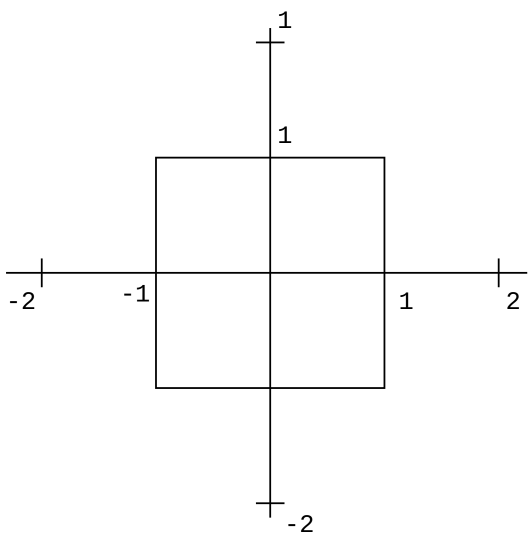
07-0: **Representing Polygons**

- We want to represent a simple polygon
  - Triangle, rectangle, square, etc
  - Assume for the moment our game only uses these simple shapes
  - No curves for the moment ...
- How could we represent one of those polygons?

07-1: **Representing Polygons**

- We want to represent a simple polygon
  - Triangle, rectangle, square, etc
  - Assume for the moment our game only uses these simple shapes
  - No curves for the moment ...
- How could we represent one of those polygons?
  - List of points, starting with some arbitrary point, moving "clockwise" around the polygon

07-2: **Representing Polygons**




`(1,1), (1,-1), (-1,-1), (-1,1)`

07-3: **Representing Polygons**

- How could we represent a simple circle?

07-4: **Representing Polygons**

- How could we represent a simple circle?
  - Center position (Vector)
  - Radius (scalar)

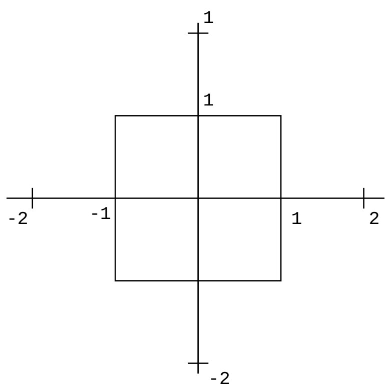
07-5: **Modifying Polygons**

- What if we wanted to modify a polygon
  - Translation, rotation, scaling, etc
- Start with an easy one – how can we translate (move) a polygon?

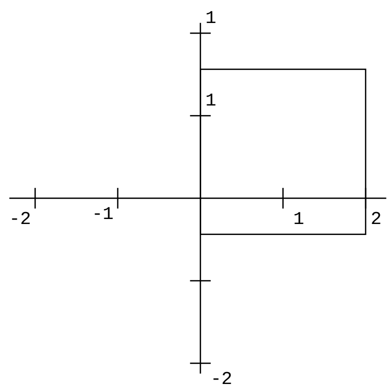
07-6: **Translating Polygon**

- To translate a polygon, all we need to do is translate each one of its points
  - Move a polygon over 1 unit, up 0.5 units
    - Add (1, 0.5) to each point
    - Points are just translations from origin

07-7: **Translating Polygon**




(1,1), (1,-1), (-1,-1), (-1,1)

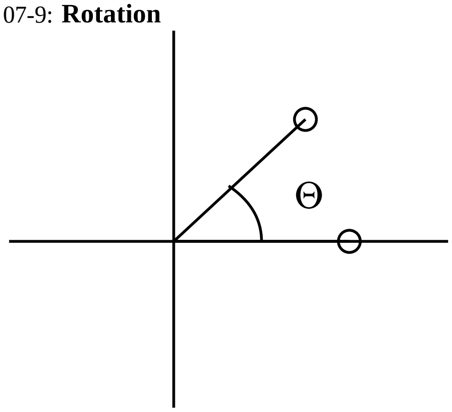



Translate over 1, up 0.5
Add (1,0.5) to each point in polygon
(2,1.5), (2,-0.5), (0,-.5), (-0,1.5)

07-8: **Rotation**

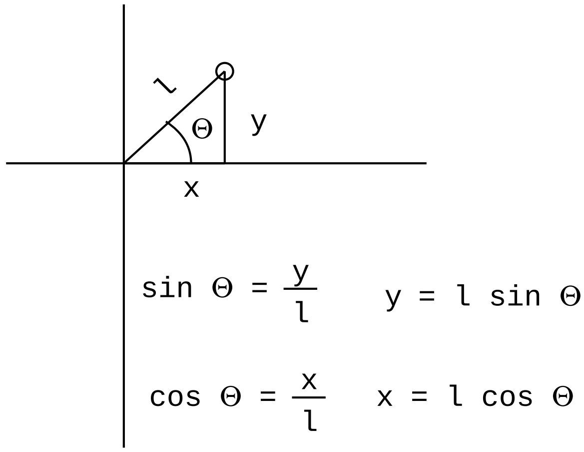
- Rotations are a bit tricky
- Start with a simnplier case
  - Rotate a point around the origin

07-9: **Rotation**



- Rotate the point (0,x) counterclockwise around the origin by $\Theta$ degrees
- What is the new point?

07-10: **Rotation**




- Rotate the point (0,x) counterclockwise around the origin by $\Theta$ degrees
- What is the new point?

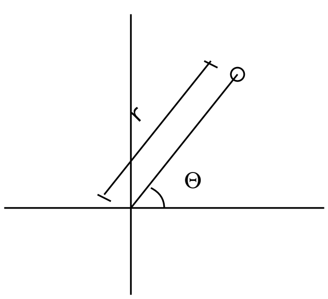
07-11: **Rotation**

- Original point is at $(x, 1)$
  - distance from the origin $l = x$
- New x position = $x \cos \Theta$
- New y postion = $x \sin \Theta$

Was easy beacuse the disance from the origin $l$ was easy to calculate. What if original point was not on an axis?

07-12: **Rotation**

- Rotating a point *not* on an axis
- Use polar coordinates!

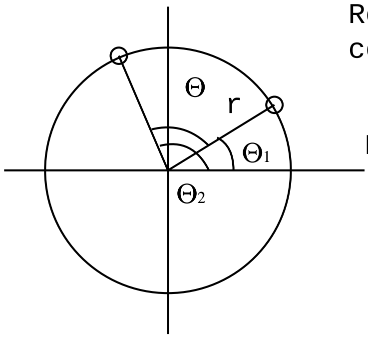



07-13: **Rotation**

- Using polar coordinates for rotation sounds kind of like cheating
  - Of course it is easy to rotate in polar coordinates!

- Translation is harder though ...
- (Probably) don't want to write all our game logic using polar coordinates
  - Depends on the game ...
- Transform into polar coordinates, do rotation, transform back. Hope everything simplifies nicely!

07-14: **Rotation**



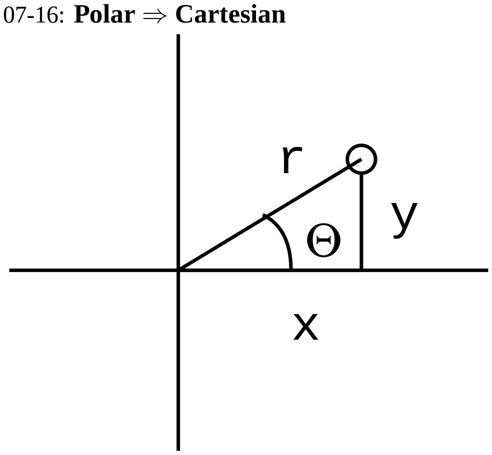

Rotate point at $(r, \Theta_1)$ $\Theta$ degrees counterclockwise

New point (r, Q2) = $(r, \Theta_1 + \Theta)$

07-15: **Rotation**

- Conversion from Polar coordinates to Cartesian coordinates
  - Given a point $(r, \Theta)$ in Polar coordinates, how can we create a point $(x, y)$ in Cartesian coordinates?

07-16: **Polar ⇒ Cartesian**




$y = r \sin \Theta$

$x = r \cos \Theta$

07-17: **Rotation**

- Point $p_1$ at $(r, \Theta_1)$, rotate $p_1$ by $\Theta$
- New point $p_2$ at $(r, \Theta_1 + \Theta)$
- In Cartesian coordinates:

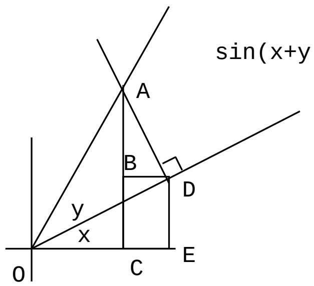
07-18: **Rotation**

- Point $p_1$ at $(r, \Theta_1)$, rotate $p_1$ by $\Theta$
- New point $p_2$ at $(r, \Theta_1 + \Theta)$
- In Cartesian coordinates:

- $x = r\cos(\Theta_1 + \Theta)$
- $y = r\sin(\Theta_1 + \Theta)$

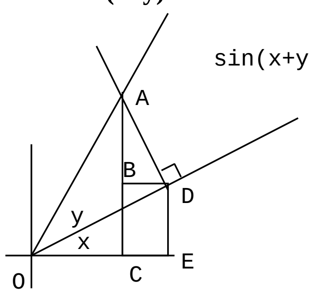
- How do we compute $\sin(\Theta_1 + \Theta)$?

07-19: **sin(x+y)**



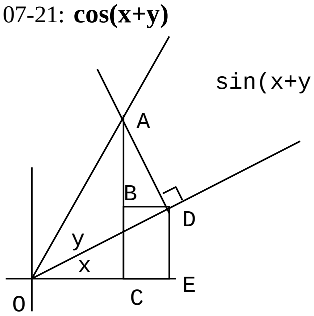

```
sin(x+y)=
```

07-20: **sin(x+y)**



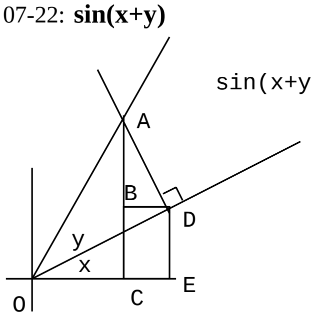

```
sin(x+y)= (AB + BC) / OA
```

07-21: **cos(x+y)**



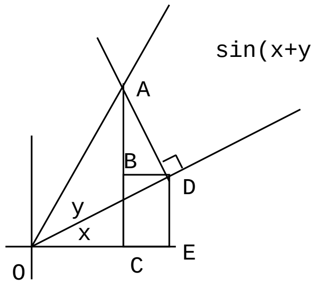

```
sin(x+y)= (AB + BC) / OA
        = (AB + DE) / OA
        = AB / OA + DE / OA
```

07-22: **sin(x+y)**



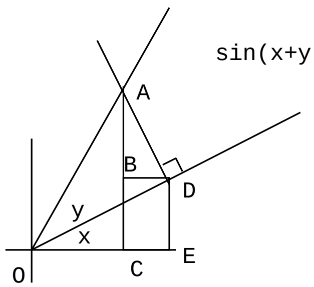

```
sin(x+y)= (AB + BC) / OA
        = (AB + DE) / OA
        = AB / OA + DE / OA
        = (AB / OA) (AD / AD) + (DE / OA)(OD / OD)
```

07-23: **sin(x+y)**



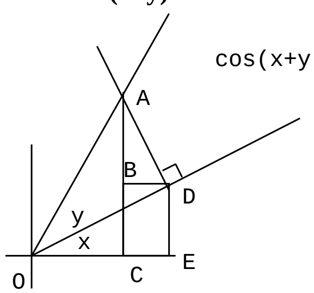

```
sin(x+y)= (AB + BC) / OA
        = (AB + DE) / OA
        = AB / OA + DE / OA
        = (AB / OA) (AD / AD) + (DE / OA)(OD / OD)
        = (AB / AD) (AD / OA) + (DE / OD)(OD / OA)
```

07-24: **sin(x+y)**



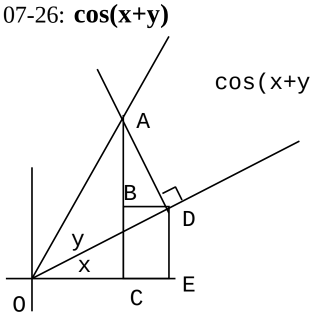

```
sin(x+y)= (AB + BC) / OA
        = (AB + DE) / OA
        = AB / OA + DE / OA
        = (AB / OA) (AD / AD) + (DE / OA)(OD / OD)
        = (AB / AD) (AD / OA) + (DE / OD)(OD / OA)
        = (cos x)(sin y) + (sin x)(cos y)
```

