# Game Engineering

## *CS420-2014S-07*

## *Homogenous Space and 4x4 Matrices*

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# 07-0: Matrices and Translations

- Matrices are great for rotations, reflections, scale
- Can't do translations
  - Matrices can only do linear transformations
  - Translations aren't linear
- Like to do *everything* with matrices
- Solution: Add a dimension

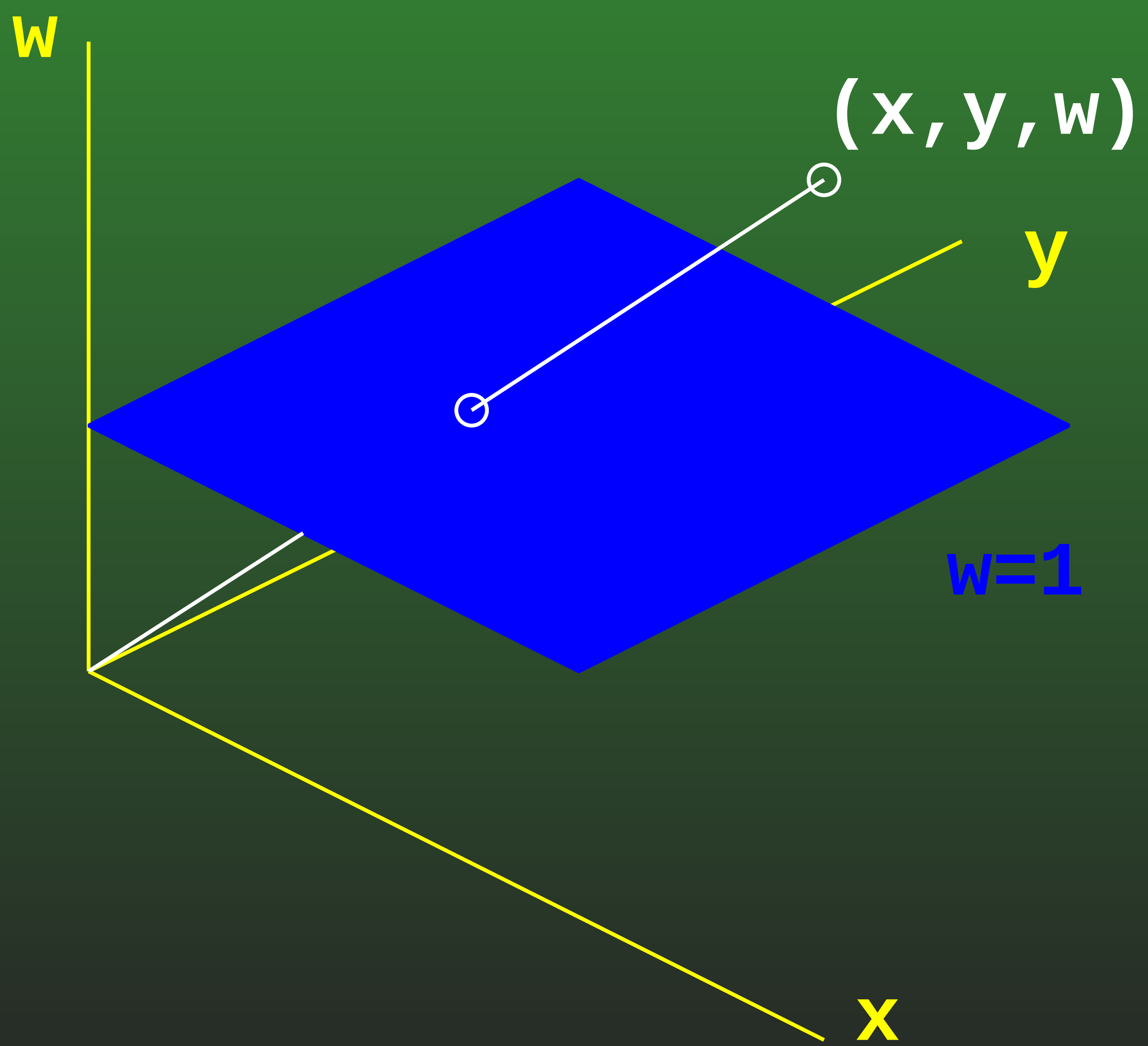
# 07-1: 4D Homogenous Space

- Extend 3D coordinates $(x, y, z)$ to 4D homogenous coordinates $(x, y, z, w)$
  - 4th dimension is *not time*
  - Start with extending 2D coordinates $(x, y)$ to 3D homogenous coordinates $(x, y, w)$

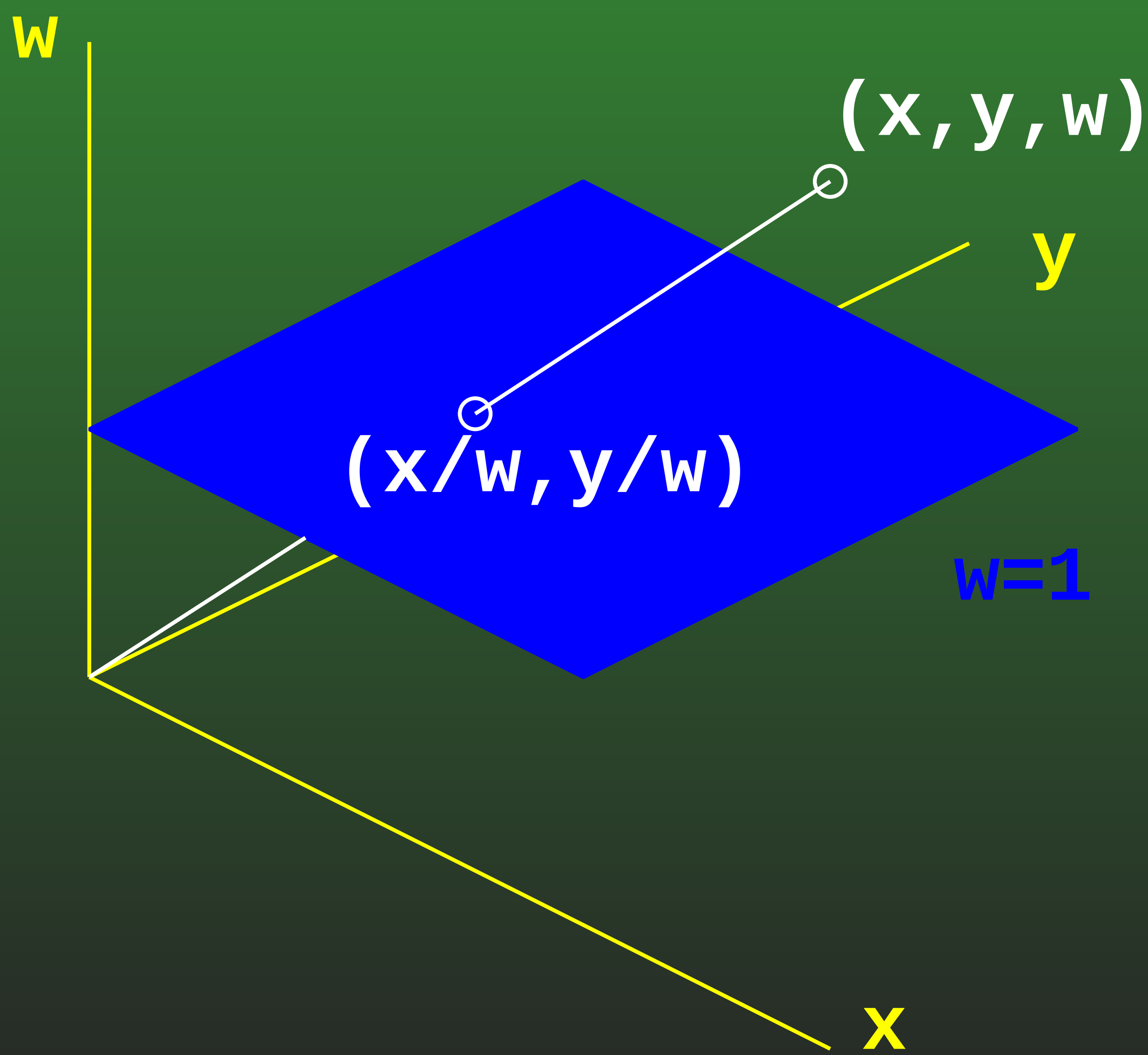
# 07-2: 3D Homogenous Space

- To convert a point $(x, y, w)$ in 3D Homogenous space into 2D $(x, y)$ space:
  - Place a plane at $w = 1$
  - $(x, y, w)$ maps to the $(x, y)$ position on the plane where the ray $(x, y, w)$ intersects the plane

# 07-3: 3D Homogenous Space



# 07-4: 3D Homogenous Space



# 07-5: 3D Homogenous Space

- Converting from a point in 3D homogenous space to 2D space is easy
  - Divide the $x$ and $y$ coordinates by $w$
  - What happens when $w = 0$?

