Game Engineering: 2D

CS420-2010F-06

2D Math

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06-0: Back to Basics

- A Vector is a displacement
- Vector has both *direction* and *length*
- Can also think of a vector as a position (really a displacement from the origin)
- Can be written as a row or column vector
  - Differnence can be important for multiplication

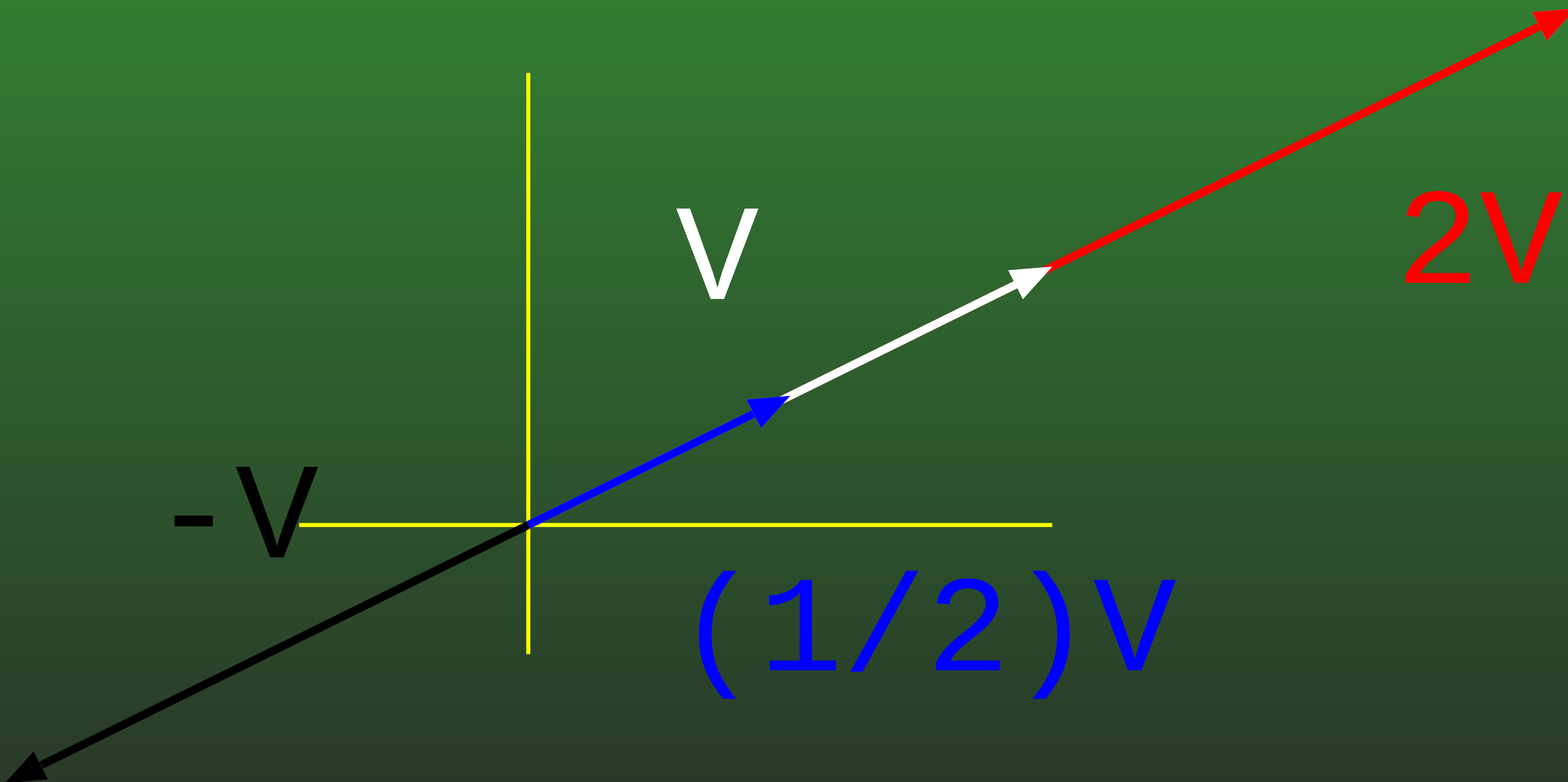
06-1: Vector Operations

- Multiplying by a scalar
  - To multiply a vector $\mathbf{v}$ by a scalar s, multiply each component of the vector by s
  - Effect is scaling the vector – multiplying by 2 maintains the direction of the vector, but makes the length twice as long

06-2: Vector Operations

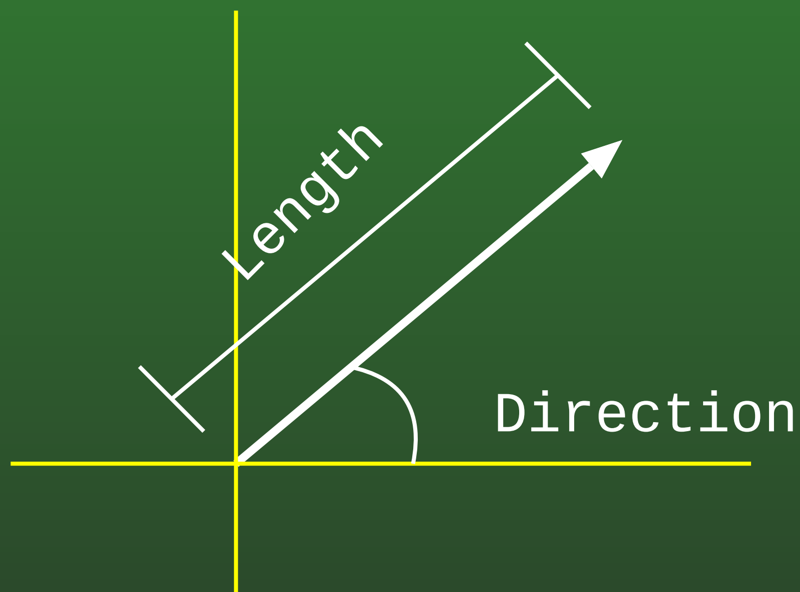
- Multiplying by a scalar
  - Multiplying a vector by -1 flips the direction of the vector
  - Multiplying a vector by -2 both flips the direction, and scales the vector