07-25: **cos(x+y)**



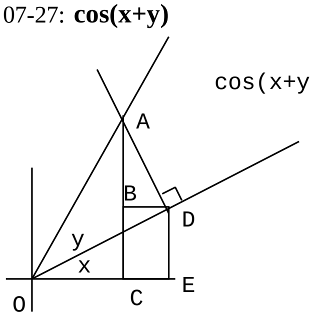

```
cos(x+y)=
```

07-26: **cos(x+y)**



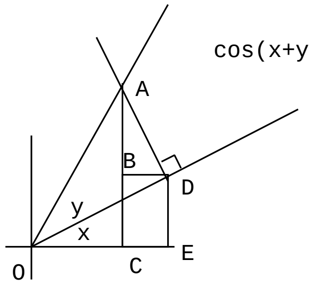

```
cos(x+y)= OC / OA
```

07-27: **cos(x+y)**



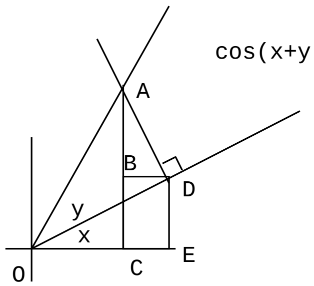

```
cos(x+y)= OC / OA
        = (OE - CE) / OA
```

07-28: **cos(x+y)**



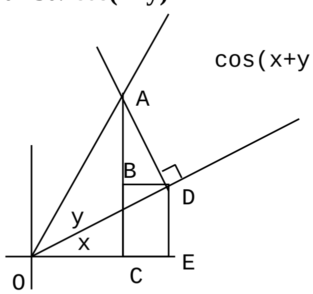

```
cos(x+y)= OC / OA
        = (OE - CE) / OA
        = (OE - BD) / OA
```

07-29: **cos(x+y)**



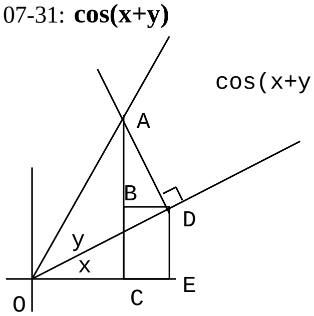

```
cos(x+y)= OC / OA
        = (OE - CE) / OA
        = (OE - BD) / OA
        = OE / OA - BD / OA
```

07-30: **cos(x+y)**




```
cos(x+y)= OC / OA
        = (OE - CE) / OA
        = (OE - BD) / OA
        = OE / OA - BD / OA
        = (OE/OA)(OD/OD) - (BD/OA)(AD/AD)
        = (OE/OD)(OD/OA) - (BD/AD)(AD/OA)
```

07-31: **cos(x+y)**




```
cos(x+y)= OC / OA
        = (OE - CE) / OA
        = (OE - BD) / OA
        = OE / OA - BD / OA
        = (OE/OA)(OD/OD) - (BD/OA)(AD/AD)
        = (OE/OD)(OD/OA) - (BD/AD)(AD/OA)
        = (cos x)(cos y) - (sin x)(sin y)
```

07-32: **Back to Rotation!**

- $x_{new} = r\cos(\Theta_1 + \Theta)$
- $x_{new} = r((\cos\Theta_1)(\cos\Theta) - (\sin\Theta_1)(\sin\Theta))$

07-33: **Back to Rotation!**

- $x_{new} = r\cos(\Theta_1 + \Theta)$
- $x_{new} = r((\cos\Theta_1)(\cos\Theta) - (\sin\Theta_1)(\sin\Theta))$

- $x_{new} = (r(\cos\Theta_1))\cos\Theta - (r(\sin\Theta_1))\sin\Theta$

07-34: **Back to Rotation!**

- $x_{new} = r\cos(\Theta_1 + \Theta)$
- $x_{new} = r((\cos\Theta_1)(\cos\Theta) - (\sin\Theta_1)(\sin\Theta))$
- $x_{new} = (r(\cos\Theta_1))\cos\Theta - (r(\sin\Theta_1))\sin\Theta$
- $x_{new} = x\cos\Theta - y\sin\Theta$

07-35: **Back to Rotation!**

- $y_{new} = r\sin(\Theta_1 + \Theta)$
- $y_{new} = r((\cos\Theta_1)(\sin\Theta) + (\sin\Theta_1)(\cos\Theta))$

07-36: **Back to Rotation!**

- $y_{new} = r\sin(\Theta_1 + \Theta)$
- $y_{new} = r((\cos\Theta_1)(\sin\Theta) + (\sin\Theta_1)(\cos\Theta))$
- $y_{new} = (r(\cos\Theta_1))\sin\Theta + (r(\sin\Theta_1))\cos\Theta$

07-37: **Back to Rotation!**

- $y_{new} = r\sin(\Theta_1 + \Theta)$
- $y_{new} = r((\cos\Theta_1)(\sin\Theta) + (\sin\Theta_1)(\cos\Theta))$
- $y_{new} = (r(\cos\Theta_1))\sin\Theta + (r(\sin\Theta_1))\cos\Theta$
- $y_{new} = x\sin\Theta + y\cos\Theta$

07-38: **Back to Rotation!**

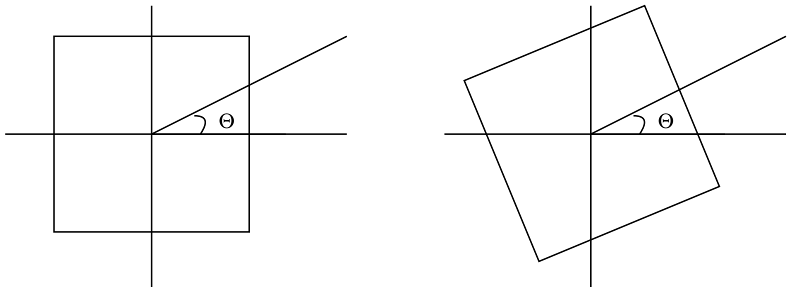
- Given a point $(x, y)$, we can rotate it around the origin as follows:
  - $x_{new} = x\cos\Theta - y\sin\Theta$
  - $y_{new} = x\sin\Theta + y\cos\Theta$
- We can do this with a matrix multiplication

07-39: **Back to Rotation!**

$$\begin{aligned} [x, y] & \begin{bmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{bmatrix} \\ = & \; [x\cos\Theta - y\sin\Theta, x\sin\Theta + y\cos\Theta] \end{aligned}$$

07-40: **Rotating Objects**

- Polygon, consisting of a list of points
- Rotate the polygon by angle $\Theta$ around origin



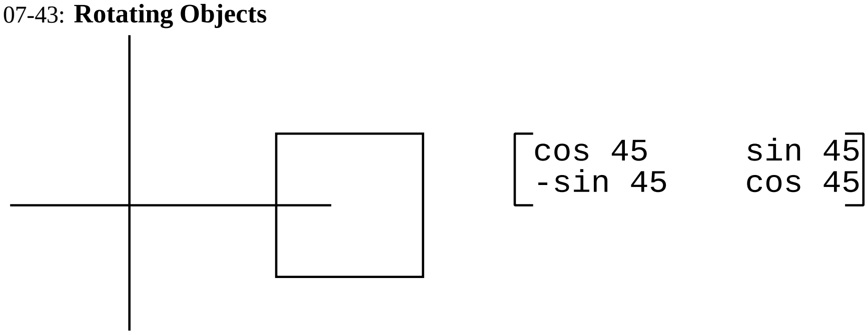

07-41: **Rotating Objects**

- Polygon, consisting of a list of points
- Rotate the polygon by angle $\Theta$ around origin
- Rotate each point individually around the origin
  - Done with a matrix multiplication

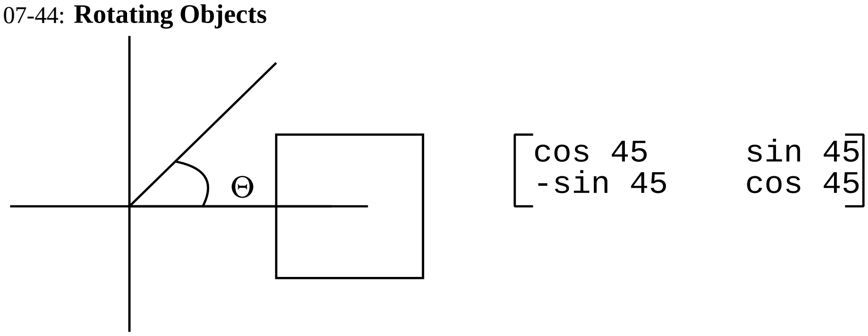
07-42: **Rotating Objects**

- Original Polygon: $p_0, p_1 \ldots p_n$
- New Polygon: $p_0 M, p_1 M, \ldots p_n M$
  - where $M = \begin{bmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{bmatrix}$

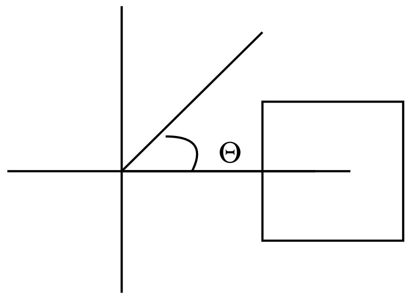
07-43: **Rotating Objects**



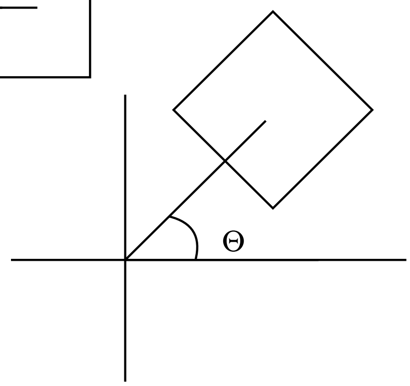

07-44: **Rotating Objects**




07-45: **Rotating Objects**

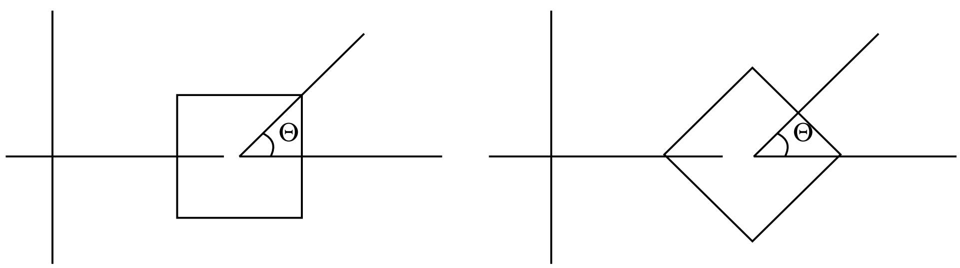



$$\begin{bmatrix} \cos 45 & \sin 45 \\ -\sin 45 & \cos 45 \end{bmatrix}$$




07-46: **Rotating Objects**

- How can we rotate an object around some point *other* than the origin?

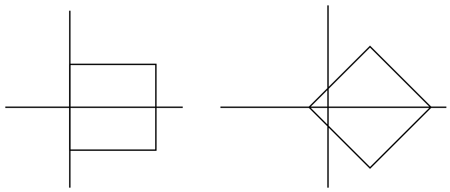



07-47: **Rotating Objects**

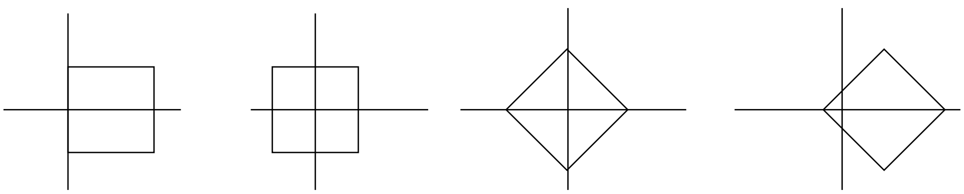
- How can we rotate an object around some point *other* than the origin?
  - Translate to the origin
  - Rotate
  - Translate back

07-48: **Rotating Objects**

**Rotate π/4 around (1,0)**



**Translate to origin, Rotate π/4 around (0,0), Translate back**



07-49: **Multiple Cooridinate Systems**

- World Space
- Camera (Screen) Space
- Inertial Space
- Object Space

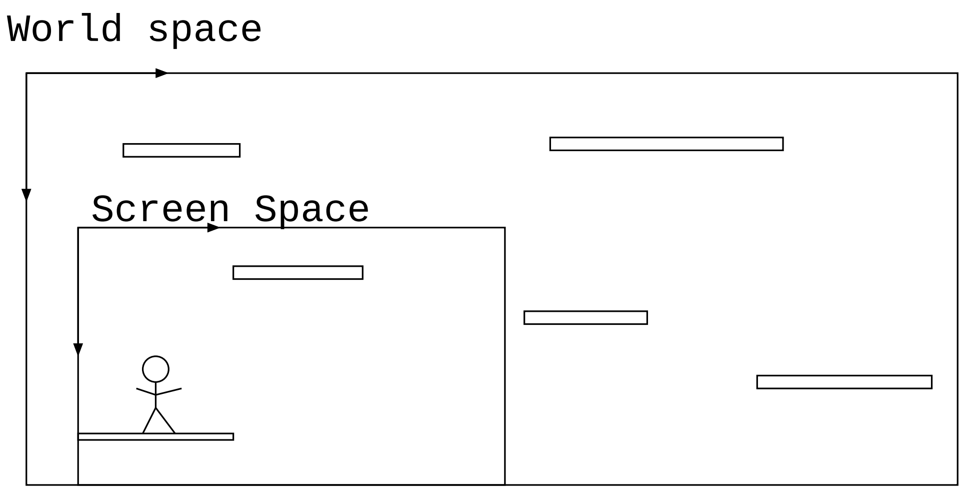
07-50: **World Space**

- Define an origin for your world
  - Could be in the middle of your world, in one corner, etc
- Define each object's position in the world as an offset from thhis point

