Graduate Algorithms

CS673-2016F-11

B-Trees

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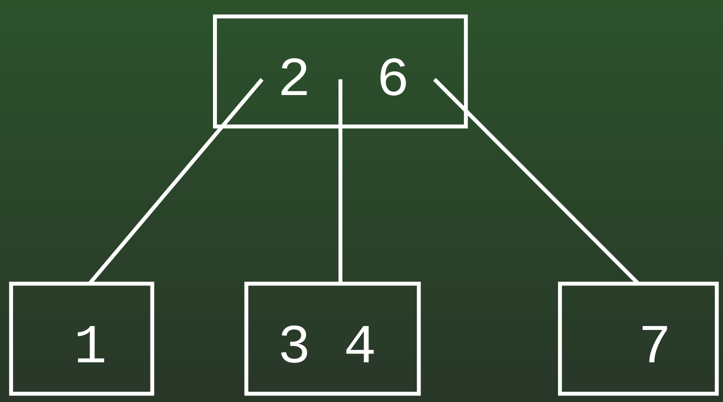
11-0: Binary Search Trees

- Binary Tree data structure
- All values in left subtree $<$ value stored in root
- All values in the right subtree $>$ value stored in root

11-1: Generalizing BSTs

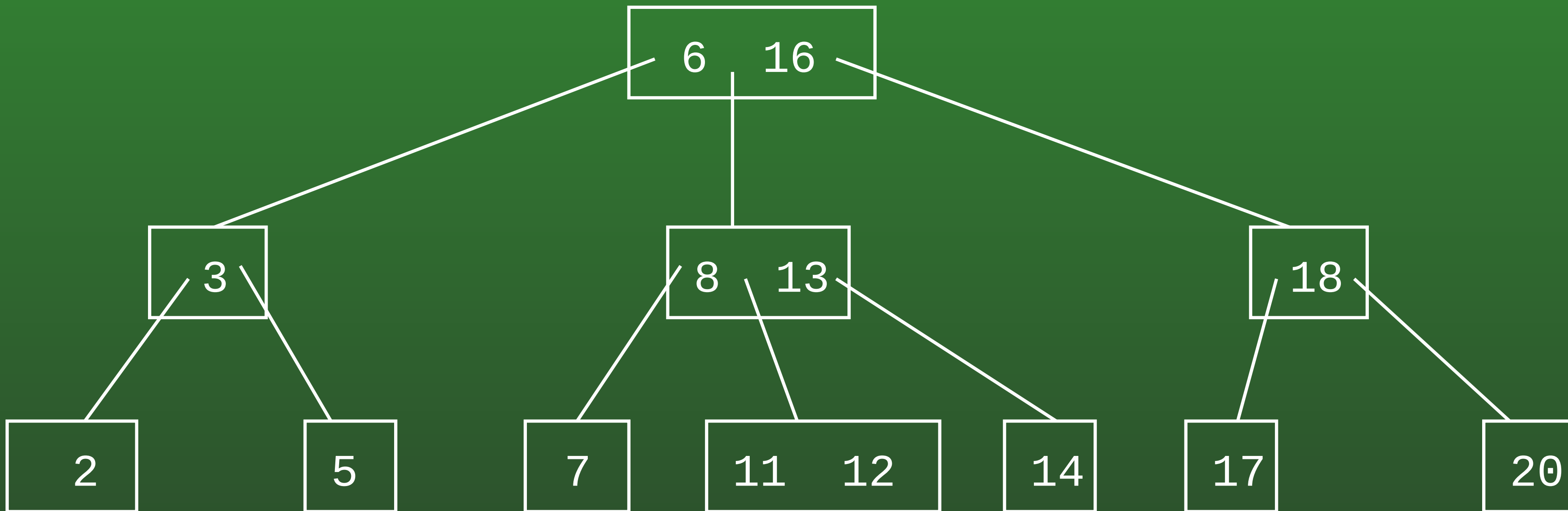
- Generalized Binary Search Trees
  - Each node can store several keys, instead of just one
  - Values in subtrees between values in surrounding keys
  - For non leaves, # of children = # of keys + 1



11-2: 2-3 Trees

- Generalized Binary Search Tree
  - Each node has 1 or 2 keys
  - Each (non-leaf) node has 2-3 children
    - hence the name, 2-3 Trees
  - All leaves are at the same depth

11-3: Example 2-3 Tree



11-4: Finding in 2-3 Trees

- How can we find an element in a 2-3 tree?