06-3: Scaling a Vector



06-4: Length

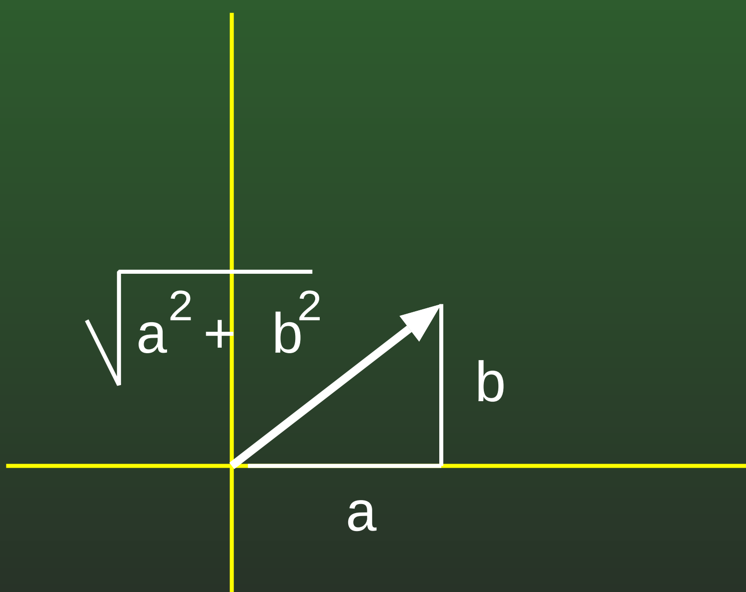
- Vector has both direction and length



06-5: Length

- Vector $\mathbf{v} = [v_x, v_y]$
- Length of $\mathbf{v}$:

$$||\mathbf{v}|| = \sqrt{v_x^2 + v_y^2}$$

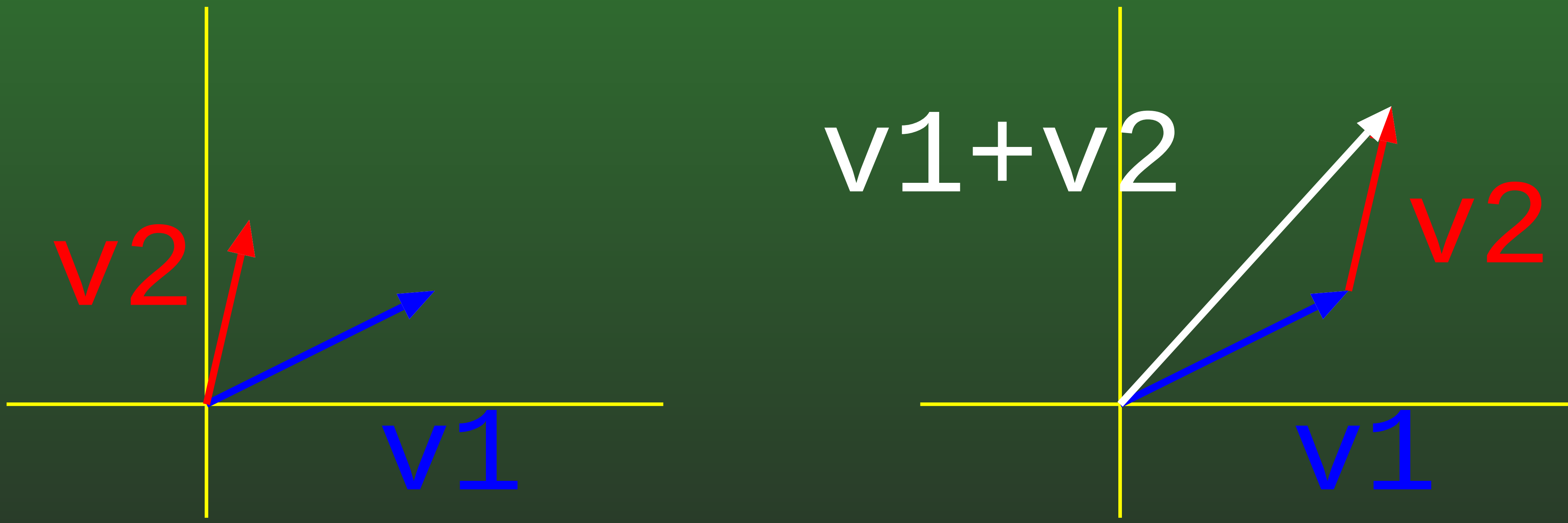


06-6: Normalizing a Vector

- Normalize a vector by setting its length to 1, but maintining its direction.
- Multiply by 1/length
- $\mathbf{v}_{norm} = \frac{\mathbf{v}}{||\mathbf{v}||}$
- Of course, v can't be the zero vector
  - Zero vector is the only vector without a direction