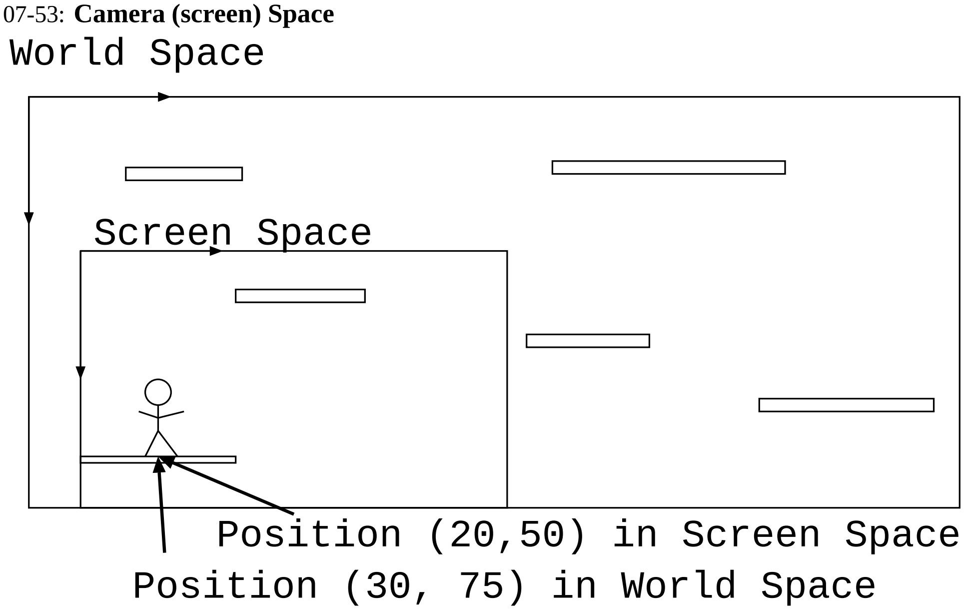
07-51: **Camera (screen) Space**

- Position that object appears on the screen
- Not always the same as world space!
  - Could have a much larger world, that screen scrolls across
  - "zoom in"

07-52: **Camera (screen) Space**




07-53: **Camera (screen) Space**




07-54: **Camera (screen) Space**

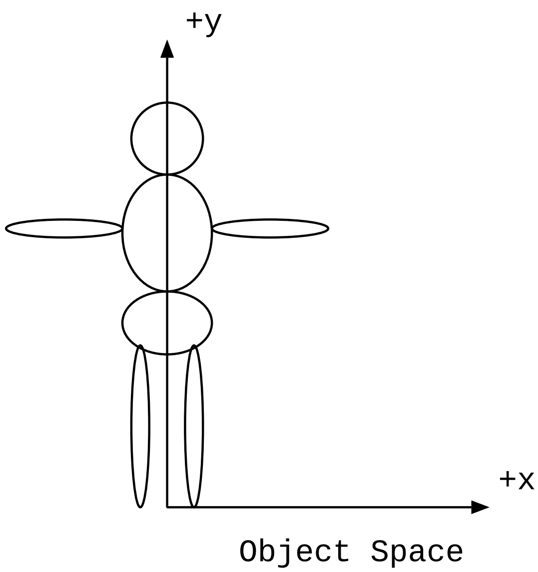
- You *could* calculate everything in Camera space ...
  - Moving camera becomes difficult – need to move all objects in the world along with the camrea
  - Objects are moving on their own, need to combine movements
  - Zooming in becomes problematic

07-55: **Object Space**

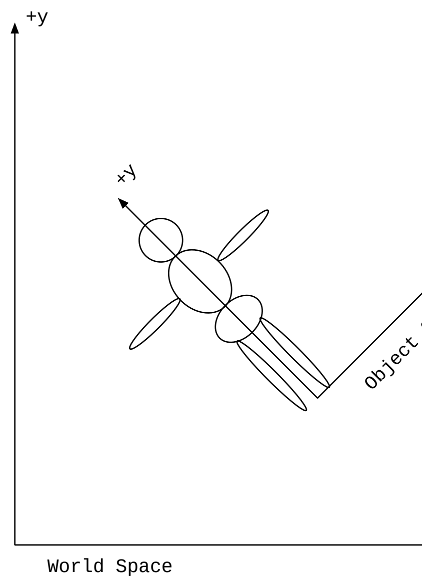
- New Coordinate system based on the object
- Origin is at the base (or center) of the objet
- Axes are nicely aligned

07-56: **Intertial Space**

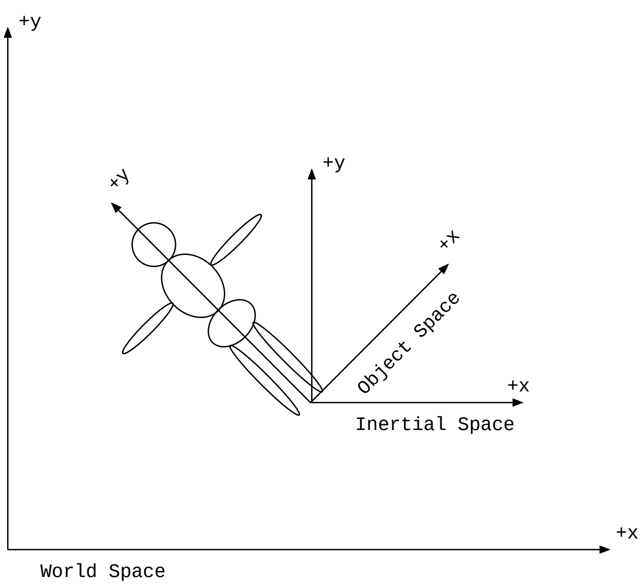
- Halfway betwen object space and world space
- Axes parallel to world space
- Origin same as object space




07-57: **Inertial Space**



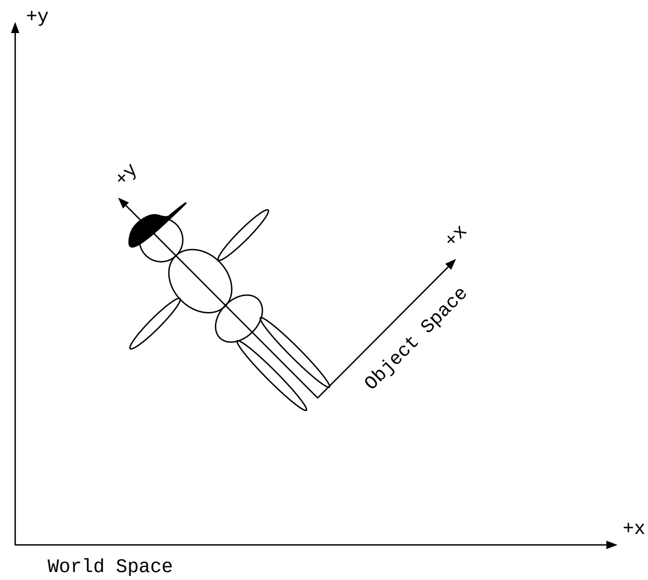

07-58: **Inertial Space**




07-59: **Inertial Space**

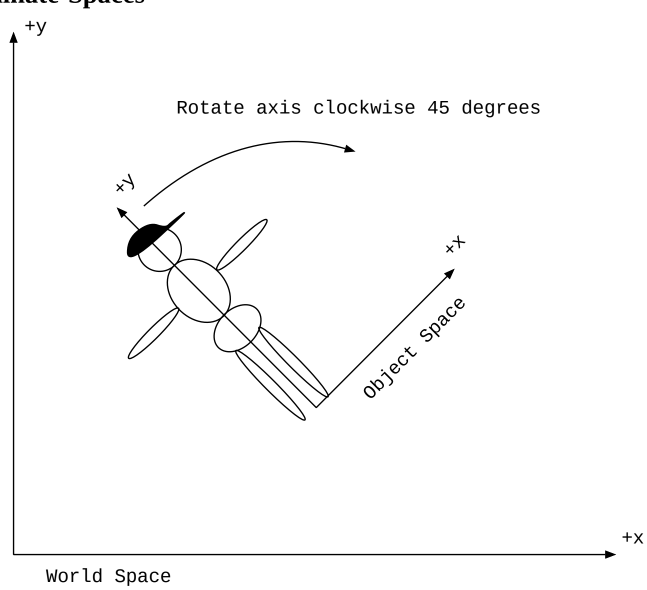
07-60: **Changing Coordinate Spaces**

- Our character is wearing a red hat
- The hat is at position (0,100) in object space
- What is the position of the hat in world space?
- To make life easier, we will think about rotating the axes, instead of moving the objects



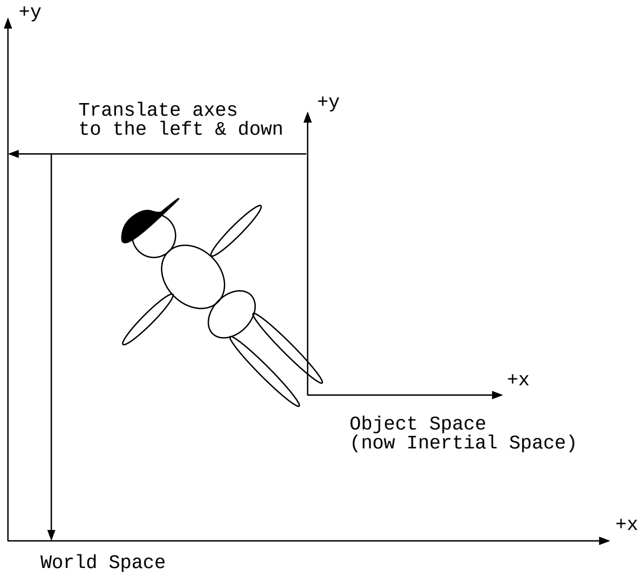

07-61: **Changing Coordinate Spaces**

07-62: **Changing Coordinate Spaces**

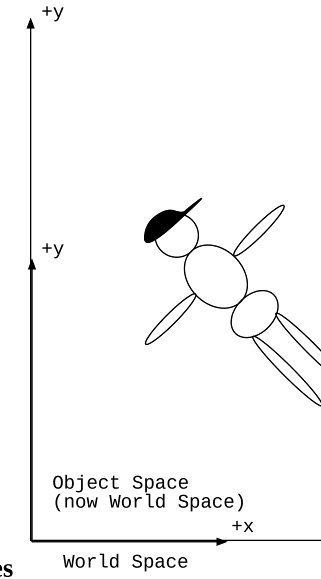



07-63: **Changing Coordinate**

**Spaces**

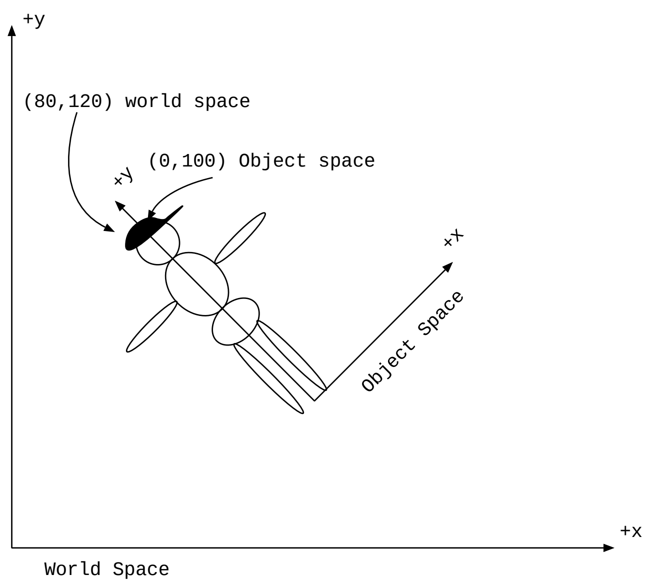



07-64: **Changing Coordinate Spaces**




07-65: **Changing Coordinate Spaces**

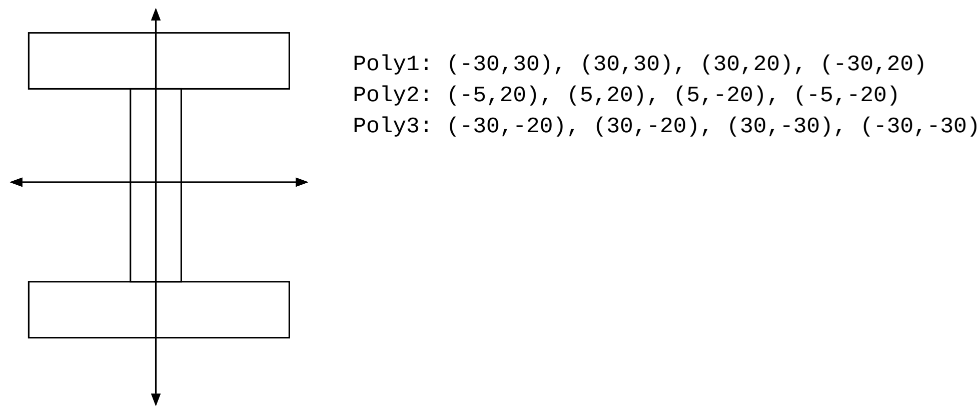
- Rotate axes to the left 45 degrees
  - Hat rotates the the right 45 degrees, from (0,100) to (-70, 70)
- Translate axes to the left 150, and down 50
  - Hat rotates to the right 150 and up 50, to (80, 120)