11-5: Finding in 2-3 Trees

- How can we find an element in a 2-3 tree?
  - If the tree is empty, return false
  - If the element is stored at the root, return true
  - Otherwise, recursively find in the appropriate subtree

11-6: Inserting into 2-3 Trees

- Always insert at the leaves
- To insert an element:
  - Find the leaf where the element would live, if it was in the tree
  - Add the element to that leaf

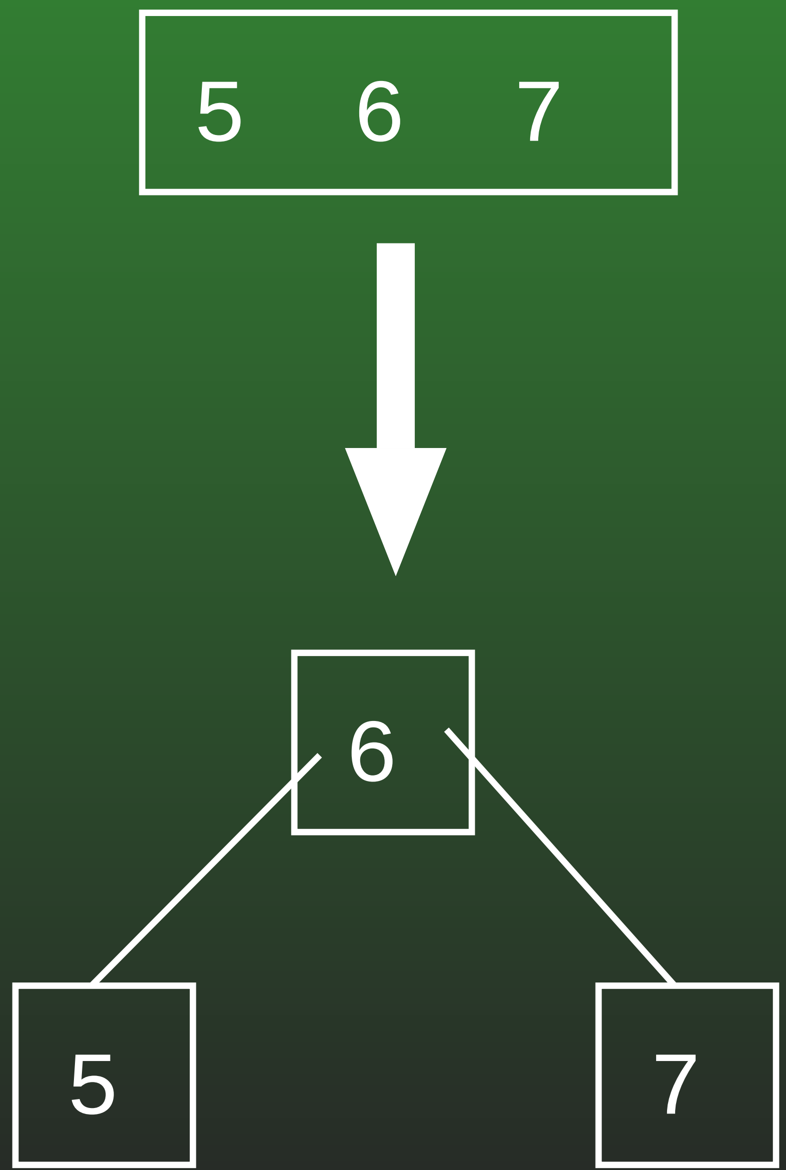
11-7: Inserting into 2-3 Trees

- Always insert at the leaves
- To insert an element:
  - Find the leaf where the element would live, if it was in the tree
  - Add the element to that leaf
    - What if the leaf already has 2 elements?