06-7: Vector Addition

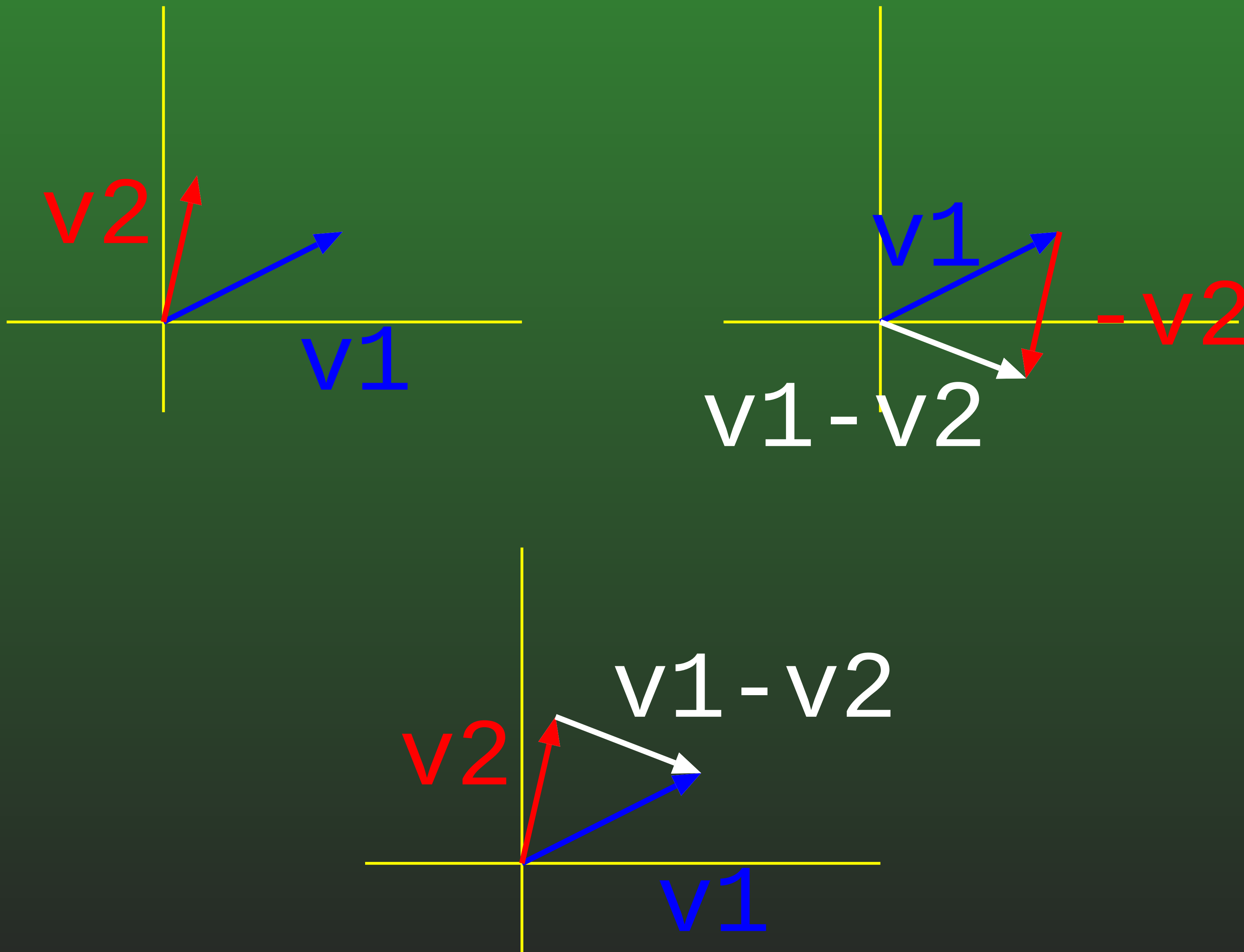
- Add two vectors by adding their components
- $[u_x, u_y] + [v_x, v_y] = [u_x + v_x, u_y + v_y]$



06-8: Vector Subtraction

- Vector subtraction is the same as multiplying by -1 and adding
- $\mathbf{v_1} - \mathbf{v_2}$ is the displacement from the point at $\mathbf{v_2}$ to the point at $\mathbf{v_1}$
  - *not* the displacement from $\mathbf{v_1}$ to $\mathbf{v_2}$

06-9: Vector Subtraction



06-10: Point Distance

- We can use subtraction and length to find the distance between two points
- Represent points as vectors – displacement from the origin
- Distance from $\mathbf{v}$ to $\mathbf{u}$ is $||\mathbf{v} - \mathbf{u}|| = ||\mathbf{u} - \mathbf{v}||$
  - Where $||\mathbf{v}||$ is the length of the vector $\mathbf{v}$.

06-11: Point Distance

- Point p_1: $[1, 7]$
- Point p_2: $[4, 3]$
- Distance between p_1 and p_2?