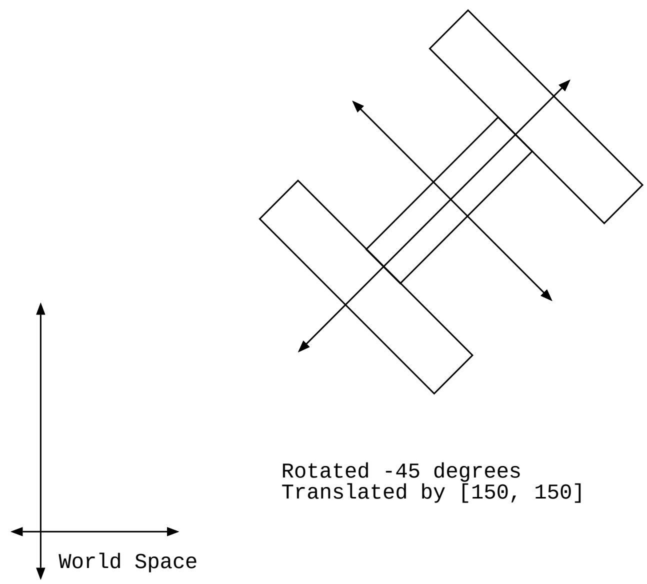

07-66: **Changing Coordinate Spaces**

07-67: **Composite Objects**

- More complicated object – made of multiple elements
  - Several polygons, circles
- Define in Object Space – origin the center of the object




07-68: **Composite Objects**




07-69: **Composite Objects**

07-70: **Composite Objects**

- We store the rotation and translation of entire object
- When we want to know the points of any sub-object in world space
  - Doing collision, for instance
- Rotate and then translate the points of the subobject

07-71: **Composite Objects**

- Want to know the position of the 4 points of the blue rectangle in world space
- Local space positions are $p_1, p_2, p_3, p_4$
- World space postions are:

07-72: **Composite Objects**

- Want to know the position of the 4 points of the blue rectangle in world space
- Local space positions are $p_1, p_2, p_3, p_4$

- World space postions are:

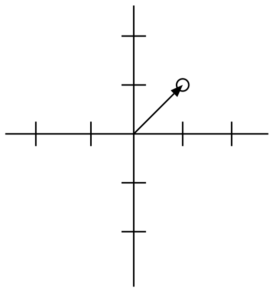
  new $p_1$ = $p_1 \begin{bmatrix} \cos(-45) & \sin(-45) \\ -\sin(-45) & \cos(-45) \end{bmatrix} + [150, 150]$

  new $p_2$ = $p_2 \begin{bmatrix} \cos(-45) & \sin(-45) \\ -\sin(-45) & \cos(-45) \end{bmatrix} + [150, 150]$

  new $p_3$ = $p_3 \begin{bmatrix} \cos(-45) & \sin(-45) \\ -\sin(-45) & \cos(-45) \end{bmatrix} + [150, 150]$

  ... etc

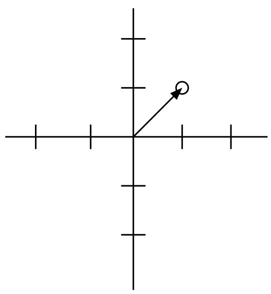
07-73: **Other Transformations**



`Transform with:`

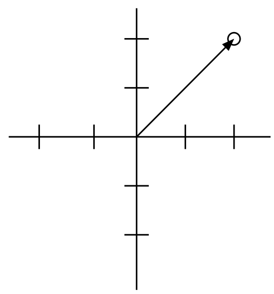
$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

07-74: **Other Transformations**

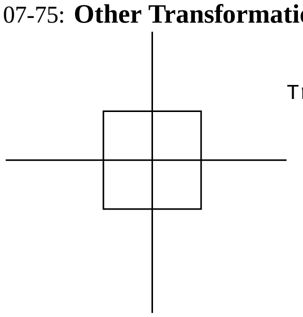


`Transform with:`

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$



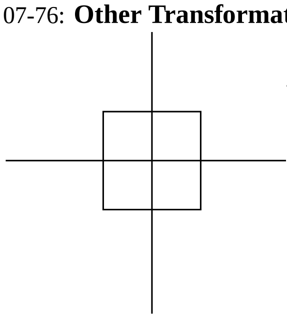
07-75: **Other Transformations**



`Transform with:`

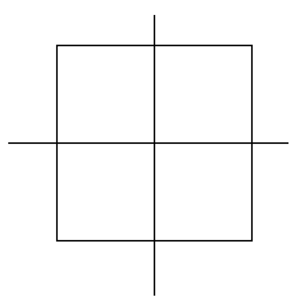
$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

07-76: **Other Transformations**



`Transform with:`

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$



07-77: **Uniform Scaling**

- A matrix of the form:

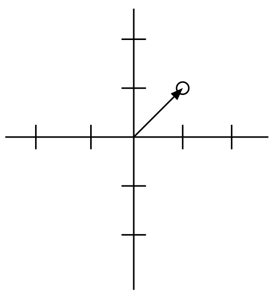
$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

will uniformly scale an object. What happens if $k = 1$? $k > 1$? $0 < k < 1$?

07-78: **Nonuniform Scaling**

- A matrix can also be used to scale in different amounts on different axes.
- Object will be stretched / distorted

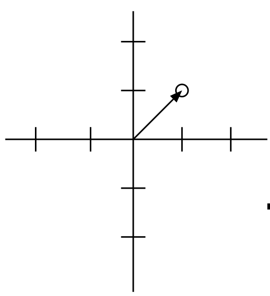
07-79: **Nonuniform Scaling**



Transform with:

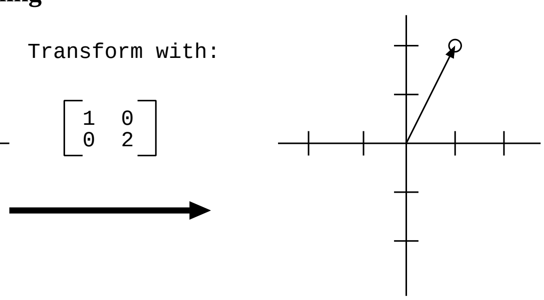
$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

07-80: **Nonuniform Scaling**

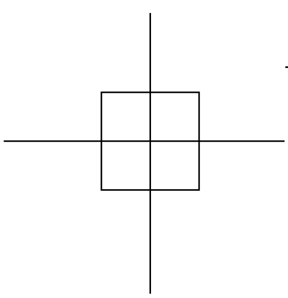


Transform with:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$



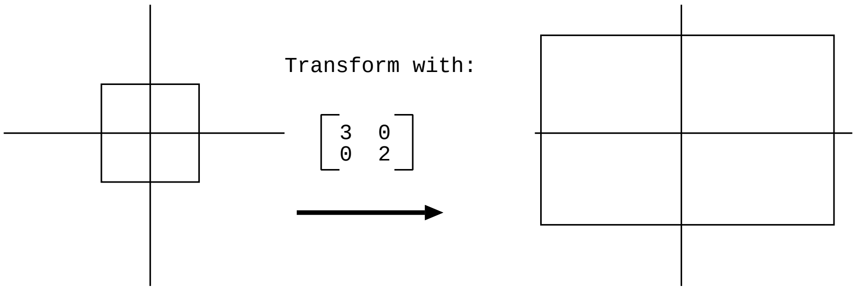
07-81: **Nonuniform Scaling**



Transform with:

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

07-82: **Nonuniform Scaling**




07-83: **Combining Transforms**

- What would happen to a point that was transformed twice?

$$[x, y]\begin{bmatrix} \cos\Theta_1 & \sin\Theta_1 \\ -\sin\Theta_1 & \cos\Theta_1 \end{bmatrix}\begin{bmatrix} \cos\Theta_2 & \sin\Theta_2 \\ -\sin\Theta_2 & \cos\Theta_2 \end{bmatrix}$$

07-84: **Combining Transforms**

- What would happen to a point that was transformed twice?

$$\left([x, y]\begin{bmatrix} \cos\Theta_1 & \sin\Theta_1 \\ -\sin\Theta_1 & \cos\Theta_1 \end{bmatrix}\right)\begin{bmatrix} \cos\Theta_2 & \sin\Theta_2 \\ -\sin\Theta_2 & \cos\Theta_2 \end{bmatrix} =$$

$$[x, y]\left(\begin{bmatrix} \cos\Theta_1 & \sin\Theta_1 \\ -\sin\Theta_1 & \cos\Theta_1 \end{bmatrix}\begin{bmatrix} \cos\Theta_2 & \sin\Theta_2 \\ -\sin\Theta_2 & \cos\Theta_2 \end{bmatrix}\right)$$

07-85: **Combining Transforms**

$$\begin{bmatrix} \cos\Theta_1 & \sin\Theta_1 \\ -\sin\Theta_1 & \cos\Theta_1 \end{bmatrix}\begin{bmatrix} \cos\Theta_2 & \sin\Theta_2 \\ -\sin\Theta_2 & \cos\Theta_2 \end{bmatrix} =$$

$$\begin{bmatrix} \cos\Theta_1\cos\Theta_2 - \sin\Theta_1\sin\Theta_2 & \cos\Theta_1\sin\Theta 2 + \sin\Theta_1\cos\Theta_2 \\ -\sin\Theta_1\cos\Theta_2 - \cos\Theta_1\sin\Theta_2 & -\sin\Theta_1\sin\Theta 2 + \cos\Theta_1\cos\Theta_2 \end{bmatrix} =$$

07-86: **Combining Transforms**

$$\begin{bmatrix} \cos\Theta_1 & \sin\Theta_1 \\ -\sin\Theta_1 & \cos\Theta_1 \end{bmatrix}\begin{bmatrix} \cos\Theta_2 & \sin\Theta_2 \\ -\sin\Theta_2 & \cos\Theta_2 \end{bmatrix} =$$

$$\begin{bmatrix} \cos\Theta_1\cos\Theta_2 - \sin\Theta_1\sin\Theta_2 & \cos\Theta_1\sin\Theta 2 + \sin\Theta_1\cos\Theta_2 \\ -\sin\Theta_1\cos\Theta_2 - \cos\Theta_1\sin\Theta_2 & -\sin\Theta_1\sin\Theta 2 + \cos\Theta_1\cos\Theta_2 \end{bmatrix} =$$

$$\begin{bmatrix} \cos(\Theta_1 + \Theta_2) & \sin(\Theta_1 + \Theta_2) \\ -\sin(\Theta_1 + \Theta_2) & \cos(\Theta_1 + \Theta_2) \end{bmatrix}$$

07-87: **Combining Transforms**

- We can also combine scaling and rotating

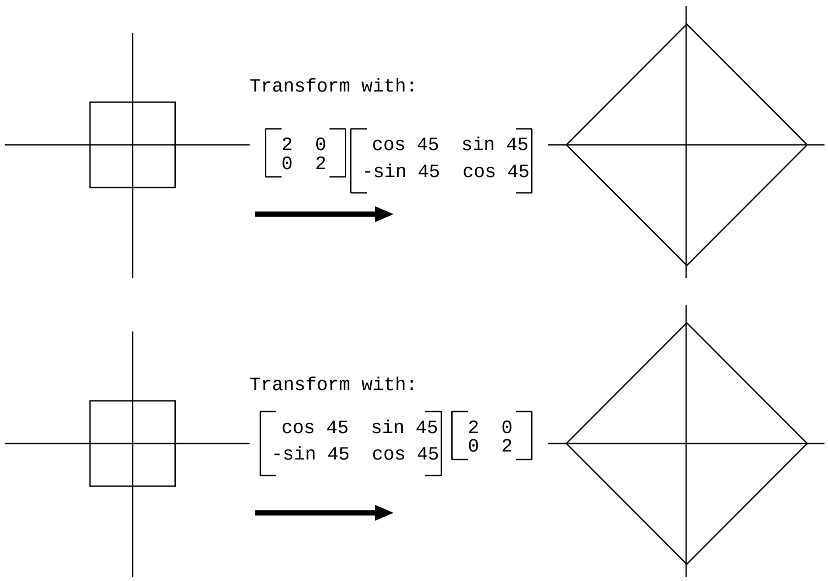
$$[x, y]\begin{bmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{bmatrix}\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

- With uniform scaling, we get the same result if we scale, then rotate as if we rotated, and then scaled.