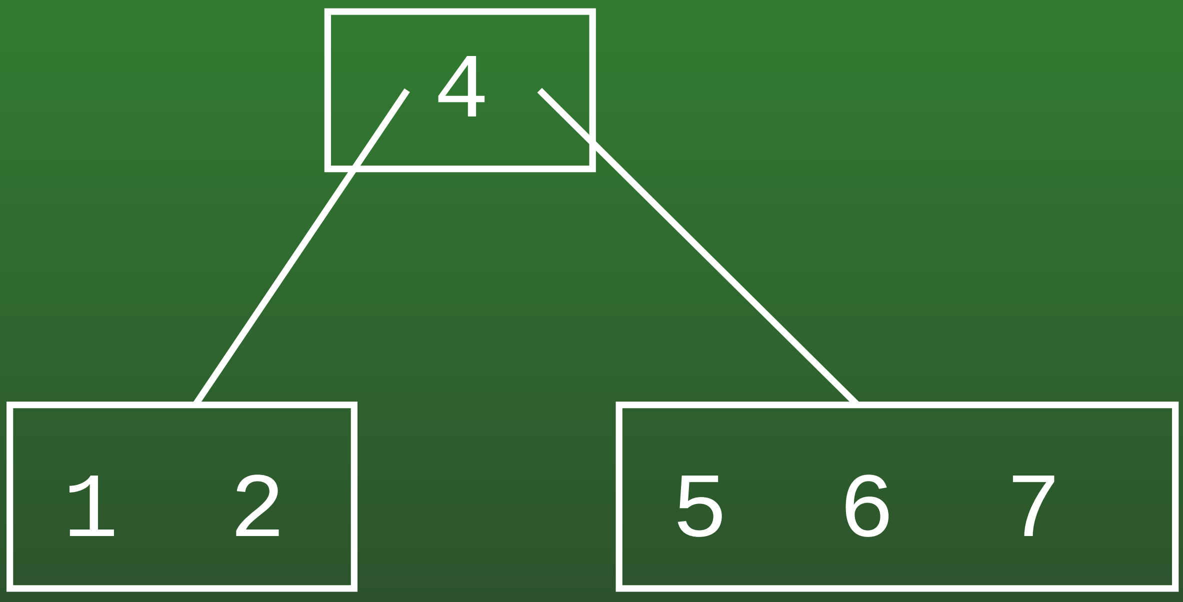
11-8: Inserting into 2-3 Trees

- Always insert at the leaves
- To insert an element:
  - Find the leaf where the element would live, if it was in the tree
  - Add the element to that leaf
    - What if the leaf already has 2 elements?
    - Split!