06-12: Point Distance

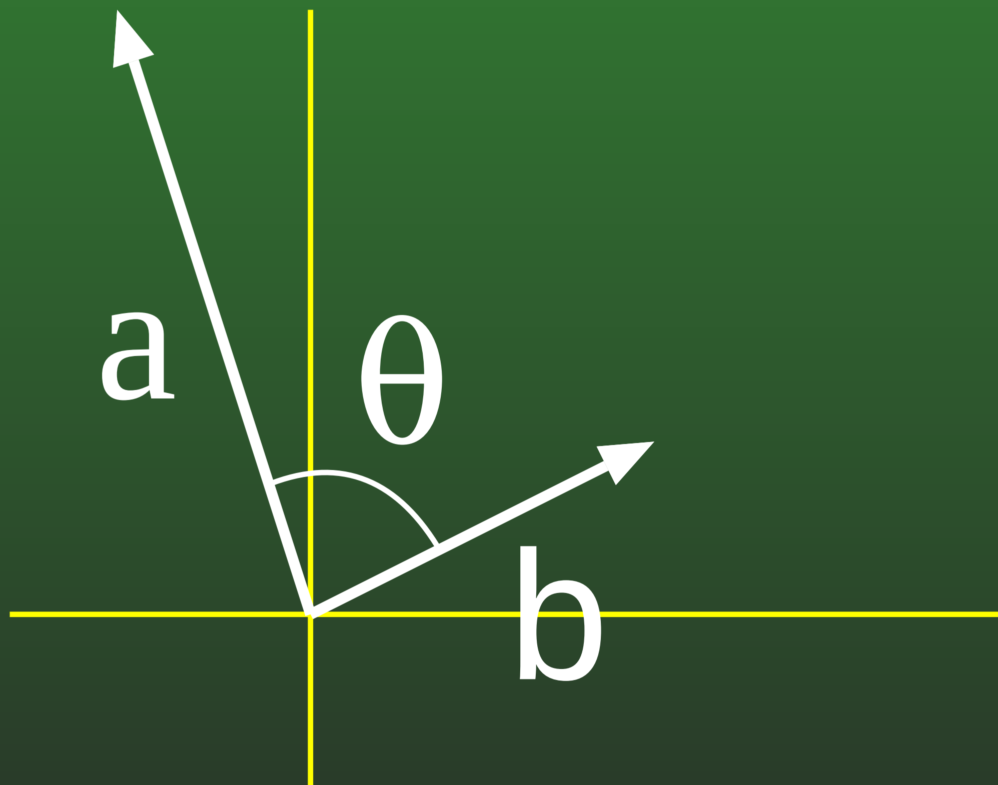
- $p_1 - p_2 = [1, 7]$
- Point p_2: $[4, 3]$
- Distance between p_1 and p_2
  - $[1, 7] - [4, 3] = [-3, 4]$
  - $\sqrt{(-3)^2 + 4^2} = 5$

06-13: Dot Product

- $a = [a_x, a_y]$
- $b = [b_x, b_y]$
- $a \cdot b = a_x * b_x + a_y * b_y$

06-14: Dot Product

$$a \cdot b = ||a|| * ||b|| * \cos\theta$$



06-15: Dot Product

$$\theta = \arccos\left(\frac{a \cdot b}{||a||||b||}\right)$$

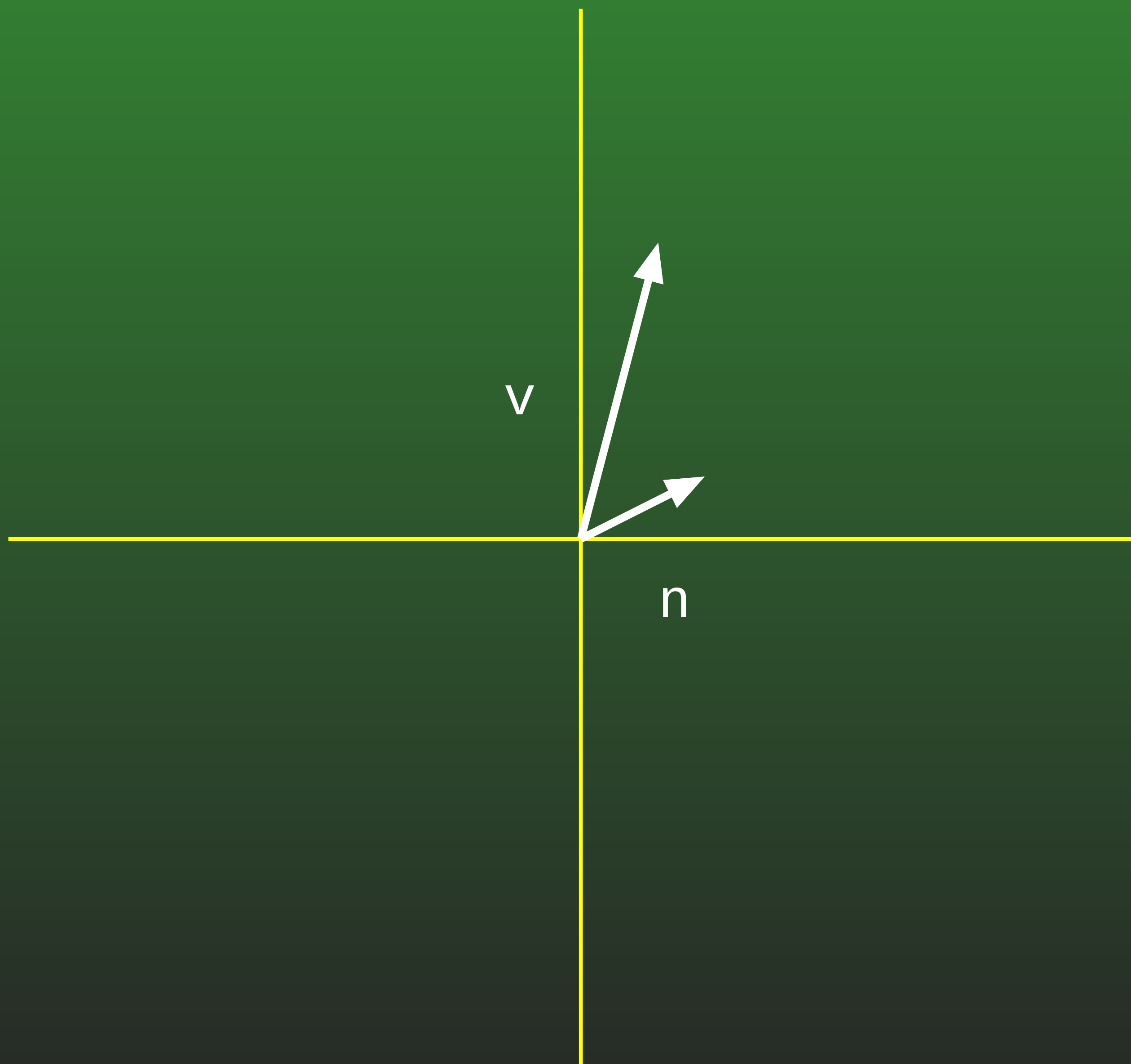
If a and b are unit vectors:

$$\theta = \arccos\left(a \cdot b\right)$$

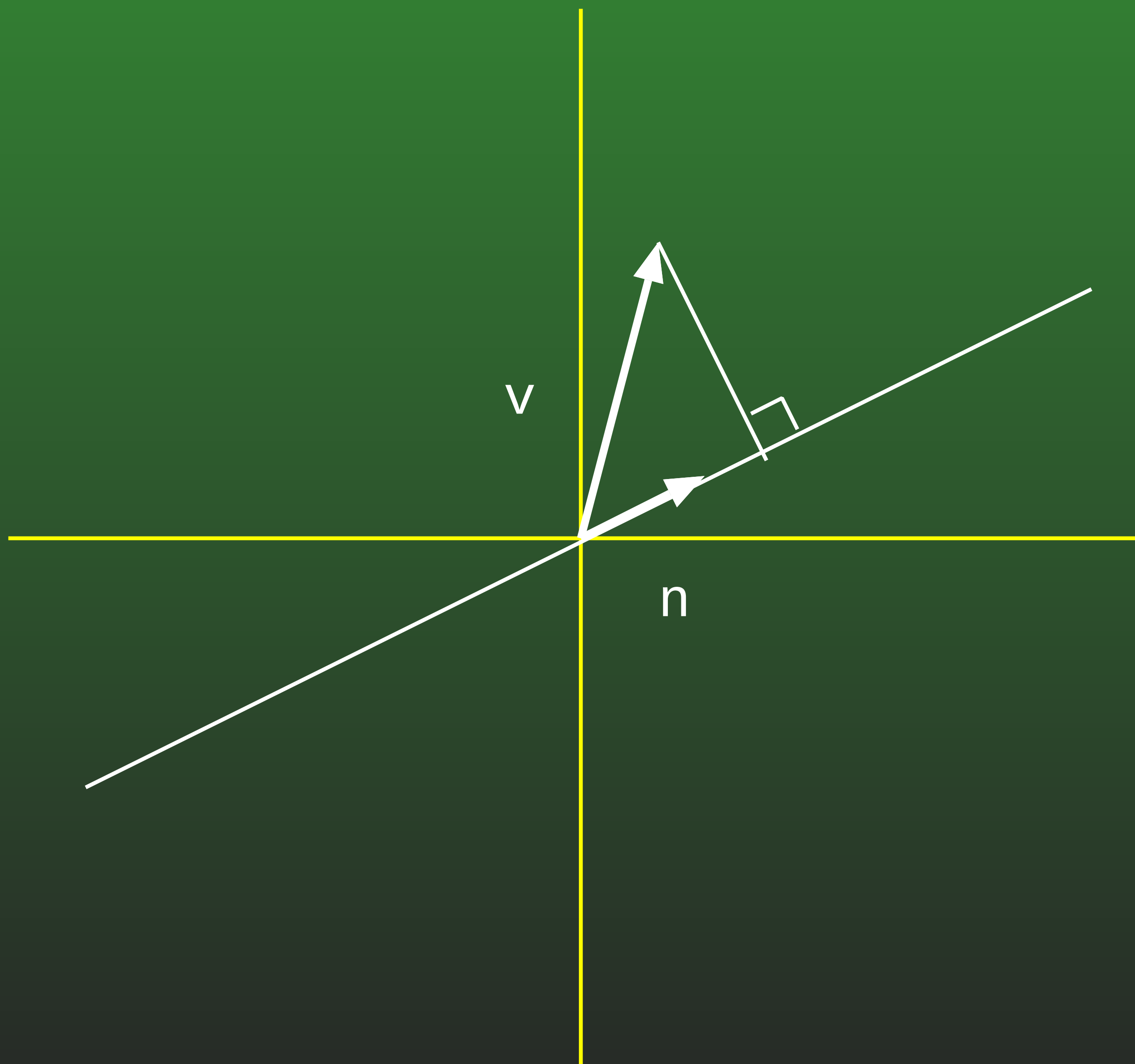
06-16: Dot Product

- If we don't need the exact angle, we can just use the sign
  - If $\theta < 90$, $\cos\theta > 0$
  - If $\theta = 90$, $\cos\theta = 0$
  - If $90 < \theta < 180$, $\cos\theta < 0$
- Since $a \cdot b = ||a||||b|| \cos\theta$:
  - If $a \cdot b > 0$, $\theta < 90(\frac{\pi}{2})$
  - If $a \cdot b = 0$, $\theta = 90(\frac{\pi}{2})$
  - If $a \cdot b < 0$, $90 < \theta < 180$
    - $\frac{\pi}{2} < \theta < \pi$