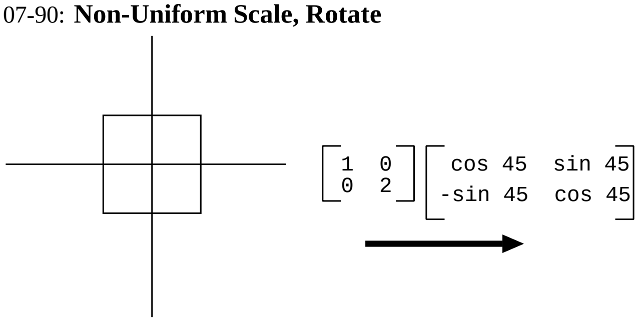
07-88: **Combining Transforms**

$$\begin{aligned}\begin{bmatrix}\cos\Theta & \sin\Theta\\ -\sin\Theta & \cos\Theta\end{bmatrix}\begin{bmatrix}k & 0\\ 0 & k\end{bmatrix} &= \begin{bmatrix}\cos\Theta & \sin\Theta\\ -\sin\Theta & \cos\Theta\end{bmatrix}\begin{bmatrix}k & 0\\ 0 & k\end{bmatrix}\\ &= \begin{bmatrix}k\cos\Theta & k\sin\Theta\\ -k\sin\Theta & k\cos\Theta\end{bmatrix}\end{aligned}$$

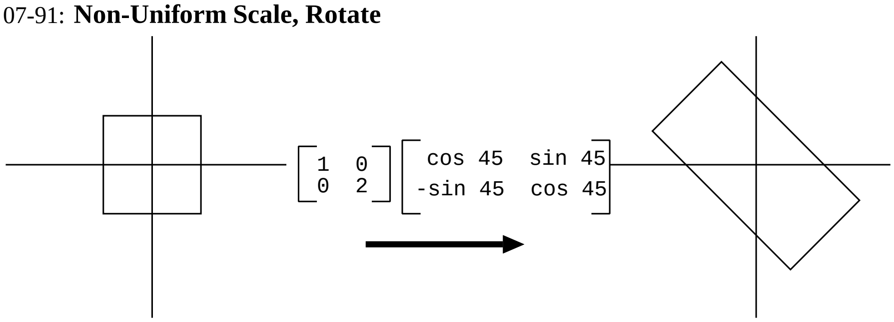
07-89: **Combining Transforms**



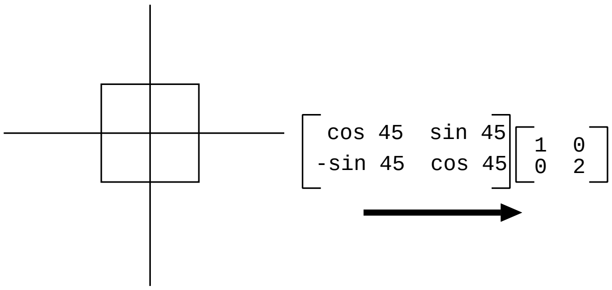

07-90: **Non-Uniform Scale, Rotate**



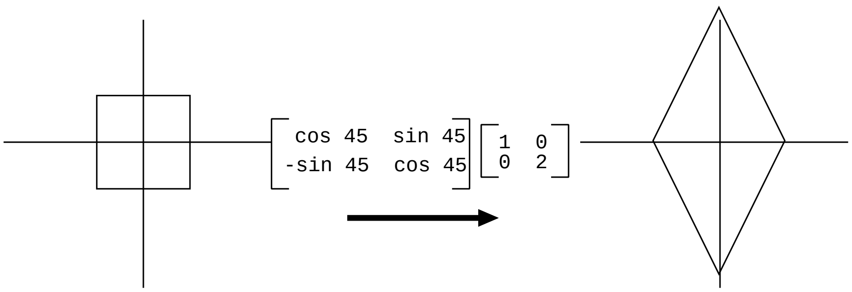

07-91: **Non-Uniform Scale, Rotate**




07-92: **Non-Uniform Scale, Rotate**




07-93: **Non-Uniform Scale, Rotate**

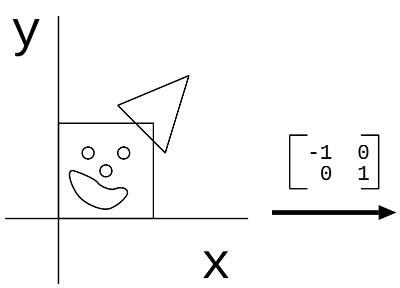



07-94: **Other Transformations**

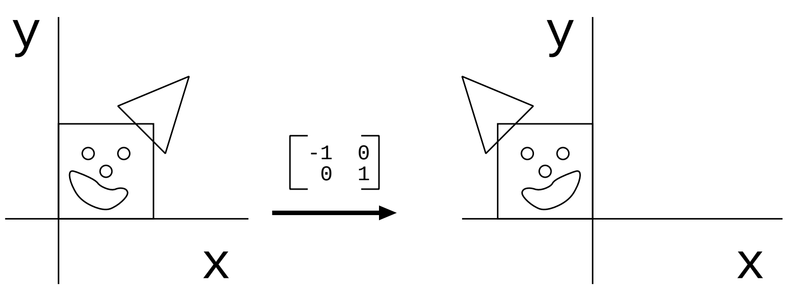
- How would the following matrix transform an object?

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

07-95: **Reflection**



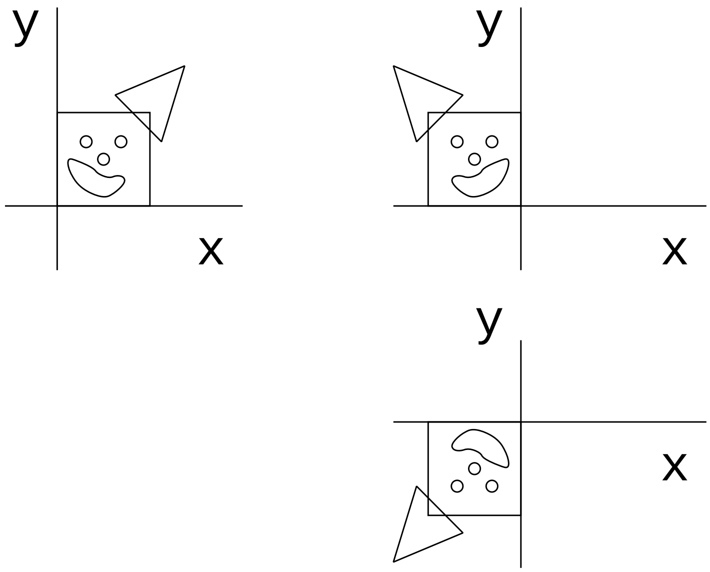

07-96: **Reflection**




07-97: **Reflection**

- Reflecting an object twice around the same axis brings and object back to its original state
- Reflecting around the y-axis and then reflecting around the x-axis is the same as ...

07-98: **Reflection**




07-99: **Reflection**

- How could we reflect an object around its own center line, instead of around the x- or y- axis?

07-100: **Reflection**

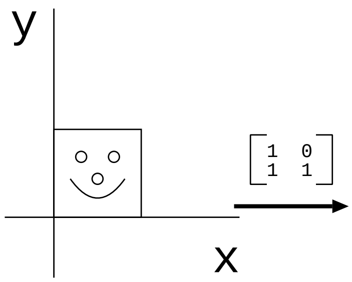
- How could we reflect an object around its own center line, instead of around the x- or y- axis?
    - Translate and rotate the object so that its own center line is the same as the x- or y- axis
    - Reflect
    - Translate, rotate back
- If the centerline of an object is either the x- or y- axis in object space, this is easier ...

07-101: **Shearing**

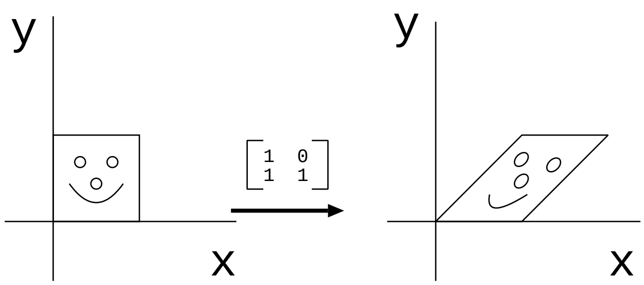
- How would the following matrix transform an object?

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

07-102: **Shearing**

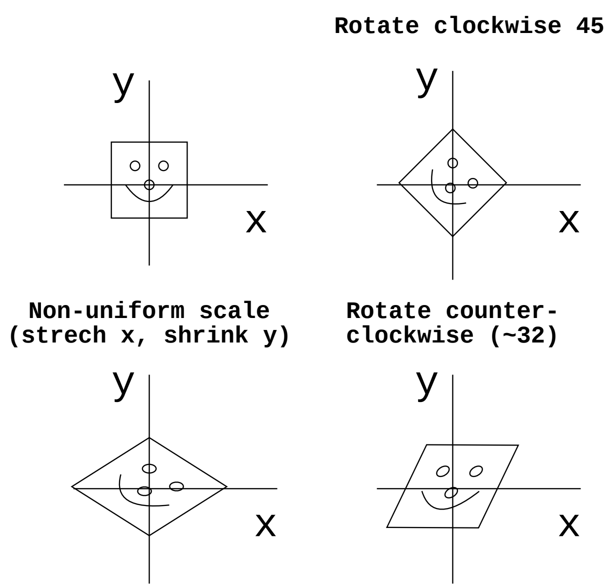



07-103: **Shearing**




07-104: **Shearing**

- Shearing an object is the same as:
  - Rotating the object
  - Non-uniform scale
  - Rotating back (not by the same angle)




07-105: **Shearing**

07-106: **Linear Transforms**

- Matrix operations represent linear transformation of objects
- Number of points in in a line before the transformation, still be in a line after the transformation
  - Line may be stretched and rotated, still be a line

07-107: **Linear Transforms**

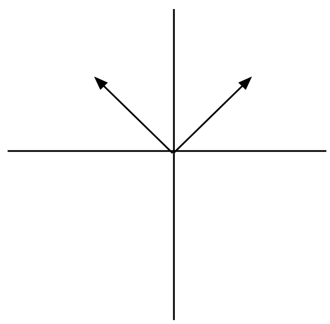
- This gives us a handy way of seeing how a matrix will transform an object
  - See how the matrix will transform the axes [1,0], [0,1]
  - Object will be transformed in the same way

How will this transform an object

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

07-108: **Linear Transforms**

How will this transform an object

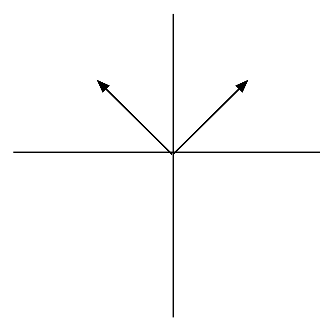


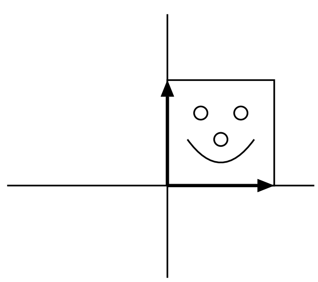
$$\begin{array}{cc} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} & \begin{array}{l} \texttt{x-axis} \\ \texttt{y-axis} \end{array} \end{array}$$

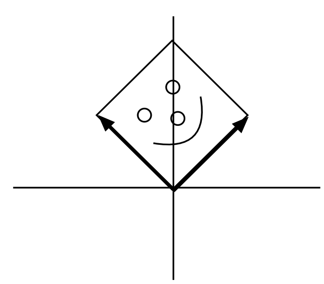
07-109: **Linear Transforms**

How will this transform an object

$$\begin{array}{cc} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} & \begin{array}{l} \texttt{x-axis} \\ \texttt{y-axis} \end{array} \end{array}$$







07-110: **Linear Transforms**

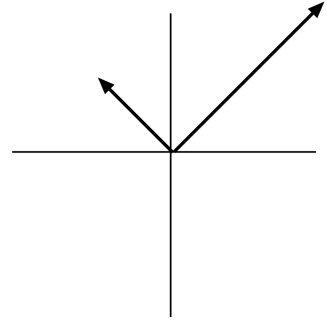
How will this transform an object

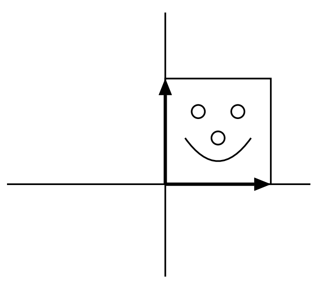
$$\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

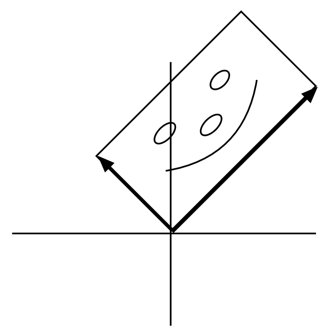
07-111: **Linear Transforms**

How will this transform an object

$$\begin{array}{cc} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} & \begin{array}{l} \text{x-axis} \\ \text{y-axis} \end{array} \end{array}$$