# 07-6: 3D Homogenous Space

- Converting from a point in 3D homogenous space to 2D space is easy
  - Divide the $x$ and $y$ coordinates by $w$
  - What happens when $w = 0$?
    - "Point at infinity"
    - Direction, but not a magnitude

# 07-7: 3D Homogenous Space

- For a given $(x, y, w)$ point in 3D Homogenous space, there is a single corresponding point in "standard" 2D space
  - Though when $w = 0$, we are in a bit of a special case
- For a single point in "standard" 2D space, there are an infinite number of corresponding points in 3D Homogenous space

# 07-8: 4D Homogenous Space

- We can now extend to 3 (4!) dimensions
- A point in 4D Homogeous space $(x, y, z, w)$ transforms to a point in 3D space by dividing $x, y$ and $z$ by $w$
- That is, where the line defined by points (0,0,0,0) and $(x, y, z, w)$ intersects the hyperplane at $w = 1$

# 07-9: 4x4 Transfromation matrices

- In the 3x3 case, a matrix is a transformation of a 3D vector
- In the 4x4 case, a matrix is a transformation of a 4D vector (which we wil then project back into 3D space)
- Let's look at what happens when we restrict $w$ to be 1:

# 07-10: 4x4 Transfromation matrices

- Given any 3x3 transformation marix, we can convert it to 4D as follows:

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} m_{11} & m_{12} & m_{13} & 0 \\ m_{21} & m_{22} & m_{23} & 0 \\ m_{31} & m_{32} & m_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# 07-11: 4x4 Transfromation matrices

- Now, take any 3D vector $\mathbf{v} = [x, y, z]$, and matrix $\mathbf{M}$
  - Convert $\mathbf{v}$ to 4D vector with $w = 1$
  - Convert $\mathbf{M}$ to 4D matrix as above
  - Transform vector using the new matrix
  - Transform back to 3D space
  - Get the same vector as if we had not gone into 4D homogenous space at all

# 07-12: 4x4 Transfromation matrices

$$[x, y, z]\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

$$= [xm_{11} + ym_{21} + zm_{31}, xm_{12} + ym_{22} + zm_{32}, xm_{13} + ym_{23} + zm_{33}]$$

$$[x, y, z, 1]\begin{bmatrix} m_{11} & m_{12} & m_{13} & 0 \\ m_{21} & m_{22} & m_{23} & 0 \\ m_{31} & m_{32} & m_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= [xm_{11} + ym_{21} + zm_{31}, xm_{12} + ym_{22} + zm_{32}, xm_{13} + ym_{23} + zm_{33}, 1]$$

# 07-13: 4x4 Transfromation matrices

- As long as the $w$ component is 1 going in, it will be 1 coming out
  - Easy to go back and forth between 3D coordinates and homogenous 4D coordinates
- We've transformed 3D problem into an equivant 4D problem
  - Why?

# 07-14: Translation

- Consider the matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \Delta x & \Delta y & \Delta z & 1 \end{bmatrix}$$

- What happens when we put a vector $[x, y, z, 1]$ through this matrix?

# 07-15: Translation

$$[x, y, z, 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \Delta x & \Delta y & \Delta z & 1 \end{bmatrix} = [x + \Delta x, y + \Delta y, z + \Delta z, 1]$$

- We can now use matrices to do translations!

# 07-16: Translation

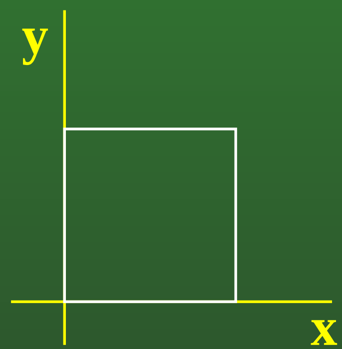
- But ...
  - We’re still just doing matrix multiplication
  - Matrix multiplication does linear transforms
  - Translation is not linear
- What’s going on?

# 07-17: Translation

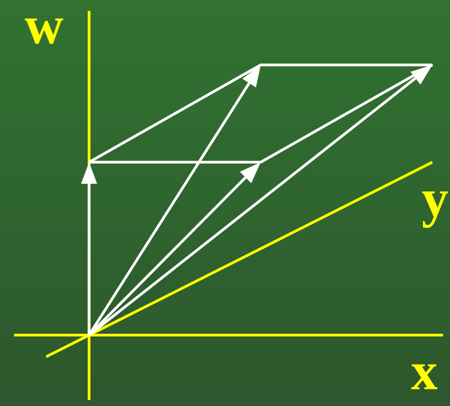
- We are still doing a linear trasformation of the 4D vector
- We are *shearing* the 4D space
- The resulting projection back to 3D is seen as a translation

# 07-18: Translation

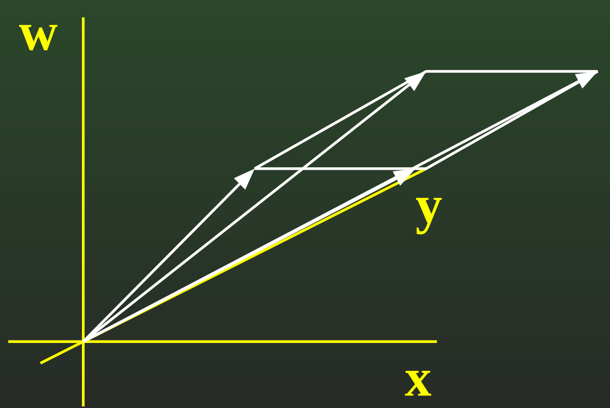
2D Shape



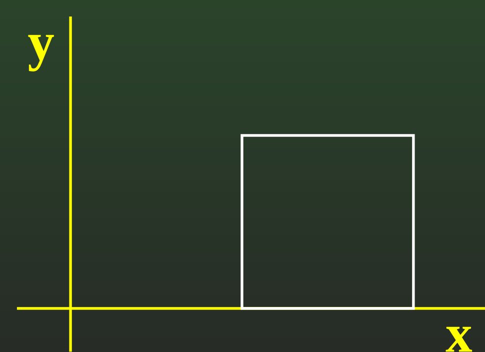

Transform to 3D Homogenous Space




Shear operation in 3D space




Back to 2D



# 07-19: Translation

- Recall our matricies for shearing in 3D:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s & t & 1 \end{bmatrix}$$