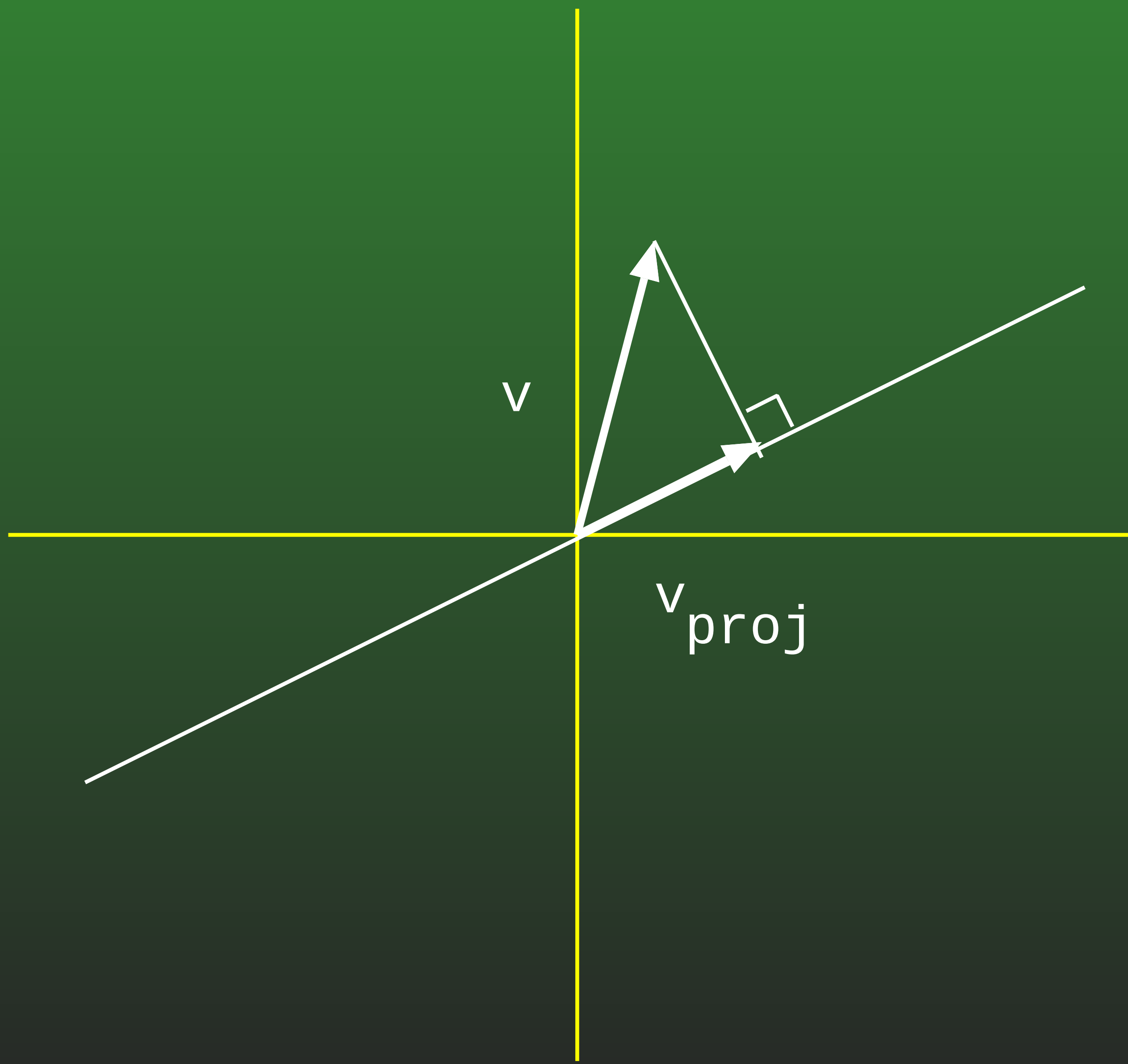
06-17: Projecting Vectors



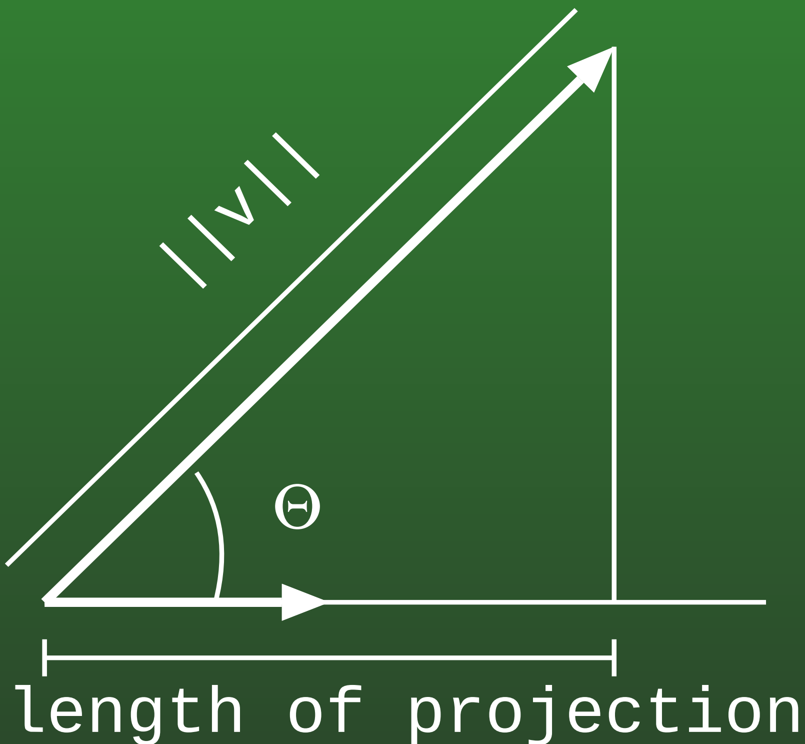
06-18: Projecting Vectors



06-19: Projecting Vectors



06-20: Projecting Vectors



$$\cos \Theta = \frac{\text{length of projection}}{||v||}$$

06-21: Projecting Vectors

- Given a vector v, and a unit vector n, find the projection of v onto n
  - length of projection l:

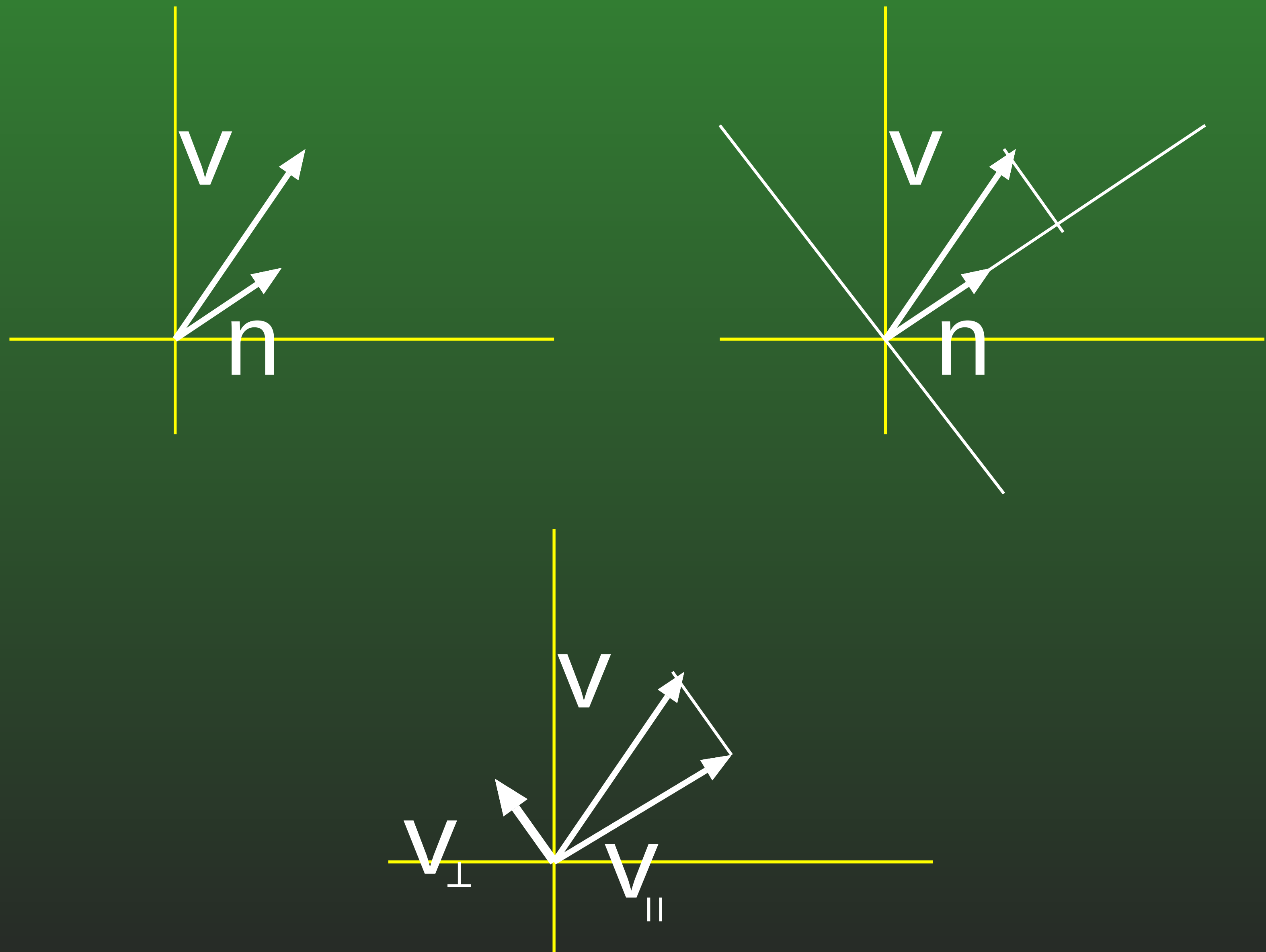
$$\begin{aligned} \cos\theta &= \frac{l}{||v||} \\ l &= \cos\theta||v|| \\ &= \cos\theta||v|| * ||n|| \\ &= v \cdot n \end{aligned}$$

  - Since n is a unit vector, projection = length * n
  - $(v \cdot n) * n$

06-22: Projecting Vectors

- Given vectors v and n, we want to decompose v into two vectors, $v_{\|}$ and $v_{\perp}$
  - $v_{\|}$ is parallel to n
  - $v_{\perp}$ is perpendicular to n
  - $v = v_{\|} + v_{\perp}$

06-23: Projecting Vectors



06-24: Projecting Vectors

- $v_{\parallel}$ is just the projection of v onto n

$$v_{\parallel} = (v \cdot n) * n$$

(if n is not a unit vector, then we will need to normalize)

$$\begin{aligned} v_{\parallel} &= \left(v \cdot \frac{n}{||n||}\right) * \frac{n}{||n||} \\ &= \frac{v \cdot n}{||n||^2} n \end{aligned}$$

06-25: Projecting Vectors

- Once we have $v_{\|}$, finding $v_{\perp}$ is easy, since $v = v_{\|} + v_{\perp}$

$$\begin{aligned} v_{\|} + v_{\perp} &= v \\ v_{\perp} &= v - v_{\|} \\ v_{\perp} &= v - n\frac{v \cdot n}{||n||^2} \end{aligned}$$

06-26: Projecting Vectors

- Sanity Check: What happens if we try to find the componets of the vector $[v_1, v_2]$ that are parallel and perpendicular to the x-axis and y-axis – what should we get?