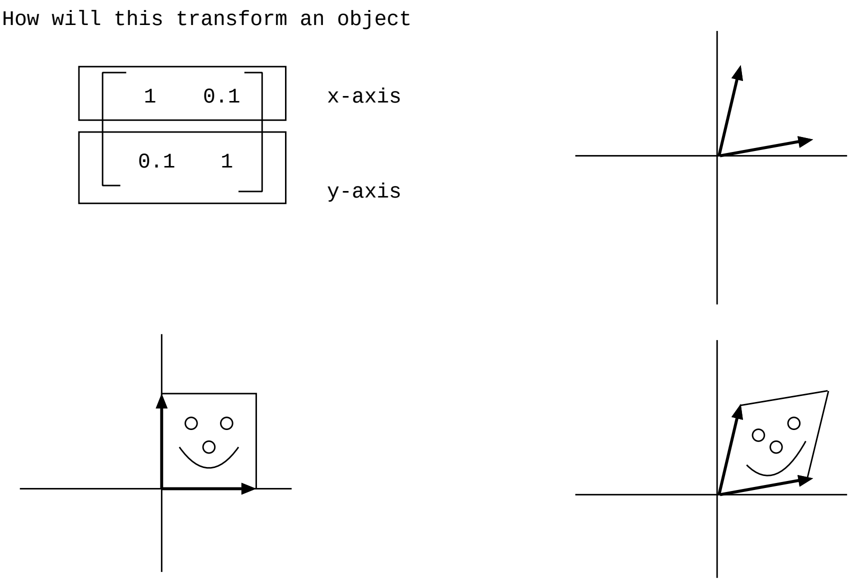


07-112: **Linear Transforms**

How will this transform an object

$$\begin{bmatrix} 1 & 0.1 \\ 0.1 & 1 \end{bmatrix}$$

07-113: **Linear Transforms**




07-114: **Linear Transforms**

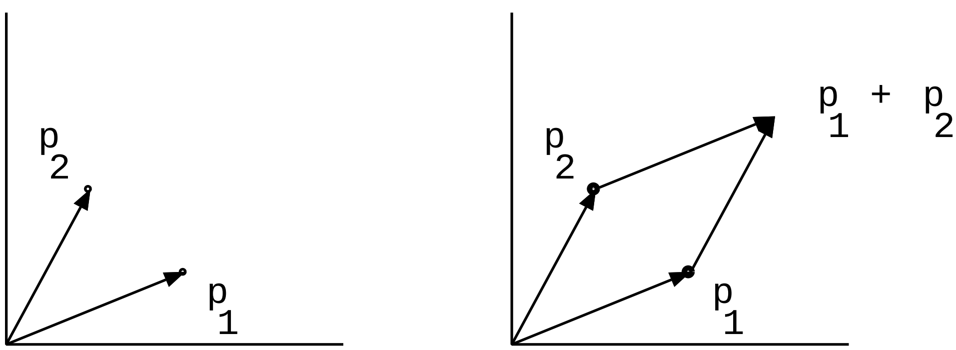
07-115: **Determinant**

- Determinant of 2x2 matrix:

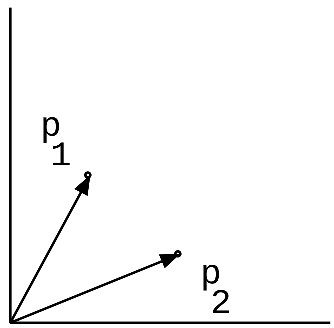
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

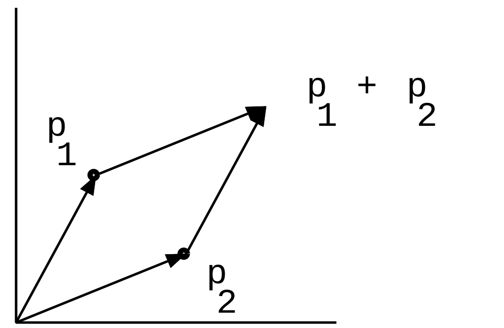
$$\begin{vmatrix} p1_x & p1_y \\ p2_x & p2_y \end{vmatrix} = \text{Area of Parallelogram}$$

07-116: **Determinant**



$$\begin{vmatrix} p1_x & p1_y \\ p2_x & p2_y \end{vmatrix} = -\text{(Area of Parallelogram)}$$




07-117: **Determinant**

07-118: **Determinant**

- Signed area of parallelogram
  - If transformation includes a reflection, then ...

07-119: **Determinant**

- Signed area of parallelogram
  - If transformation includes a reflection, then
    - Determinant is negative

07-120: **Determinant**

- Signed area of parallelogram
  - If is a pure rotation (no scale, no shear, no rotation) ...

07-121: **Determinant**

- Signed area of parallelogram
  - If is a pure rotation (no scale, no shear, no rotation) ...
    - Determinant = 1
    - (Other direction is not always true..)

07-122: **Translation**

- We can implement rotation / scale / reflection / shearing using matrix operations
- What about translation?

07-123: **Translation**

- Can’t translate an object using a 2x2 matrix

- Consider a point at the origin
    - How will a point a the origin be modified by a matrix?

07-124: **Translation**

- Can't translate an object using a 2x2 matrix
- Consider a point at the origin
    - How will a point a the origin be modified by a matrix?
    - Not changed by *any* matrix!

07-125: **Translation**

- Matrices can only do linear transformations
- Translation is not a linear transformation
    - Can't use matrices to do translation
    - ... Unless we cheat a little!

07-126: **Translation**

- We can use matrices to do translation – as long as we use something different than 2x2 matrices
    - Add a dummy value to the end of all points (always 1)
    - Add a new row / column to matrix

$$[x, y, 1]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [x, y, 1]$$

07-127: **Translation**

$$[x, y, 1]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \delta_x & \delta_y & 1 \end{bmatrix} =$$

07-128: **Translation**

$$[x, y, 1]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \delta_x & \delta_y & 1 \end{bmatrix} = [x + \delta_x, y + \delta_y, 1]$$

07-129: **Translation**

- Adding rotation

$$[x, y]\begin{bmatrix} a & b \\ c & d \end{bmatrix} = [xa + yc, xb + yd]$$

$$[x, y, 1]\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} = [xa + yc, xb + yd, 1]$$

07-130: **Translation**

$$[x, y, 1]\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ \delta_x & \delta_y & 1 \end{bmatrix} = [xa + yc + \delta_x, xb + yd + \delta_y, 1]$$

- So, we can use a matrix to do both rotation and translation

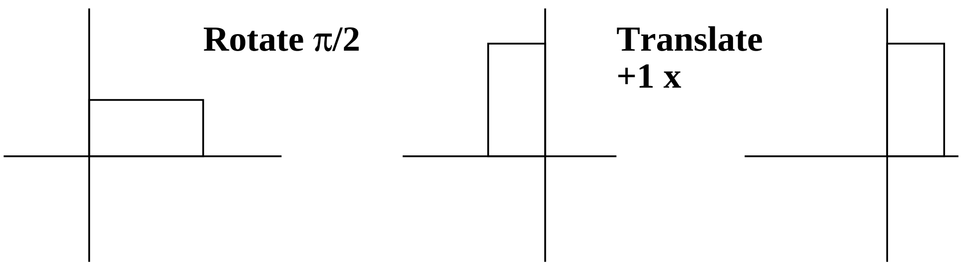
07-131: **Translation**

$$\begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ \delta_x & \delta_y & 1 \end{bmatrix}$$

- First, rotate counter-clockwise by $\Theta$
- Then translate by $[\delta_x, \delta_y]$

07-132: **Combining Transforms**

- Let's look at an example
  - First rotate by $\pi/2$ (90 degrees) counterclockwise
  - Then translate $x$ by +1

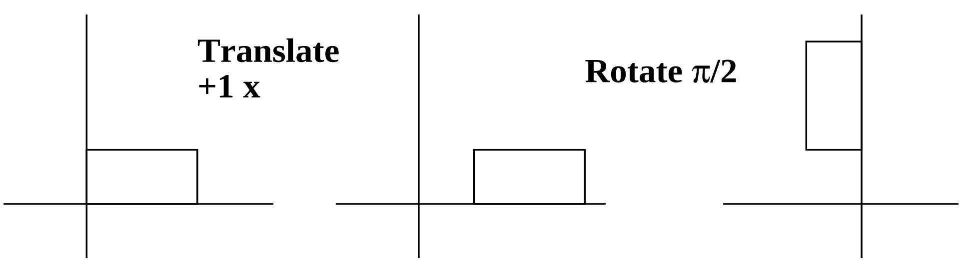



07-133: **Combining Transforms**

$$\begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

07-134: **Combining Transforms**

- Another example
  - First translate $x$ by +1
  - Then rotate by $\pi/2$ (90 degrees) counterclockwise

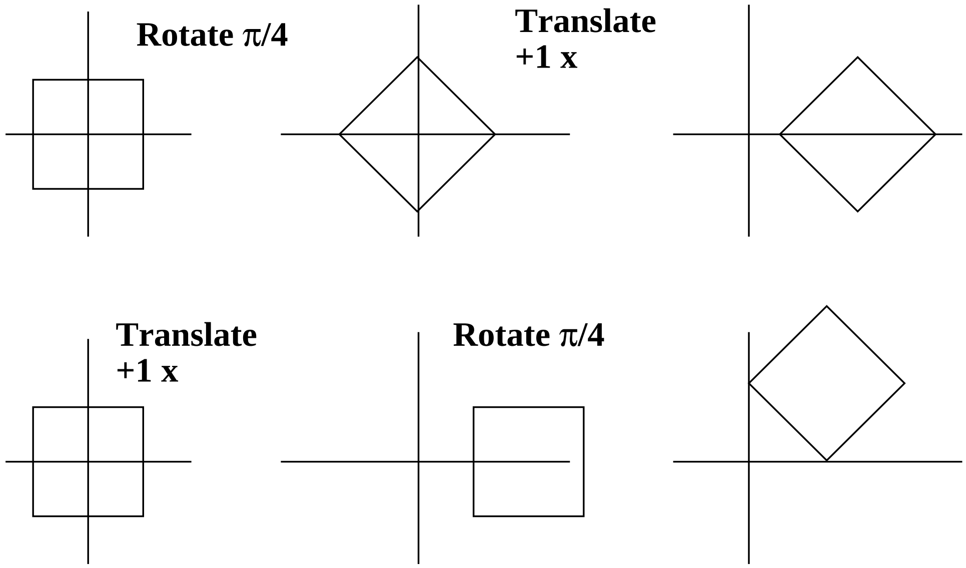



07-135: **Combining Transforms**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ \cos\Theta & \sin\Theta & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

- Same as rotating, and then moving up $+y$

07-136: **Combining Transforms**




07-137: **Combining Transforms**

- Rotating by $\pi/4$, then translating 1 unit $+x$

$$\begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

07-138: **Combining Transforms**

- Translating 1 unit $+x$, then rotating by $\pi/4$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ \cos\Theta & \sin\Theta & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \end{bmatrix}$$

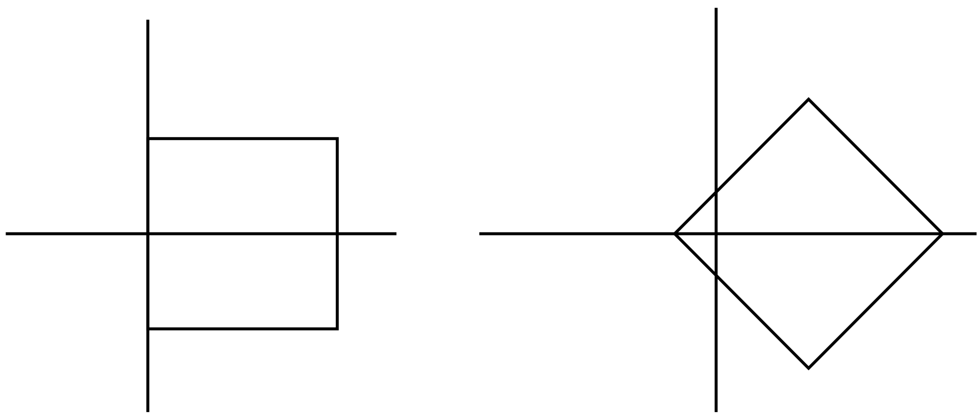
- Same as rotating $\pi/4$ counterclockwise, and then translating over (+x) $1/\sqrt{2}$ and up (+y) $1/\sqrt{2}$

07-139: **Non-Standard Axes**

- We want to rotate around an axis that does not go through the origin
- Rotate around point at 1,0
- Create the approprate 3x3 vector

07-140: **Non-Standard Axes**

# Rotate π/4 around (1,0)



07-141: **Non-Standard Axes**

- First, translate to the origin
- Then, do the rotation
- Finally, translate back

07-142: **Non-Standard Axes**

- First, translate to the origin

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

- Then, do the rotation
- Finally, translate back

07-143: **Non-Standard Axes**

- First, translate to the origin
- Then, do the rotation

$$\begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Finally, translate back

07-144: **Non-Standard Axes**

- First, translate to the origin
- Then, do the rotation

- Finally, translate back

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

07-145: **Non-Standard Axes**

- Final matrix:

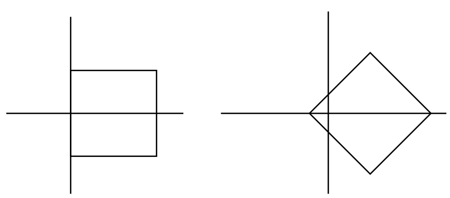
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & -1/\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

