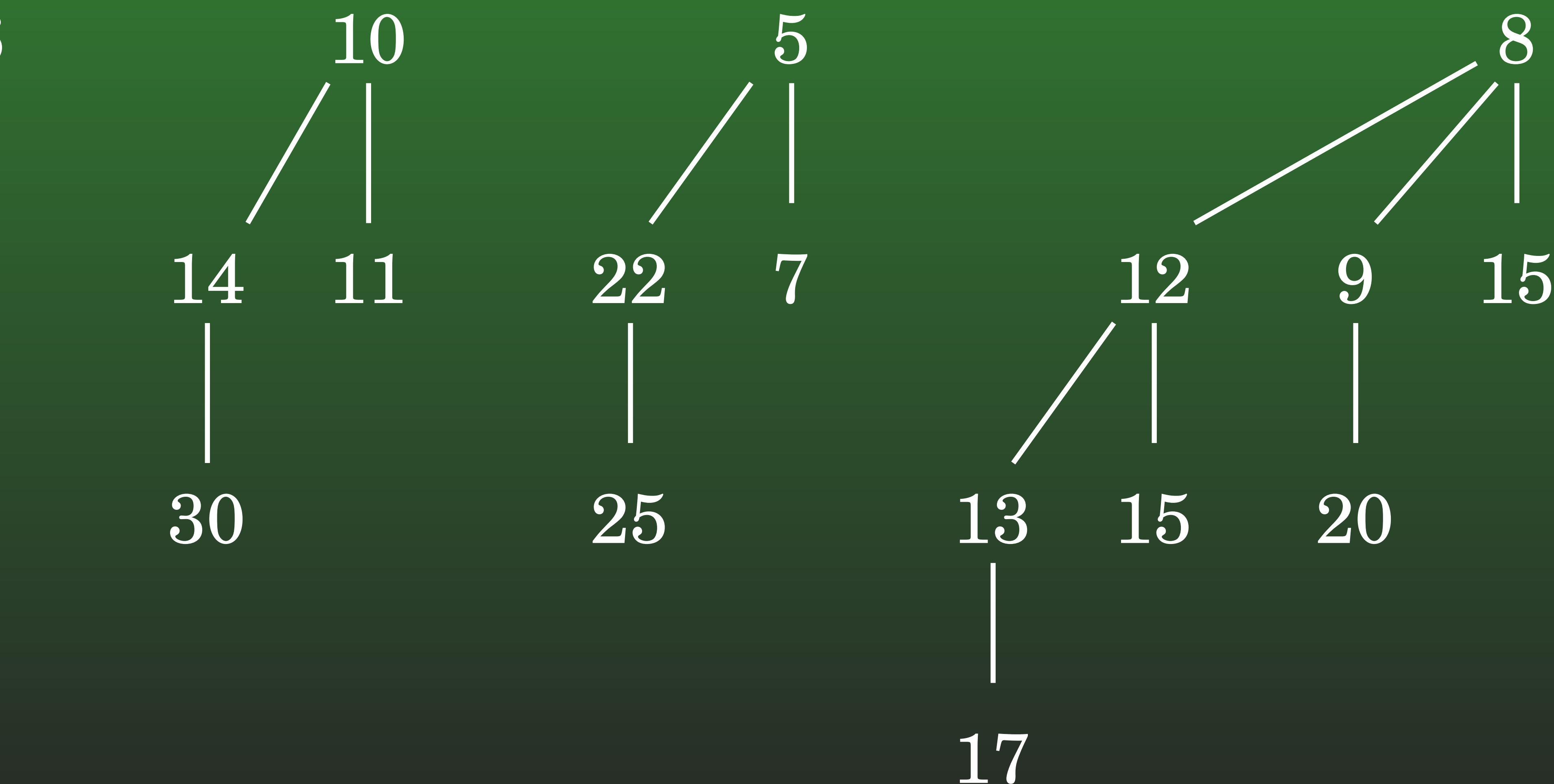
Binomial Heaps

- Removing minimum element



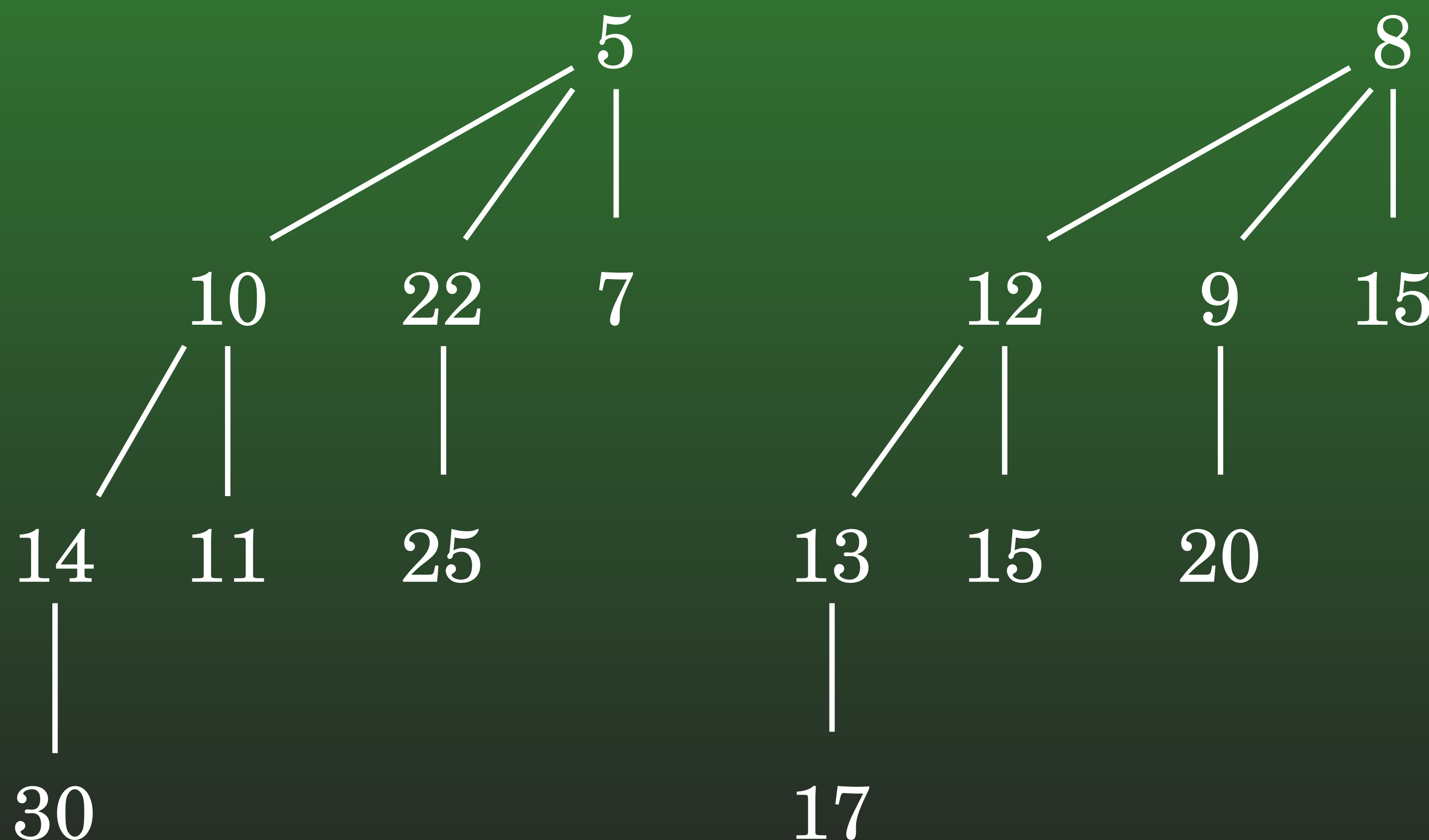
Binomial Heaps

- Removing minimum element