Projecting Vectors

- Sanity Check: What happens if we try to find the componets of the vector $[v_x, v_y]$ that are parallel and perpendicular to the x-axis and y-axis – what should we get?
  - $[v_x, 0]$ and $[0, v_y]$
- Let's make sure that's what we get!
- Start with the component parallel to x-axis

06-28: Projecting Vectors

- Component of $[v_x, v_y]$ that is parallel to the x-axis:
  - Length of component parallel to x-axis:
    - $v \cdot n = [v_x, v_y] \cdot [1, 0] = v_x * 1 + v_y * 0 = v_x$
  - Component parallel to x-axis:
    - $(v \cdot n) * n = v_x * [1, 0] = [v_x, 0]$

06-29: Projecting Vectors

- Component of $[v_x, v_y]$ that is perpendicular to the x-axis:
  - $v_\perp = v - v_\parallel$
  - $v_\perp = [v_x, v_y] - [v_x, 0]$
  - $v_\perp = [0, v_y]$

06-30: Matrices

- A 2x2 matrix M:

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

06-31: Matrices

- A diagonal matrix is a square matrix with non-diagonal elements equal to zero

$$\mathbf{M} = \begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix}$$

Matrices

- The $Identity\ Matrix$ is a diagonal matrix with all diagonal elements = 1

$$\mathbf{I_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

06-33: Matrices

- Matrices and vectors
  - Vectors are a special case of matrices
  - Row vectors (as we've seen so far) $[x, y]$
  - Column vectors : $\begin{bmatrix} x \\ y \end{bmatrix}$

06-34: Matrices

- Transpose
  - Written $\mathbf{M}^{\mathbf{T}}$
  - Exchange rows and colums

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Transpose

- The transpose of a row vector is a column vector
- For any matrix M, $(M^T)^T = M$
- For a diagonal matrix D, $D^T = ?$

06-36: Matrix Multiplication

- Multiplying a Matrix by a scalar
  - Multiply each element in the Matrix by the scalar
  - Just like multiplying a vector by a scalar

$$k\mathbf{M} = k \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} km_{11} & km_{12} \\ km_{21} & km_{22} \end{bmatrix}$$

06-37: Matrix Multiplication

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

06-38: Matrix Multiplication

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$c_{11} = a_{11} b_{11} + a_{12} b_{21}$$

06-39: Matrix Multiplication

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$c_{21} = a_{21} b_{11} + a_{22} b_{21}$$

06-40: Matrix Multiplication

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$c_{12} = a_{11} b_{12} + a_{12} b_{22}$$

06-41: Matrix Multiplication

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$c_{22} = a_{21} b_{12} + a_{22} b_{22}$$

06-42: Matrix Multiplication

- Vectors are special cases of matrices
- Multiplying a vector and a matrix is just like multiplying two matrices

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} =$$

$$\begin{bmatrix} xm_{11} + ym_{21} & xm_{12} + ym_{22} \end{bmatrix}$$

Matrix Multiplication

- Vectors are special cases of matrices
- Multiplying a vector and a matrix is just like multiplying two matrices

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xm_{11} + ym_{12} \\ xm_{21} + ym_{22} \end{bmatrix}$$

06-44: Matrix Multiplication

- Note that the following multiplications are not legal:

$$\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix}$$

06-45: Matrix Multiplication

- Matrix Multiplicaton is not commutative: $AB \neq BA$ (at least not for all A and B – is it true for at least one A and B?)
- Matrix Multiplication is associative: $(AB)C = A(BC)$
- Transposing product is the same as the product of the transpose, in reverse order: $(AB)^T = B^T A^T$

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

06-46: Matrix Multiplication

- Identity Matrix I:
  - $AI = A$ (for appropriate I)
  - $IA = A$ (for appropriate I)

$$\begin{bmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{bmatrix}$$

Matrix Multiplication

- Identity Matrix I:
  - $AI = A$ (for appropriate I)
  - $IA = A$ (for appropriate I)

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

06-48: Matrix Multiplication

- Identity Matrix I:
  - $AI = A$ (for appropriate I)
  - $IA = A$ (for appropriate I)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

06-49: Row vs. Column Vectors

- A vector can be reresented as a row vector or a column vector
- This makes a difference when using matrices
  - Row: $\mathbf{vA}$, Column $\mathbf{Av}$
- It gets even more fun when using matrices to do several transformations of a vector:
  - Row $\mathbf{vABC}$, Column $\mathbf{CBAv}$ (note that to get the same transformation, you need to take the transpose of A, B, and C when swapping between row and column vectors

06-50: Row vs. Column Vectors

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} =$$

$$\begin{bmatrix} xa + yc & xb + yd \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} =$$

$$\begin{bmatrix} (xa + yc)e + (xb + yd)g & (xa + yc)f + (xb + yd)h \end{bmatrix}$$

$$\begin{bmatrix} e & g \\ f & h \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} =$$

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} ax + cy \\ bx + dy \end{bmatrix} =$$

$$\begin{bmatrix} e(ax + cy) + g(ax + cy) \\ f(ax + cy) + h(ax + yd) \end{bmatrix}$$