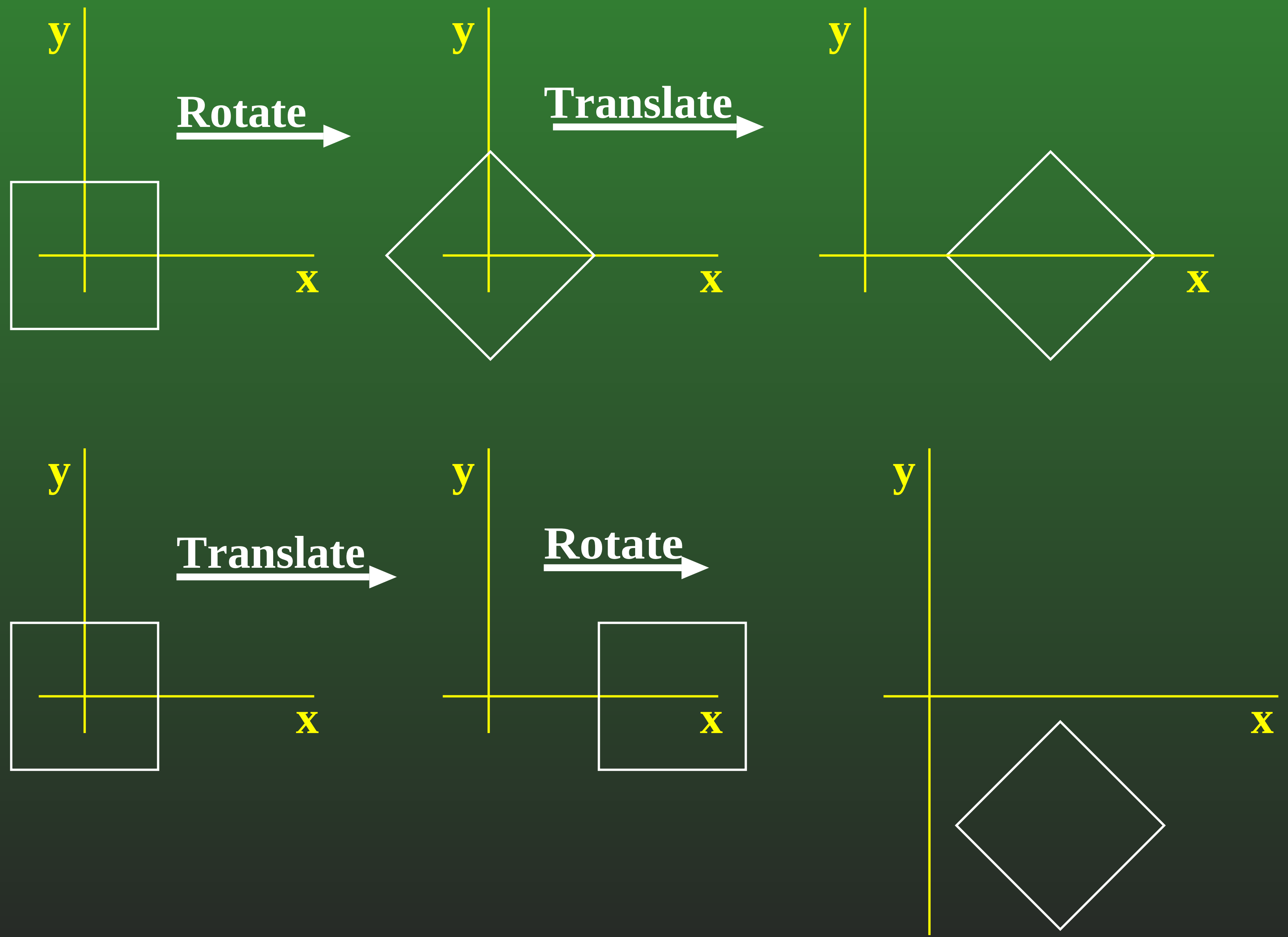
- This is precisely what we are doing when translating!

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \Delta x & \Delta y & \Delta z & 1 \end{bmatrix}$$

# 07-20: Combining Transforms

- Since matrix multiplication is associative, we can combine translation and rotation into a single matrix
- First do a rotation, and then a translation
  - Order is important!
  - Why?

# 07-21: Combining Transforms



# 07-22: Combining Transforms

- First rotate, and then translate
- $(\mathbf{v}\mathbf{M}_R)\mathbf{M}_T = \mathbf{v}(\mathbf{M}_r\mathbf{M}_T)$
- What is $\mathbf{M}_r\mathbf{M}_T$?

$$\mathbf{M}_R = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \Delta x & \Delta y & \Delta z & 1 \end{bmatrix}$$

# 07-23: Combining Transforms

- First rotate, and then translate
- $(\mathbf{v}\mathrm{M}_R)\mathrm{M}_T = \mathbf{v}(\mathrm{M}_r\mathrm{M}_T)$
- What is $\mathrm{M}_r\mathrm{M}_T$?

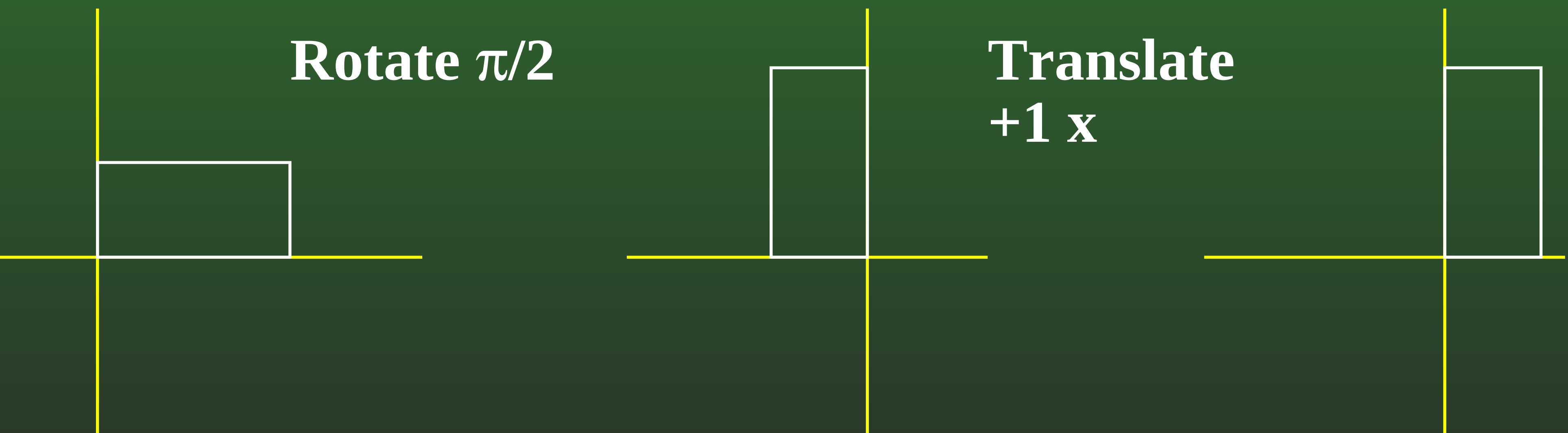
$$\mathbf{M}_R = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \Delta x & \Delta y & \Delta z & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ \Delta x & \Delta y & \Delta z & 1 \end{bmatrix}$$

# 07-24: Combining Transforms

- Any 4x4 Homogenous matrix can be split into a rotational component and a translation component
  - Upper 3x3 matrix is rotation (which is done first)
  - Bottom row is translation (done second)
- But wait – rotation is not always done first!
  - True, but any series of rotations and translations is equivalent to a single rotation followed by a single translation

# 07-25: Combining Transforms

- Let's look at an example
  - First rotate by $\pi/2$ (90 degrees) counterclockwise
  - Then translate $x$ by +1