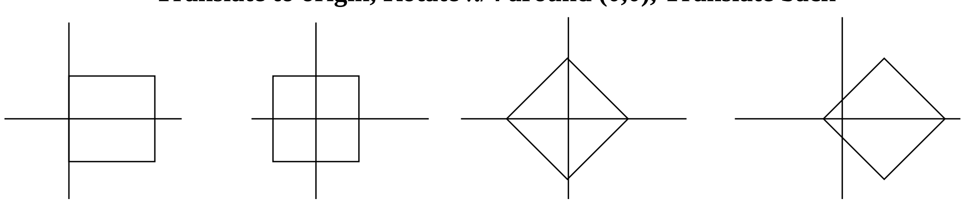
$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1-1/\sqrt{2} & -1/\sqrt{2} & 1 \end{bmatrix}$$

07-146: **Non-Standard Axes**

**Rotate π/4 around (1,0)**

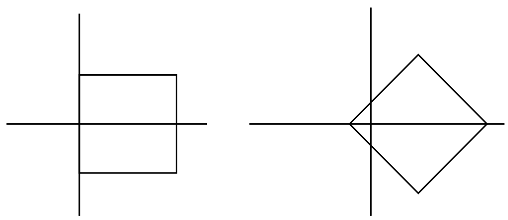


**Translate to origin, Rotate π/4 around (0,0), Translate back**

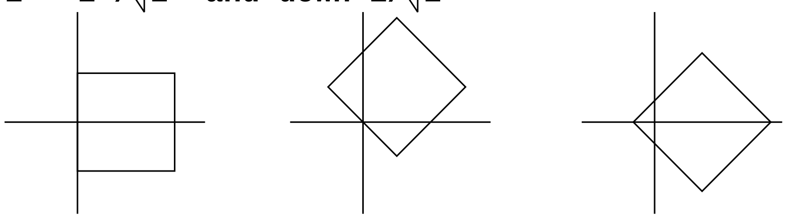


07-147: **Non-Standard Axes**

**Rotate $\pi/4$ around (1,0)**



**Rotate p/4 around (0,0), then translate over**
**1 - 1 /$\sqrt{2}$ and down 1/$\sqrt{2}$**



07-148: **Non-Standard Axes**

- Note that the *rotation* component (upper left 2x2 matrix) is the same as if we were rotating around the origin
- Only the *position* component is altered.
- In general, whenever we do a rotation and a number of translations, the rotation component will be unchanged

07-149: **Linear Transforms?**

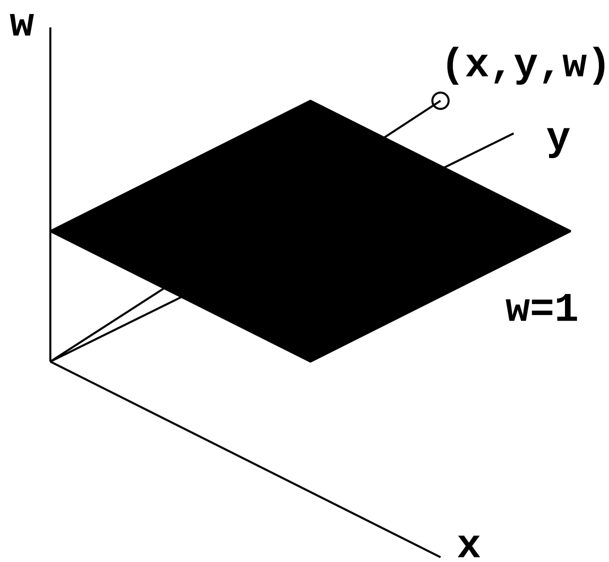
- Matricies can only do linear transformations
- Translation is *not* a linear transform
- ... but we are using matrices to do translation
  - What's up?

07-150: **Homogeneous Space**

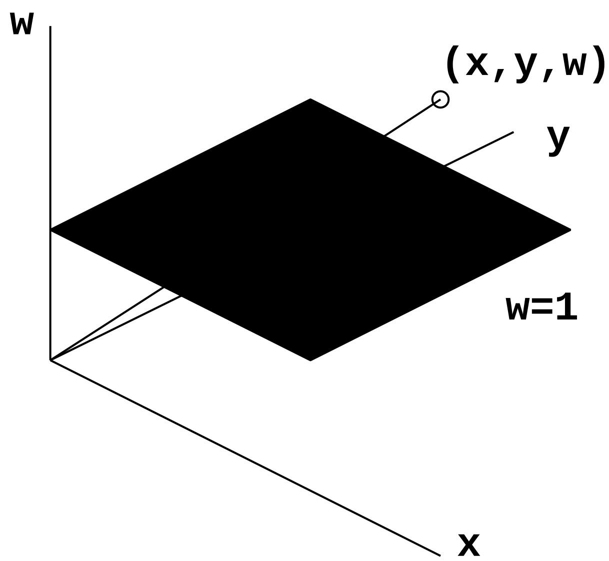
- We are no longer working in 2D, we are now working in 3D
  - Extra 3rd parameter, that is always == 1
- We can extend our definition to allow the 3rd parameter to be some value other than 1
  - Need to be able to convert back to 2D space

07-151: **3D Homogeneous Space**

- To convert a point $(x, y, w)$ in 3D Homogeneous space into 2D $(x, y)$ space:
  - Place a plane at $w = 1$
  - $(x, y, w)$ maps to the $(x, y)$ position on the plane where the ray $(x, y, w)$ intersects the plane




07-152: **3D Homogeneous Space**




07-153: **3D Homogeneous Space**

07-154: **3D Homogeneous Space**

- Converting from a point in 3D homogeneous space to 2D space is easy
  - Divide the $x$ and $y$ coordinates by $w$
  - What happens when $w = 0$?

07-155: **3D Homogeneous Space**

- Converting from a point in 3D homogeneous space to 2D space is easy
  - Divide the $x$ and $y$ coordinates by $w$
  - What happens when $w = 0$?
    - “Point at infinity”

- Direction, but not a magnitude

07-156: **3D Homogeneous Space**

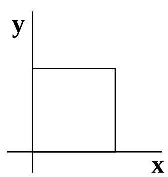
- For a given $(x, y, w)$ point in 3D Homogeneous space, there is a single corresponding point in “standard” 2D space
  - Though when $w = 0$, we are in a bit of a special case
- For a single point in “standard” 2D space, there are an infinite number of corresponding points in 3D Homogeneous space

07-157: **Translation**

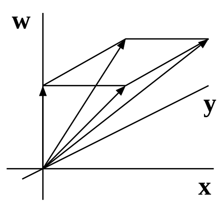
- We are still doing a linear trasformation of the 3D vector
- We are *shearing* the 3D space
- The resulting projection back to 2D is seen as a translation

07-158: **Translation**

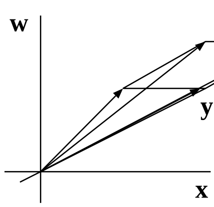
**2D Shape**



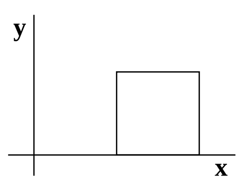

**Transform to 3D Homogenous Space**




**Shear operation in 3D space**




**Back to 2D**




07-159: **Inverse**

- Finding inverse of 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

07-160: **Inverse**

- A matrix is signular if it does not have an inverse
  - determinant = 0

- Why does this make sense, geometrically?

07-161: **Orthogonal Matrices**

- A matrix $\mathbf{M}$ is orthogonal if:
  - $\mathbf{M}\mathbf{M}^T = \mathbf{I}$
  - $\mathbf{M}^T = \mathbf{M}^{-1}$
- Orthogonal matrices are handy, because they are easy to invert
- Is there a geometric interpretation of orthogonality?

07-162: **Orthogonal Matrices**

$$\begin{aligned} \mathbf{M}\mathbf{M}^T &= \mathbf{I} \\ \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Do all the multiplications ...

07-163: **Orthogonal Matrices**

$$\begin{aligned} m_{11}m_{11} + m_{12}m_{12} &= 1 \\ m_{11}m_{21} + m_{12}m_{22} &= 0 \\ m_{21}m_{11} + m_{22}m_{12} &= 0 \\ m_{21}m_{21} + m_{22}m_{22} &= 1 \end{aligned}$$

- Hmmm... that doesn't seem to help much

07-164: **Orthogonal Matrices**

- Recall that rows of matrix are basis after rotation
  - $\mathbf{v}_x = [m_{11}, m_{12}]$
  - $\mathbf{v}_y = [m_{21}, m_{22}]$
- Let's rewrite the previous equations in terms of $\mathbf{v}_x$ and $\mathbf{v}_y$ ...

07-165: **Orthogonal Matrices**

$$\begin{aligned} m_{11}m_{11} + m_{12}m_{12} &= 1 & \mathbf{v}_x \cdot \mathbf{v}_x = 1 \\ m_{11}m_{21} + m_{12}m_{22} &= 0 & \mathbf{v}_x \cdot \mathbf{v}_y = 0 \\ m_{21}m_{11} + m_{22}m_{12} &= 0 & \mathbf{v}_y \cdot \mathbf{v}_x = 0 \\ m_{21}m_{21} + m_{22}m_{22} &= 1 & \mathbf{v}_y \cdot \mathbf{v}_y = 1 \end{aligned}$$

07-166: **Orthogonal Matrices**

- What does it mean if $\mathbf{u} \cdot \mathbf{v} = 0$?

- (assuming both $\mathbf{u}$ and $\mathbf{v}$ are non-zero)
- What does it mean if $\mathbf{v} \cdot \mathbf{v} = 1$?

07-167: **Orthogonal Matrices**

- What does it mean if $\mathbf{u} \cdot \mathbf{v} = 0$?
  - (assuming both $\mathbf{u}$ and $\mathbf{v}$ are non-zero)
  - $\mathbf{u}$ and $\mathbf{v}$ are perpendicular to each other (orthogonal)
- What does it mean if $\mathbf{v} \cdot \mathbf{v} = 1$?
  - $||\mathbf{v}|| = 1$
- So, transformed basis vectors must be mutually perpendicular unit vectors

07-168: **Orthogonal Matrices**

- If a transformation matrix is orthogonal,
  - Transformed basis vectors are mutually perpendicular unit vectors
- What kind of transformations are done by orthogonal matrices?

07-169: **Orthogonal Matrices**

- If a transformation matrix is orthogonal,
  - Transformed basis vectors are mutually perpendicular unit vectors
- What kind of transformations are done by orthogonal matrices?
  - Rotations & Reflections

07-170: **Orthogonal Matrices**

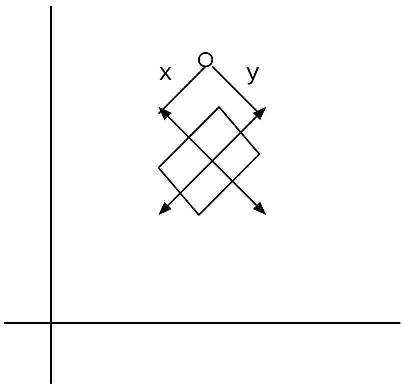
- Sanity check: Rotational matrices are orthoginal:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{aligned} A = \begin{bmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{bmatrix}, A^{-1} &= \begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix} \\ &= \begin{bmatrix} \cos(-\Theta) & \sin(-\Theta) \\ -\sin(-\Theta) & \cos(-\Theta) \end{bmatrix} \end{aligned}$$

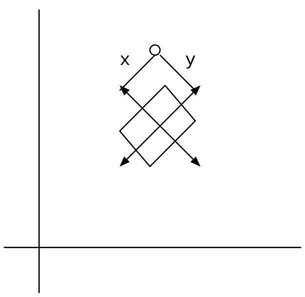
07-171: **Examples**

- An object's position has a rotational matrix $M$ and a position $p$. A point $o_1 = [o_{1x}, o_{1y}]$ is in *object space* for the object, What is the position of the point in world space?




07-172: **Examples**

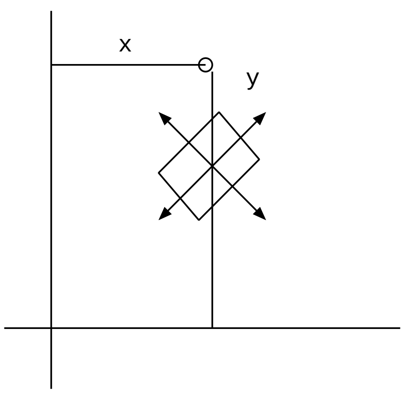
- An object's position has a rotational matrix $M$ and a position $p$. A point $o_1 = [o_{1x}, o_{1y}]$ is in *object space* for the object, What is the position of the point in world space?