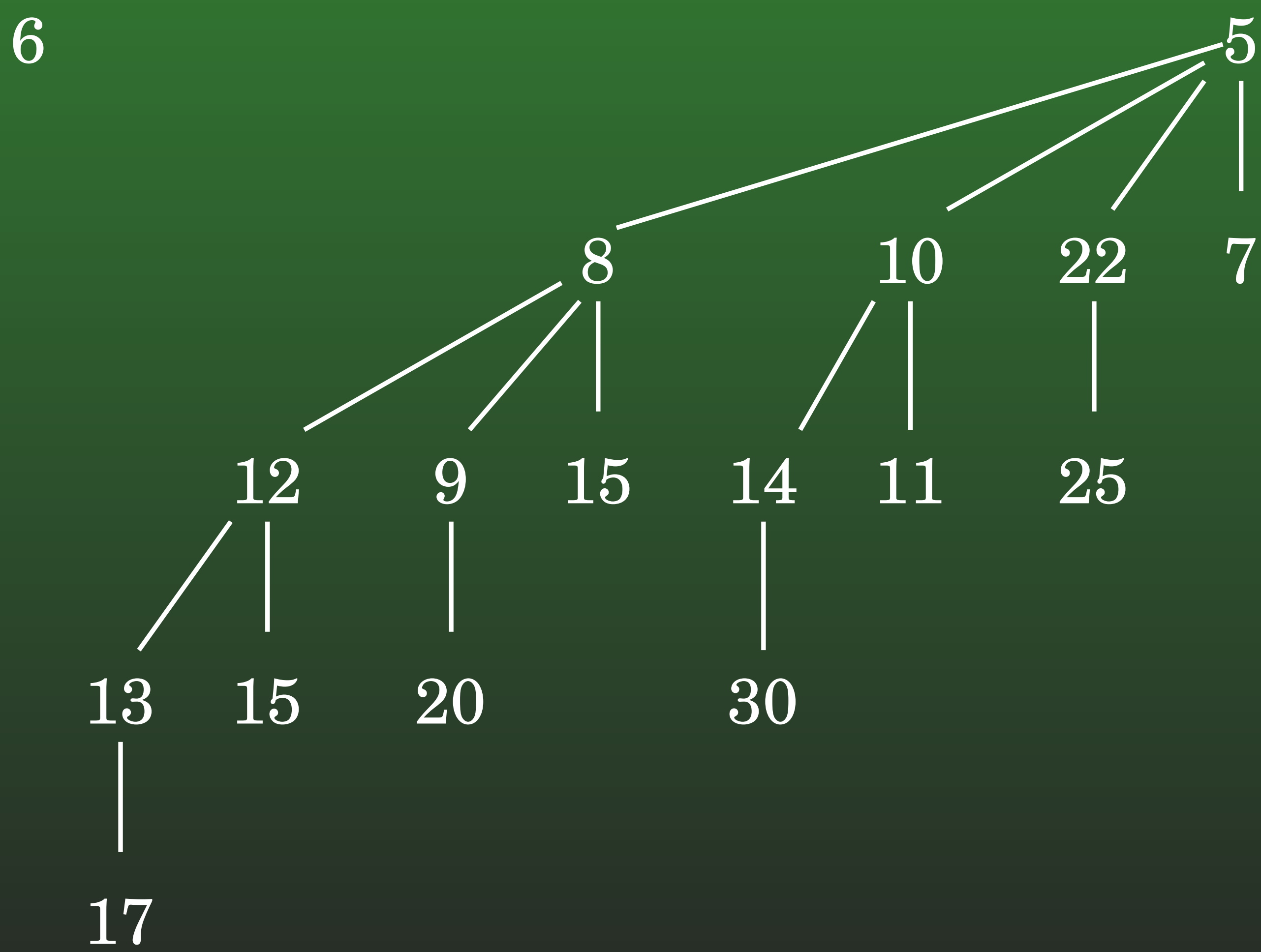
FR-238: Binomial Heaps

- Removing minimum element



FR-239: Binomial Heaps

- Removing minimum element



FR-240: Binomial Heaps

- Removing minimum element
  - Time?
    - Find the smallest element: $O(\lg n)$
    - Reverse list of children $O(\lg n)$
    - Merge heaps $O(\lg n)$