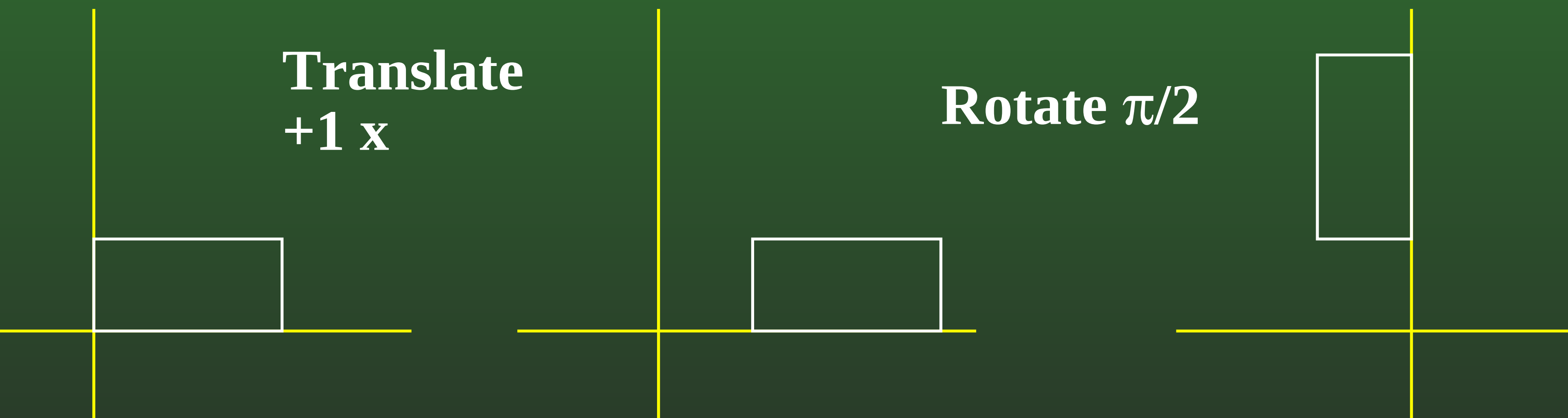


# 07-26: Combining Transforms

$$\begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & cos\Theta & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

# 07-27: Combining Transforms

- Another example
  - First translate $x$ by +1
  - Then rotate by $\pi/2$ (90 degrees) counterclockwise

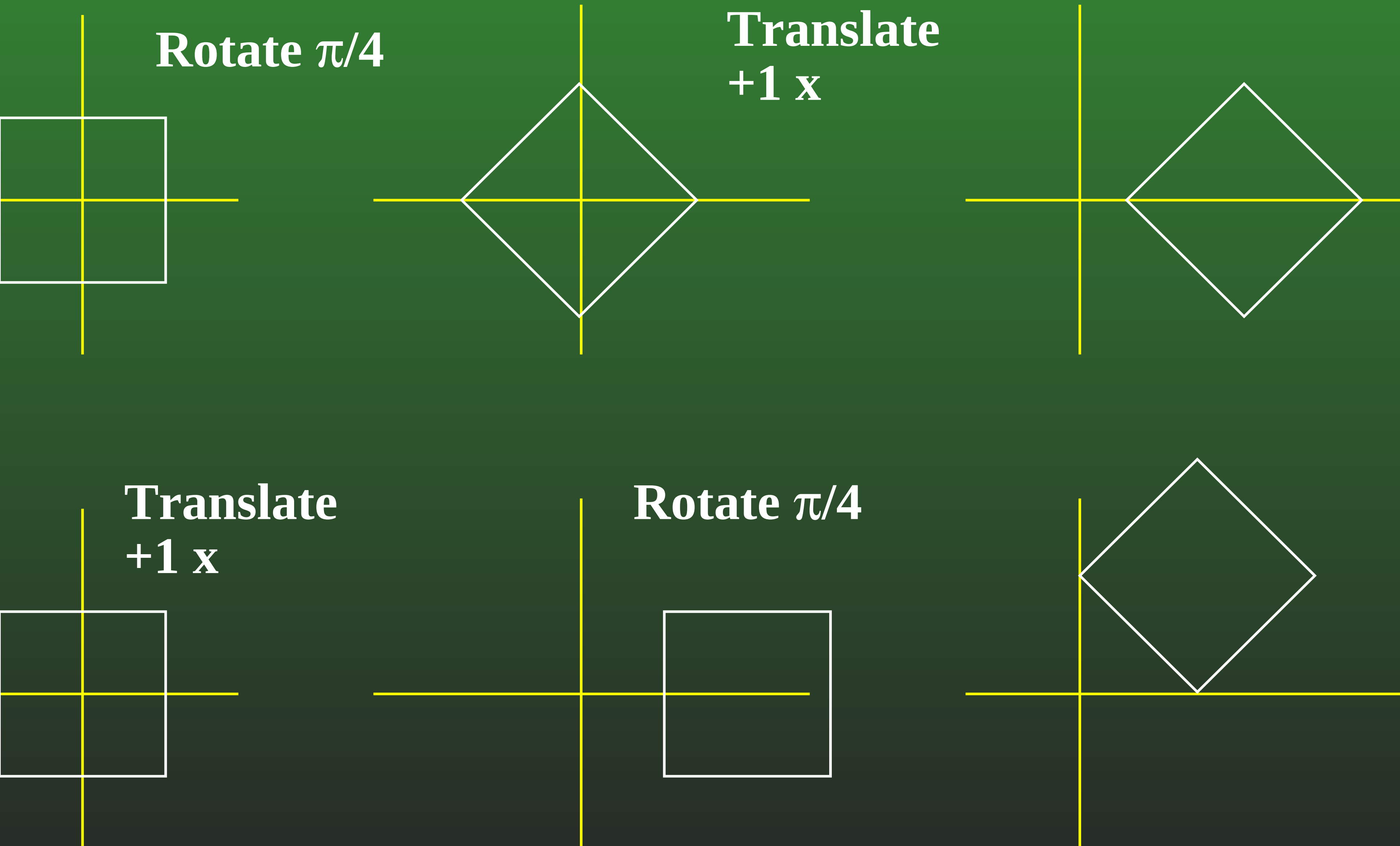


# 07-28: Combining Transforms

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ \cos\Theta & \sin\Theta & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

- Same as rotating, and then moving up $+y$

# 07-29: Combining Transforms



# 07-30: Combining Transforms

- Rotating by $\pi/4$, then translating 1 unit $+x$

$$\begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & cos\Theta & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

# 07-31: Combining Transforms

- Translating 1 unit $+x$, then rotating by $\pi/4$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ \cos\Theta & \sin\Theta & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \end{bmatrix}$$

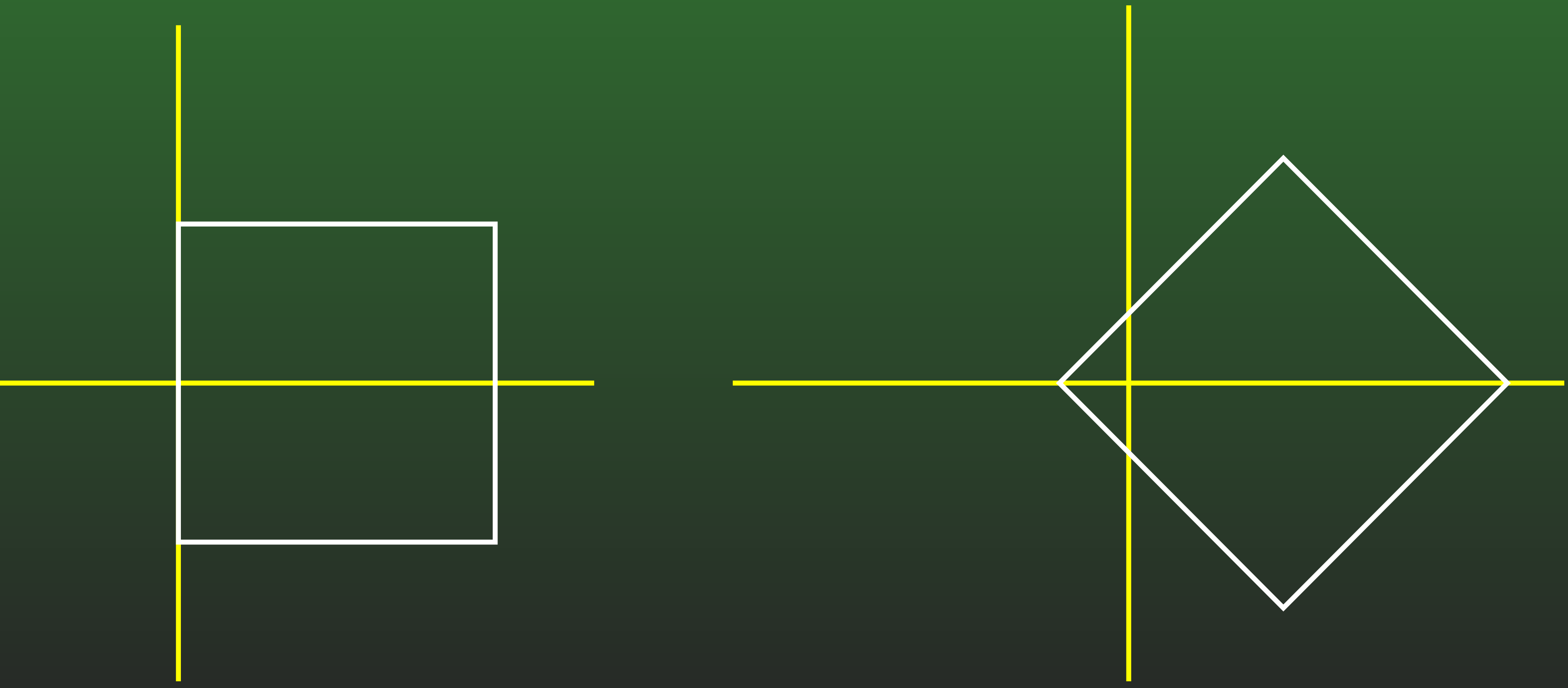
- Same as rotating $\pi/4$ counterclockwise, and then translating over (+x) $1/\sqrt{2}$ and up (+y) $1/\sqrt{2}$