Position in world space: $o_1 M + p$

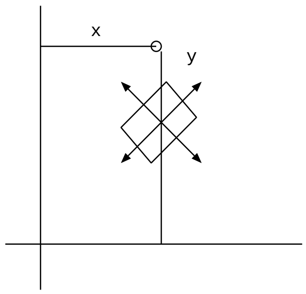
07-173: **Examples**

- An object's position has a rotational matrix $M$ and a position $p$. A point $w_1 = [w_{1x}, w_{1y}]$ is in *world space.* What is the position of the point in objet space




07-174: **Examples**

- An object's position has a rotational matrix $M$ and a position $p$. A point $w_1 = [w_{1x}, w_{1y}]$ is in *world space.* What is the position of the point in objet space

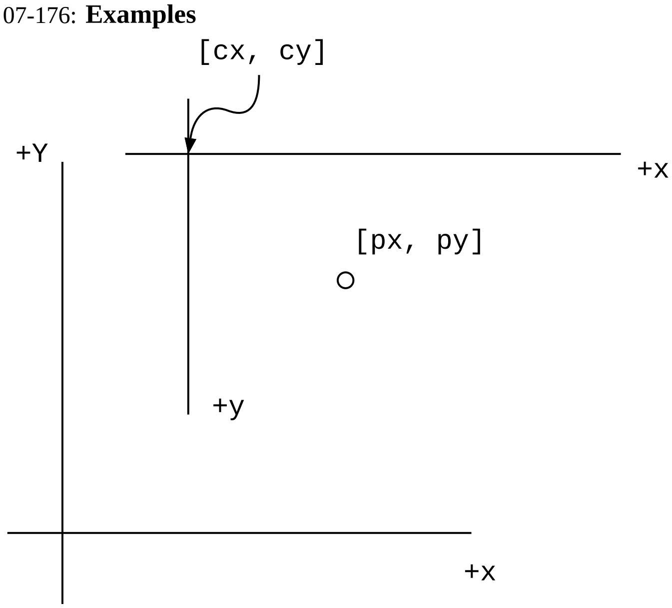



Position in object space: $(w_1 - p)M^T$

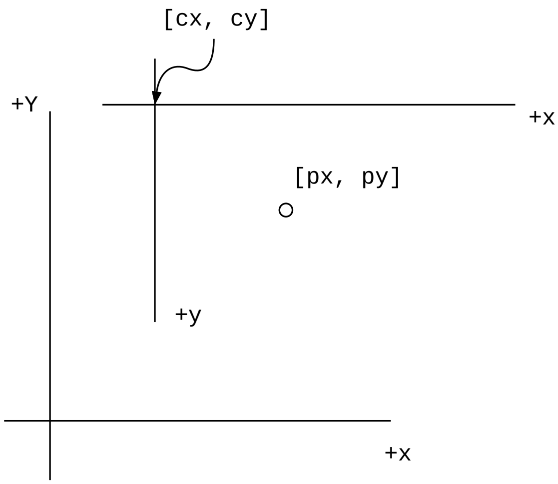
07-175: **Examples**

- Origin of screen is at position $[c_x, c_y]$ in *world space*
- Object is a point $[p_x, p_y]$ in world space
- World has $+x$ to right, $+y$ up
- Screen has $+x$ to right, $+y$ down
- What is the position of $p$ in screen space?

07-176: **Examples**

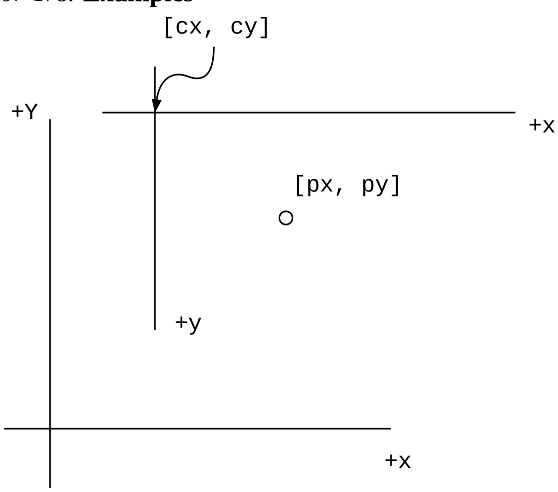



07-177: **Examples**




$[p_x, p_y] - [c_x, c_y] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

07-178: **Examples**

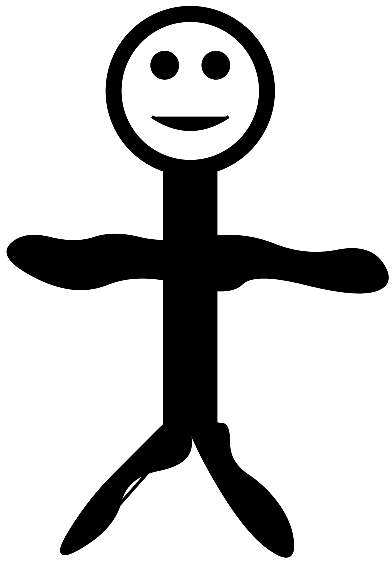



$[p_x - c_x, c_y - p_y]$

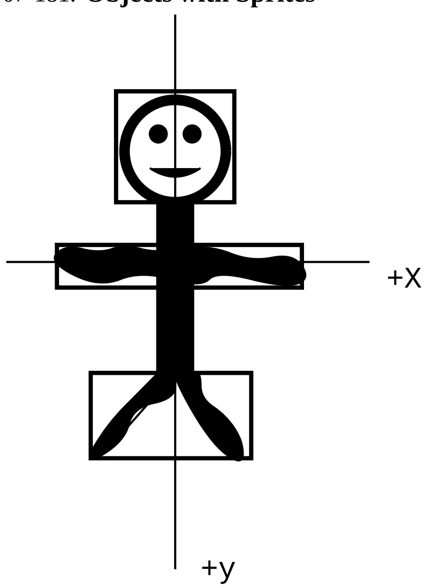
07-179: **Examples**

- Coversion from +y up to +y down
    - Points and rectangles easy to convert
    - Sprites would require a reflection
        - Can use SpriteEffects to reflect sprites
    - Best to stay in “reflected” space

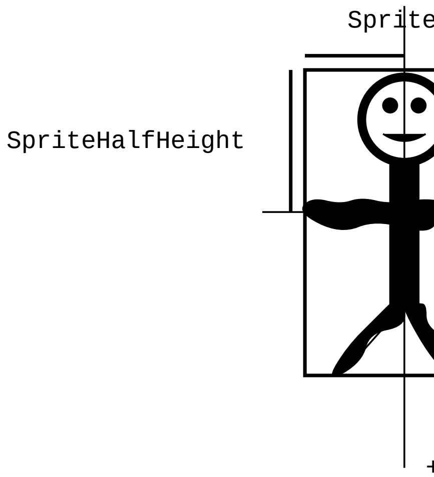
07-180: **Objects with Sprites**



07-181: **Objects with Sprites**



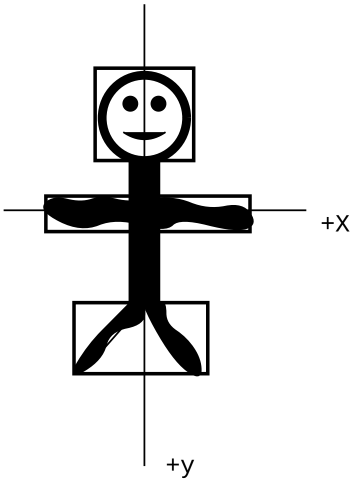

07-182: **Objects with Sprites**



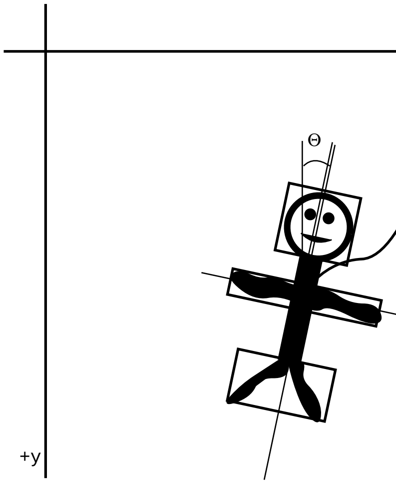

- Center of the sprite is at [SpriteHalfWidth,SpriteHalfHeight] from top left corner of sprite

07-183: **Objects with Sprites**

- Boundary box locations are stored as edge points of each rectangle (4 points per)




07-184: **Objects with Sprites**




- Object is located at position $[x, y]$ and has rotation $\Theta$ *clockwise* (since +x is right, +y is down)

07-185: **Objects with Sprites**

- When dealing with the object in game logic (collistions, etc), we need to know the positions of each of the points of each rectangle in world space.
- If a vertex has position $p$ in local (object) space, what is its position in world space?

07-186: **Objects with Sprites**

- When dealing with the object in game logic (collistions, etc), we need to know the positions of each of the points of each rectangle in world space.
- If a vertex has position $p$ in local (object) space, what is its position in world space?
  - $p \begin{bmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{bmatrix} + [x,y]$

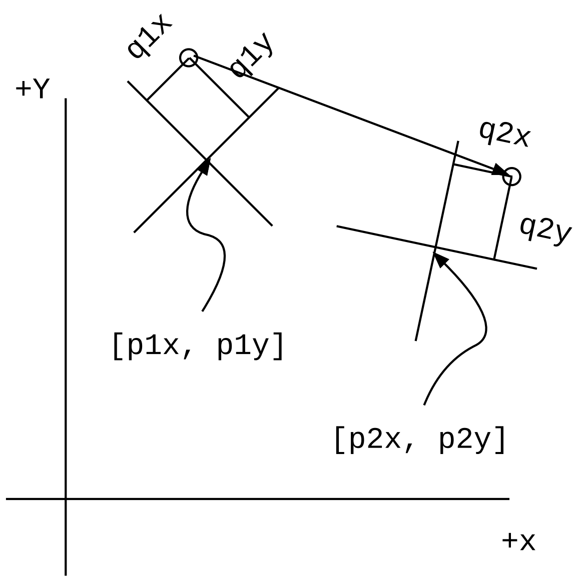
07-187: **Objects with Sprites**

- To draw the sprite on the screen:

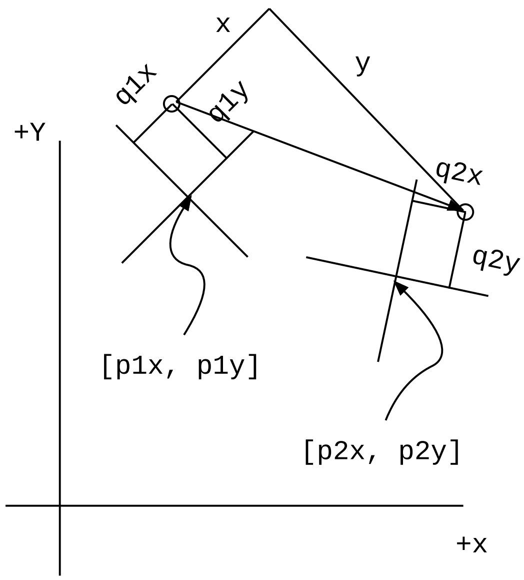
```
SpriteBatch sb
Vector2 pos = new Vector2(x,y);
Vector2 center = new Vector2(SpriteHalfWidh, SpriteHalfHeihgt);
sb.Draw(texture, pos, null, Color.White, theta, center, 1.0f,
        SpriteEffects.None, 0.0f);
```

07-188: **Examples**

- Object1 has rotational matrix $M_1$ and position $p_1$
- Object2 has rotational matrix $M_2$ and position $p_2$
- Point $q_1$ is at postion $[{q_1}_x, {q_1}_y]$ in the object space of Object1
- Point $q_2$ is at positio $[{q_2}_x, {q_2}_y]$ in the object space of Object 2
- What is the vector from $q_1$ to $q_2$ in the object space of $q_1$?




07-189: **Examples**




07-190: **Examples**

07-191: **Examples**

- First, find the position of point $q_2$ in the local space of Object1.
- What's the best way to do this?

07-192: **Examples**

- First, find the position of point $q_2$ in the local space of Object1.
- What's the best way to do this?
  - Go through the world space
  - Find the position of $q_2$ in world space, translate to object space

07-193: **Examples**

- Position of $q_2$ in world space:
  $q_2M_2 + p_2$
- Position of $q_2$ in Object1's local space
  $(q_2(global) - p_1)M_1^T = (q_2M_2 + p_2 - p_1)M_1^T$
- Vector from $q_1$ to $q_2$ in local space of Object 1:

07-194: **Examples**

- Position of $q_2$ in world space:
  $q_2M_2 + p_2$
- Position of $q_2$ in Object1's local space
  $(q_2(global) - p_1)M_1^T = (q_2M_2 + p_2 - p_1)M_1^T$
- Vector from $q_1$ to $q_2$ in local space of Object 1:
  $(q_2M_2 + p_2 - p_1)M_1^T - q_1$

07-195: **Examples**

- Given a transformation matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- How can we determine if this is a “pure rotation” – no scale, shear, reflection

07-196: **Examples**

- Matrix is pure rotation if:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- a = d
- -c = b
- sin(arccos(a)) = b
- What if we don’t have access to arccos, sin?

07-197: **Examples**

- Matrix is pure rotation if:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- ac + bd = 0 [a,b] and [c,d] are perpendicular
- a*a + b*b = 1 [a,b] is unit vector
- c*c + d*d = 1 [c,d] is unit vector
- a*d - c*b = 1 No reflection

07-198: **Examples**

- Spaceship, in local space looks down x axis
- Position $[p_x, p_y]$
- Orientation: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- Place an enemy 10 units directly in front of spaceship, pointing straight back at it
- What is position and orientation of the new spaceship?

07-199: **Examples**

- Spaceship, in local space looks down x axis
- Position $[p_x, p_y]$

- Orientation: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- Place an enemy 10 units directly in front of spaceship, pointing straight back at it
- What is position and orientation of the new spaceship?
  - Position = $[p_x, p_y] + 10 * [a, b]$
  - Orientation: $\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$

07-200: **Examples**