# 07-32: Non-Standard Axes

- We want to rotate around an axis that does not go through the origin
- 2D Case: Rotate around point at 1,0
- Create the appropriate 3x3 vector

# 07-33: Non-Standard Axes

## Rotate $\pi/4$ around (1,0)



# 07-34: Non-Standard Axes

- First, translate to the origin
- Then, do the rotation
- Finally, translate back

# 07-35: Non-Standard Axes

- First, translate to the origin

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

- Then, do the rotation
- Finally, translate back

# 07-36: Non-Standard Axes

- First, translate to the origin
- Then, do the rotation

$$\begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Finally, translate back

# Non-Standard Axes

- First, translate to the origin
- Then, do the rotation
- Finally, translate back

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

# 07-38: Non-Standard Axes

- Final matrix:

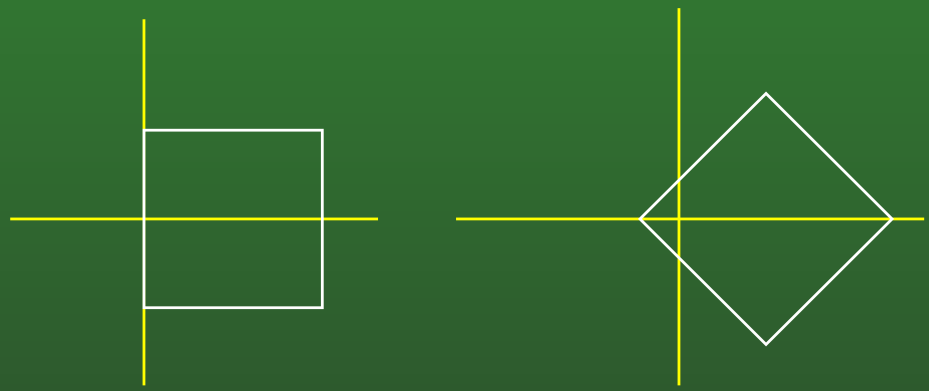
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & -1/\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

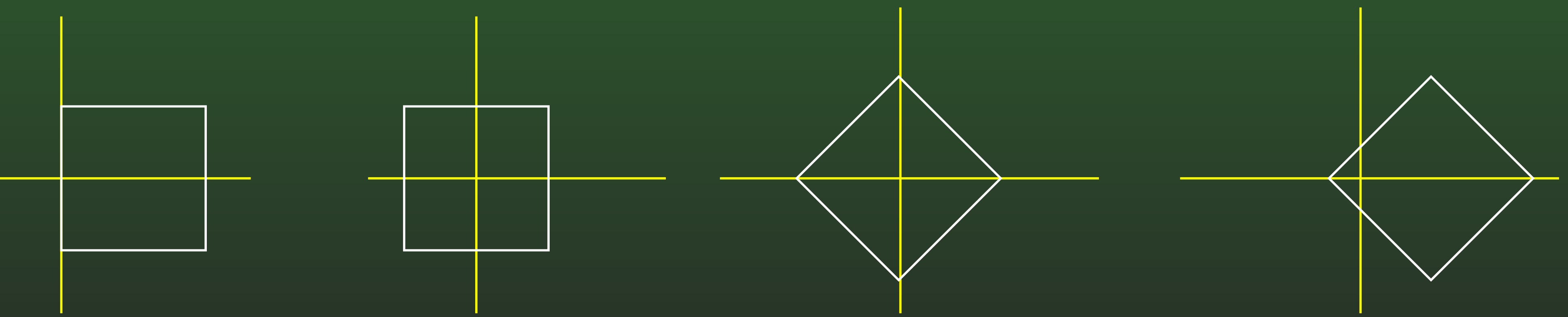
$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1-1/\sqrt{2} & -1/\sqrt{2} & 1 \end{bmatrix}$$

# 07-39: Non-Standard Axes

**Rotate π/4 around (1,0)**

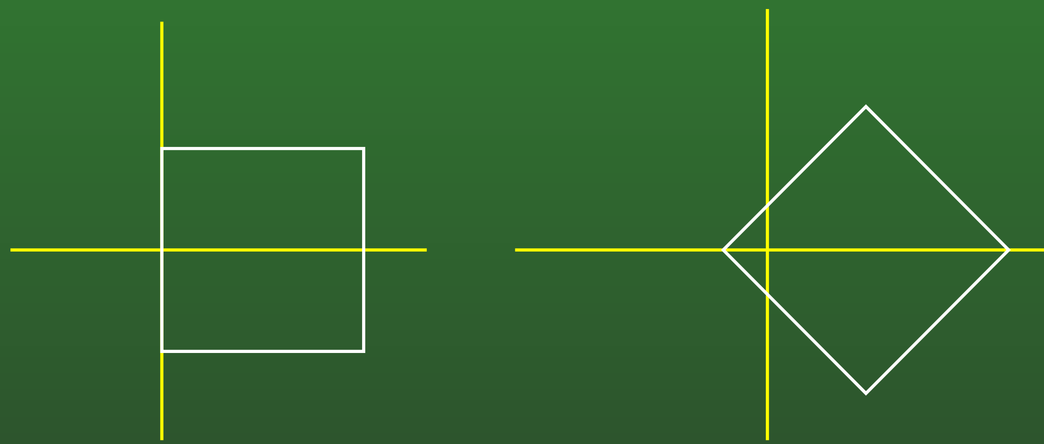


**Translate to origin, Rotate π/4 around (0,0), Translate back**

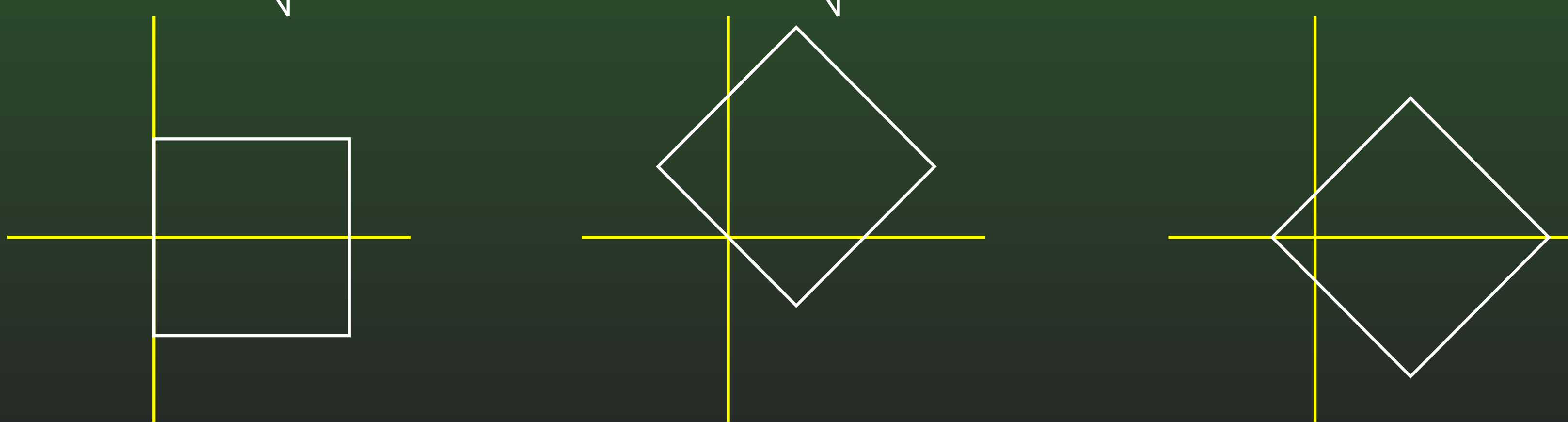


# 07-40: Non-Standard Axes

**Rotate $\pi/4$ around (1,0)**



**Rotate p/4 around (0,0), then translate over $1 - 1/\sqrt{2}$ and down $1/\sqrt{2}$**



# 07-41: Non-Standard Axes

- Note that the *rotation* component (upper right 2x2 matrix) is the same as if we were rotating around the origin
- Only the *position* component is altered.
- In general, whenever we do a rotation and a number of translations, the rotation component will be unchanged

# Non-Standard Axes 3D

- To rotate in 3D around an axis whose center point does not go through the origin
  - Let $\mathbf{p} = [p_x, p_y, p_z]$ be some point on the axis of rotation
  - Let $R_{3x3}$ be a 3x3 matrix that does the rotation, assuming the axis goes through the origin
- We can write the rotation as $TR_{4x4}T^{-1}$, where $T$, $R_{4x4}$, and $T^{-1}$ are defined as:

# 07-43: Non-Standard Axes 3D

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -p_x & -p_y & -p_z & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{p} & 1 \end{bmatrix}$$

$$\mathbf{R}_{4x4} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{3x3} & 0 \\ 0 & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p_x & p_y & p_z & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{p} & 1 \end{bmatrix}$$

# 07-44: Non-Standard Axes 3D

$$TRT^{-1} = \begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{p} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_{3x3} & 0 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{p} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{R}_{3x} & 0 \\ -\mathbf{p}\mathbf{R}_{3x3} + \mathbf{p} & 1 \end{bmatrix}$$

# 07-45: Non-Standard Axes 3D

- Let's take a closer look:
  - First, rotate around axis that goes through origin (this will rotate the object's position through space – we want to undo this)
  - Move the object from its new (rotated) position back to the origin
  - Translate back to the original position

$$
\begin{aligned}
TRT^{-1} &= \begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{p} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_{3x3} & 0 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{p} & 1 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{R}_{3x} & 0 \\ -\mathbf{p}\mathbf{R}_{3x3} + \mathbf{p} & 1 \end{bmatrix}
\end{aligned}
$$

# 07-46: Non-Standard Axes 3D

- This doesn't just work for rotating – it works for any linear transform (scaling, reflecting, shearing, etc)
  - Move object to origin
  - Do the transformation
  - Move the object back

# 07-47: Non-Standard Axes 3D

- This doesn’t just work for rotating – it works for any linear transform (scaling, reflecting, shearing, etc)
  - Do the transformation, assuming axis runs through origin
  - Move the object to the origin (using transformed position)

# Non-Standard Axes 3D

- This doesn't just work for rotating – it works for any linear transform (scaling, reflecting, shearing, etc)
  - Do the transformation, assuming axis runs through origin
  - Move the object to the origin (using transformed position)
  - Move the object back to the original position

$$
\begin{aligned}
TRT^{-1} &= \begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{p} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_{3x3} & 0 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{p} & 1 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{R}_{3x} & 0 \\ -\mathbf{p}\mathbf{R}_{3x3} + \mathbf{p} & 1 \end{bmatrix}
\end{aligned}
$$

# 07-49: Homogenous Dimension = 0

- Consider a vector in homogenous 4-space
  - $[x, y, z, w]$
- What happens when $w = 0$?

# 07-50: Homogenous Dimension = 0

- Consider a vector in homogenous 4-space
  - $[x, y, z, w]$
- What happens when $w = 0$?
  - $x$, $y$, and $z$ components are divided by $w$
  - “Point at infinity”
  - Direction only, not magnitude

# 07-51: Homogenous Dimension = 0

- What happens when multiply a vector with $w = 0$ by a transform that contains no translation?

$$[x, y, z, 0] \begin{bmatrix} m_{11} & m_{12} & m_{13} & 0 \\ m_{21} & m_{22} & m_{23} & 0 \\ m_{31} & m_{32} & m_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$[xm_{11} + ym_{21} + zm_{31}, xm_{12} + ym_{22} + zm_{32}, xm_{13} + ym_{23} + zm_{33}, 0]$$

- Standard transformation – just as if $w = 1$

# 07-52: Homogenous Dimension = 0

- What happens when multiply a vector with $w = 0$ by a transform that does contain translation?

$$[x, y, z, 0] \begin{bmatrix} m_{11} & m_{12} & m_{13} & 0 \\ m_{21} & m_{22} & m_{23} & 0 \\ m_{31} & m_{32} & m_{33} & 0 \\ \Delta x & \Delta y & \Delta z & 1 \end{bmatrix} =$$

# 07-53: Homogenous Dimension = 0

- What happens when multiply a vector with $w = 0$ by a transform that does contain translation?

$$[x, y, z, 0] \begin{bmatrix} m_{11} & m_{12} & m_{13} & 0 \\ m_{21} & m_{22} & m_{23} & 0 \\ m_{31} & m_{32} & m_{33} & 0 \\ \Delta x & \Delta y & \Delta z & 1 \end{bmatrix} =$$

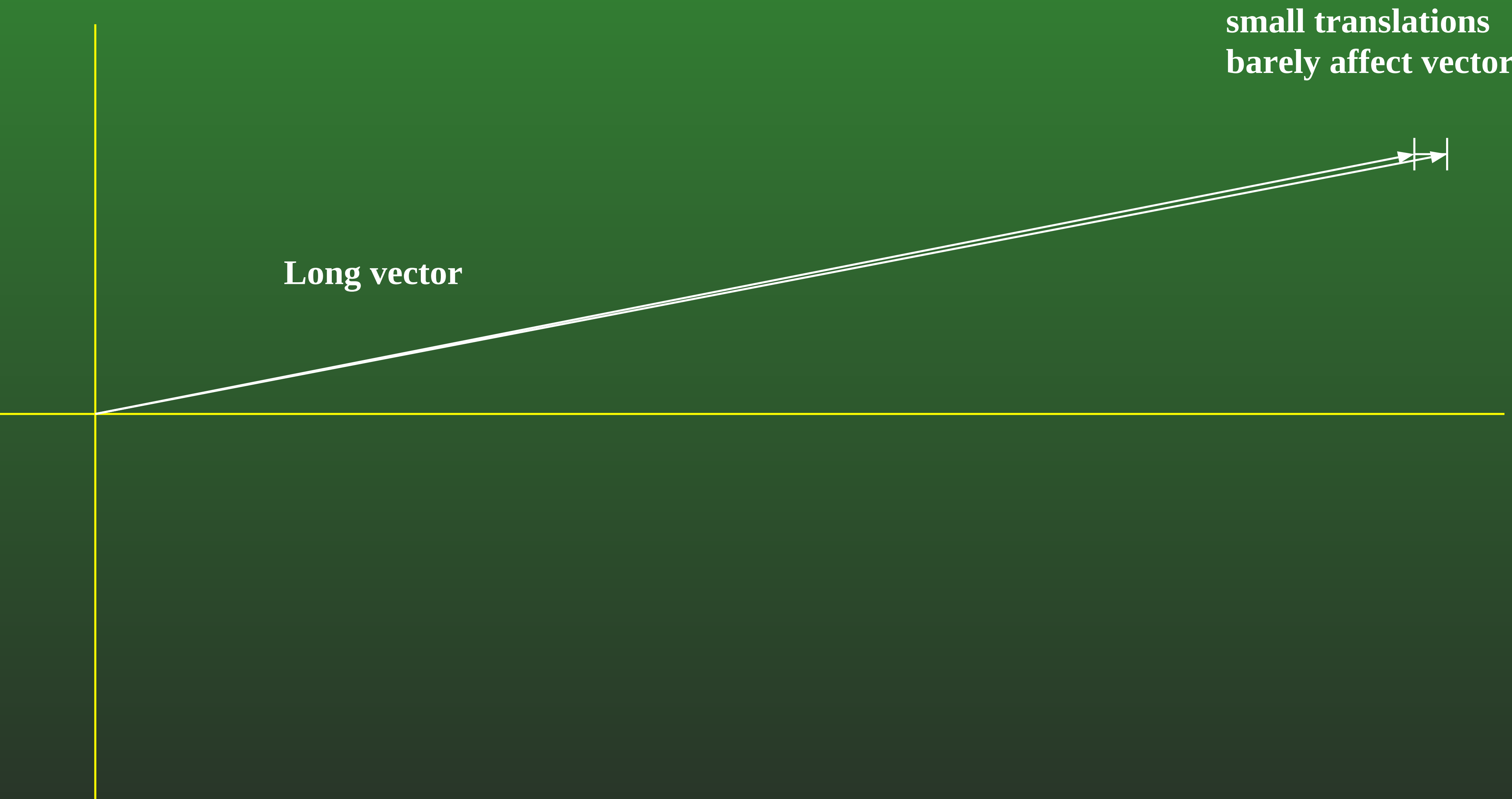
$$[xm_{11} + ym_{21} + zm_{31}, xm_{12} + ym_{22} + zm_{32}, xm_{13} + ym_{23} + zm_{33}, 0]$$

- Rotation occurs as before – but translation is ignored

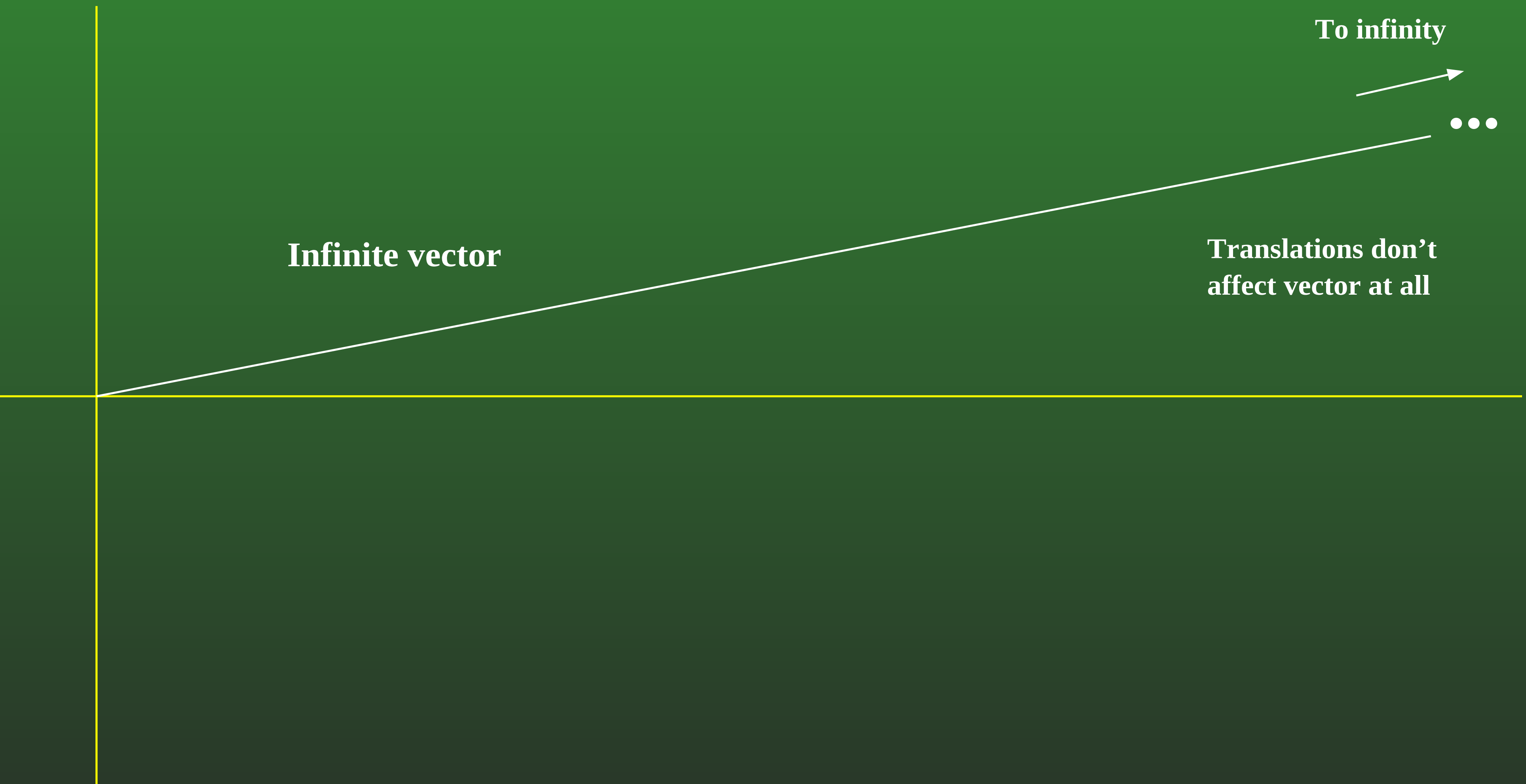
# 07-54: "Point at Infinity"

- If we have "point at infinity", then having the vector be affected by rotation (and non-uniform scaling, and shearing, etc.), but not translation makes sense

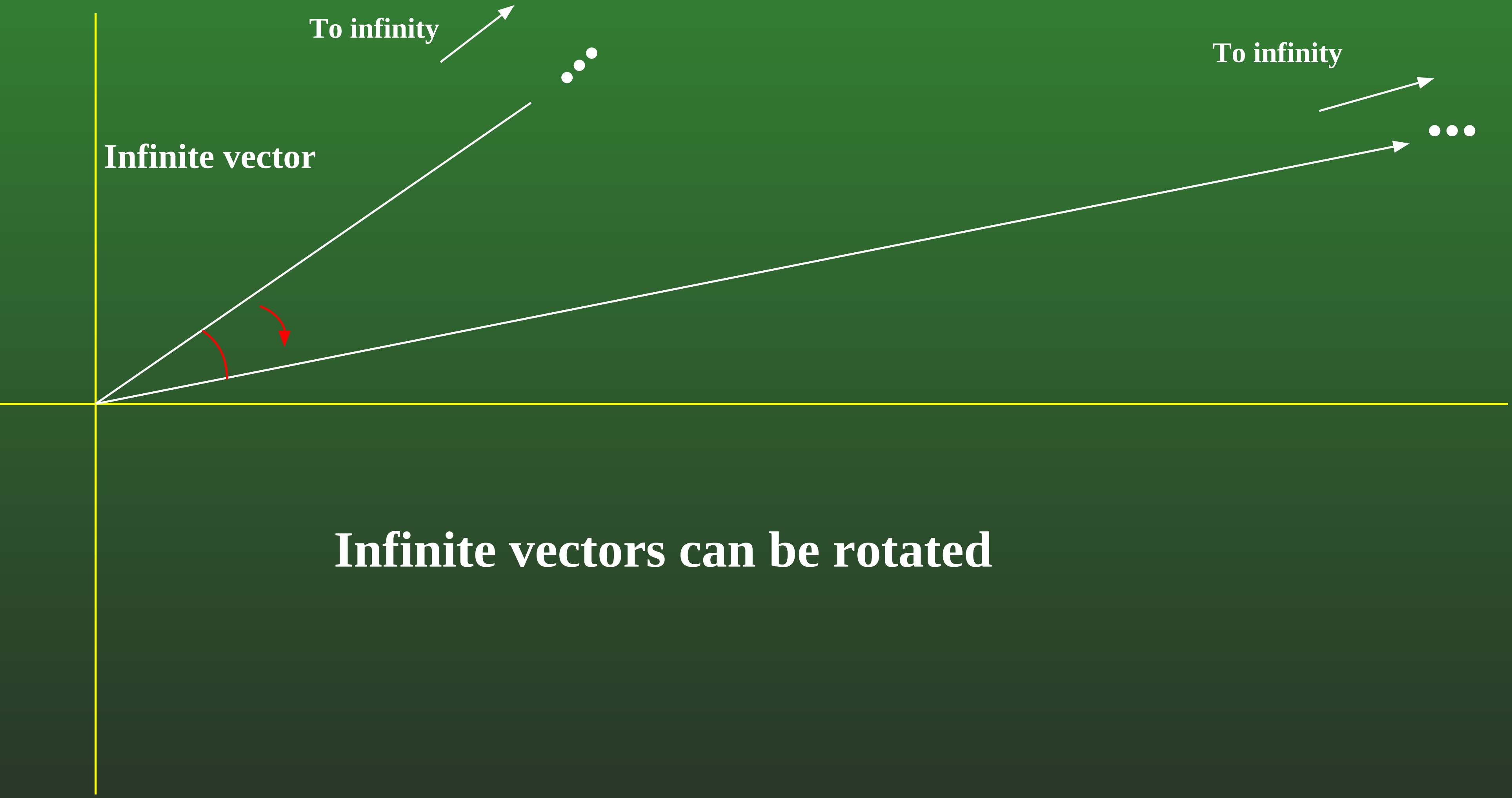
# 07-55: "Point at Infinity"



# 07-56: "Point at Infinity"



# 07-57: "Point at Infinity"




Infinite vectors can be rotated

# 07-58: Homogenous Dimension = 0

- We can “Turn off” translation by setting $w = 0$
- Handy for when we want direction only, not position
  - Surface normals are an excellent example of when we want rotation to affect the vector, but not translation

# 07-59: Review

- We can describe the orientation of an object using a rotation matrix
  - Describes how to transform (rotate) points in the object from object space to inertial space
  - Example: Rotate 45 degrees around the Z-axis

$$\begin{bmatrix} \cos\pi/4 & \sin\pi/4 & 0 \\ -\sin\pi/4 & \cos\pi/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# 07-60: Review

- We have a point at position $[x_1, y_1, z_1]$ in object space (That's how points in the mesh are stored)
- We need to know the position of the point in world space before rendering (assume no translation yet – our model is at the origin)
- We can do a simple multiply:

$$[x_1, y_1, z_1] \begin{bmatrix} \cos \pi/4 & \sin \pi/4 & 0 \\ -\sin \pi/4 & \cos \pi/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# 07-61: Review

- So, our rotation matrix gives us a way to transform points from object space into world space
- Rotation matrix also tells us where our object is facing in world space, and what the up vector of our object is in world space
- How?

# Review

- So, our rotation matrix gives us a way to transform points from object space into world space
- Rotation matrix also tells us where our object is facing in world space, and what the up vector of our object is in world space
  - We know the direction our object is facing in local space: $[0, 0, 1]$
  - If we transfrom this by a matrix, what do we get?

# 07-63: Review

- So, our rotation matrix gives us a way to transform points from object space into world space
- Rotation matrix also tells us where our object is facing in world space, and what the up vector of our object is in world space
  - We know the direction our object is facing in local space: $[0, 0, 1]$
  - If we transfrom this by a matrix we get the bottom row of the matrix

# 07-64: Review

- Of course, our objects are not always at the origin
- In addition to the rotational matrix, we also have a position – location of the center of the model
- Now, to transform a point, we first rotate it, and then translate it
  - Rotation matrix for our model: $M_R$
  - Position of our object (displacement from the origin): $\mathbf{pos} = [x_m, y_m, z_m]$
- How can we transform a point $[x, y, z]$ in the object space of this model into world space?

# 07-65: Review

- Rotation matrix for our model: $M_R$
- Position of our object (displacement from the origin): $\mathbf{pos} = [x_m, y_m, z_m]$
  - How can we transform a point $\mathbf{p_O} = [x, y, z]$ in the object space of this model into world space?

World space $\mathbf{p_w} = \mathbf{p_O M_R} + \mathbf{pos}$

# Review

- As a mathematical trick, we can combine our rotation matrix and position into a single entity

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & 0 \\ m_{21} & m_{22} & m_{23} & 0 \\ m_{31} & m_{32} & m_{33} & 0 \\ x & y & z & 1 \end{bmatrix}$$

- Now, to transform a point, convert it to a 4-element vector (by adding a 1 at the end), multiply by this matrix, look at first 3 elements of the vector

# Review

- A 4x4 matrix represents a rotation, followed by a translation
- We can combine multiple transformations by multiplying matrices together
- Result is a single matrix, which represents a single rotation, followed by a single translation.

# 07-68: Review

- Example: Finding the end of a tank barrel
  - Tank has a location and rotation in world space (represented by a position vector and 3x3 rotation matrix)
  - Barrel has a location and rotation (represented by a position vector and 3x3 rotation matrix – reltaive to the center of the tank
  - End of the tank barrel is at location $[0, 0, 3]$ in barrel space
- What is the location of the end of the tank barrel in world space? (do both 3x3 matrices & positions, and 4x4 matrices)

# Review

- Given:
  - A bullet position in world space $\mathbf{p}_b = [b_x, b_y, b_z]$
  - A bullet position in world space $\mathbf{v}_b = [bv_x, bv_y, bv_z]$
  - A rotation matrix for a tank $\mathrm{M}_T$, and a position for a tank $\mathbf{p}_T$
  - What is the position and velocity of the bullet in tank space?
  - Why might that be a useful thing to have?

# Row vs. Column Vectors

- Row Vectors
  - Rows of the matrix represent transform of object (1st row is x, 2nd row is y, 3rd row is z)
  - To transform a vector $\mathbf{v}$ by first $\mathbf{A}$, then $\mathbf{B}$, then $\mathbf{C}$: $\mathbf{vABC}$
- Column Vectors
  - Columns of the matrix represent transform of object (1st col is x, 2nd col is y, 3rd col is z)
  - To transform a vector $\mathbf{v}$ by first $\mathbf{A}$, then $\mathbf{B}$, then $\mathbf{C}$: $\mathbf{CBAv}$

# 07-71: Row vs. Column Vectors

- 4x4 Matrx using Row vectors:

$$\begin{bmatrix} x_1 & x_2 & x_3 & 0 \\ y_1 & y_2 & y_3 & 0 \\ z_1 & z_2 & z_3 & 0 \\ \Delta x & \Delta y & \Delta z & 1 \end{bmatrix}$$

- 4x4 Matrx using Column vectors:

$$\begin{bmatrix} x_1 & y_1 & z_1 & \Delta x \\ x_2 & y_2 & z_2 & \Delta y \\ x_3 & y_3 & z_3 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# 07-72: Row vs. Column Vectors

- Ogre & OpenGL use column vectors
- Direct3D uses row vectors
- How does Ogre do both?
  - Does everything in column vectors
  - Multiplies matrices together using column vector convetion
  - When it’s time to send a matrix to D3D, does a quick transpose first

# 07-73: Rotational Matrix Trick

- To remember how to create rotational matrices for the cardinal axes, you just need to remember: cos, sin, -sin, cos
  - If you forget, do the 2D case
- Create 3x3 rotational matrix with the non-rotating vector in the correct location
- From the one in the non-rotating vector, go down and right, and fill in cos,sin,-sin,cos
  - Wrap around as necessary
  - Examples (column major and row major)