11-9: Splitting Nodes

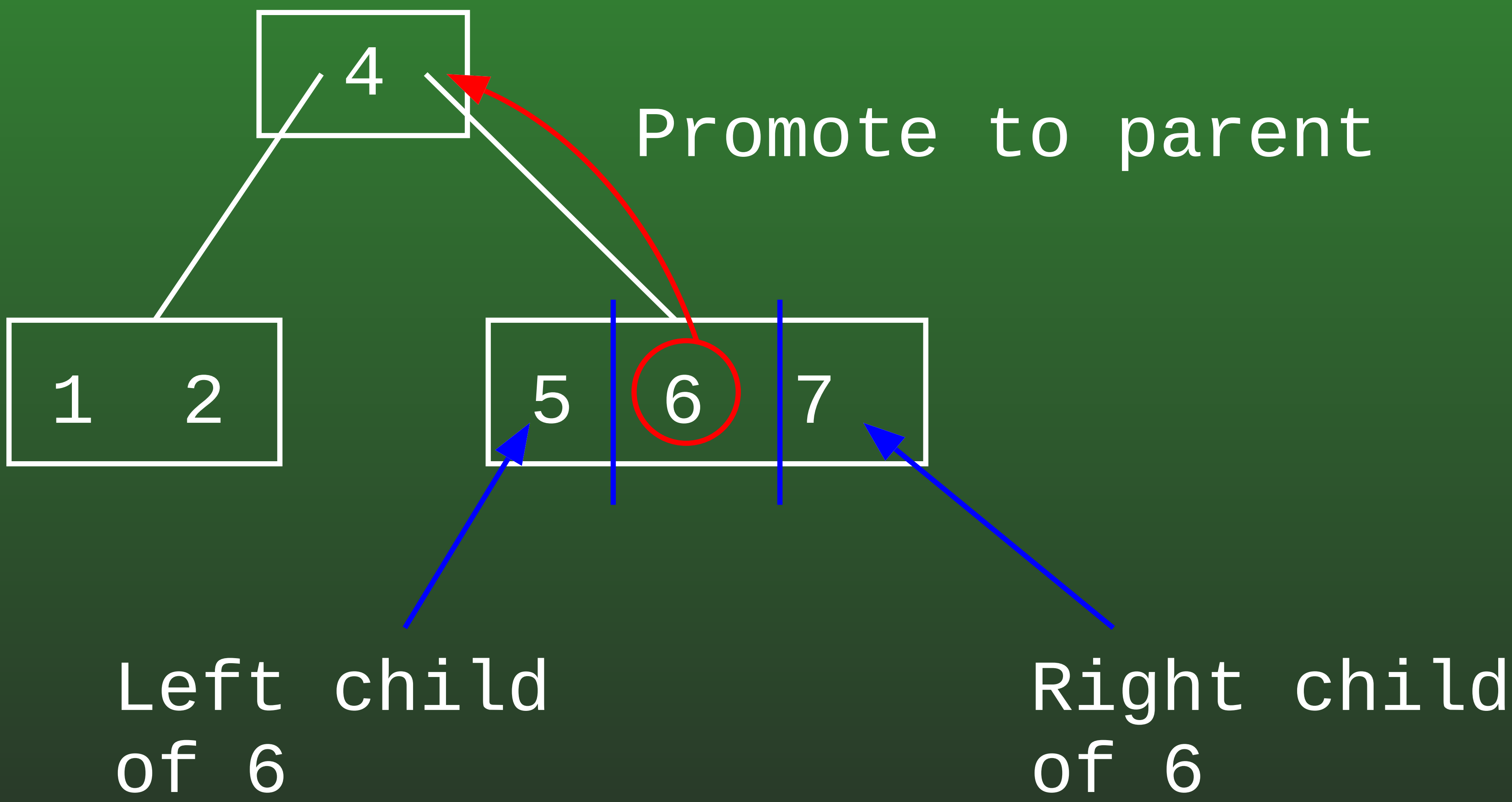


11-10: Splitting Nodes

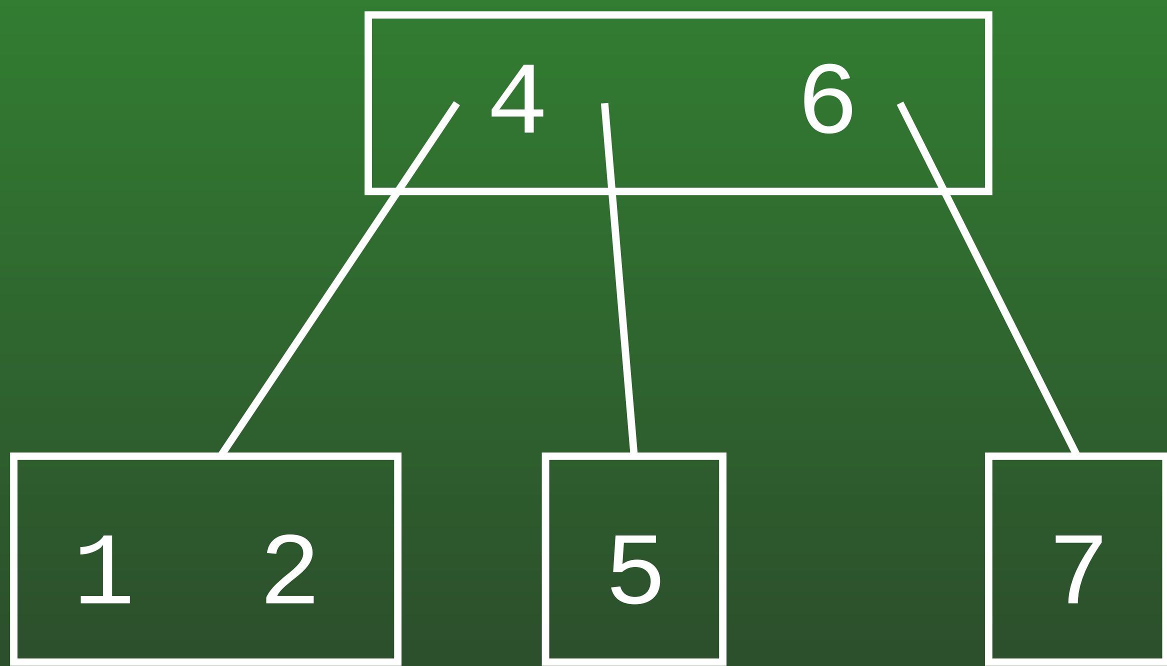


Too many elements

Splitting Nodes



11-12: Splitting Nodes



11-13: Splitting Root

- When we split the root:
  - Create a new root
  - Tree grows in height by 1

11-14: 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree

11-15: 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree

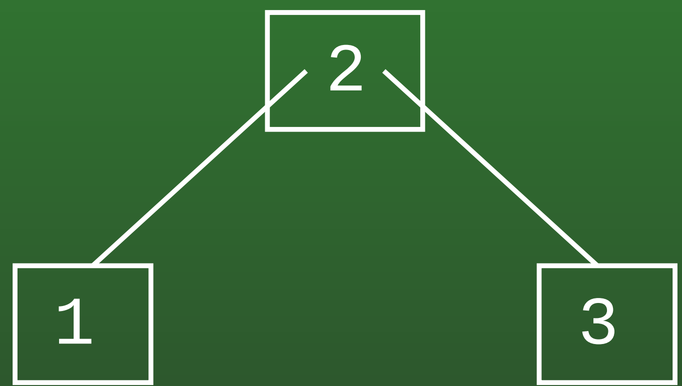
11-16: 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree

Too many keys,
need to split

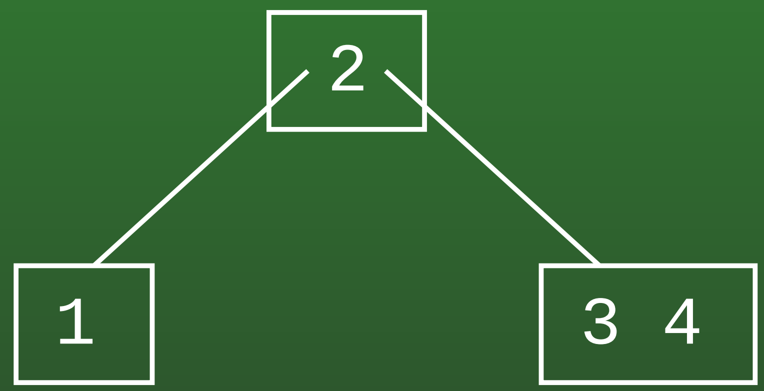
11-17: 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree



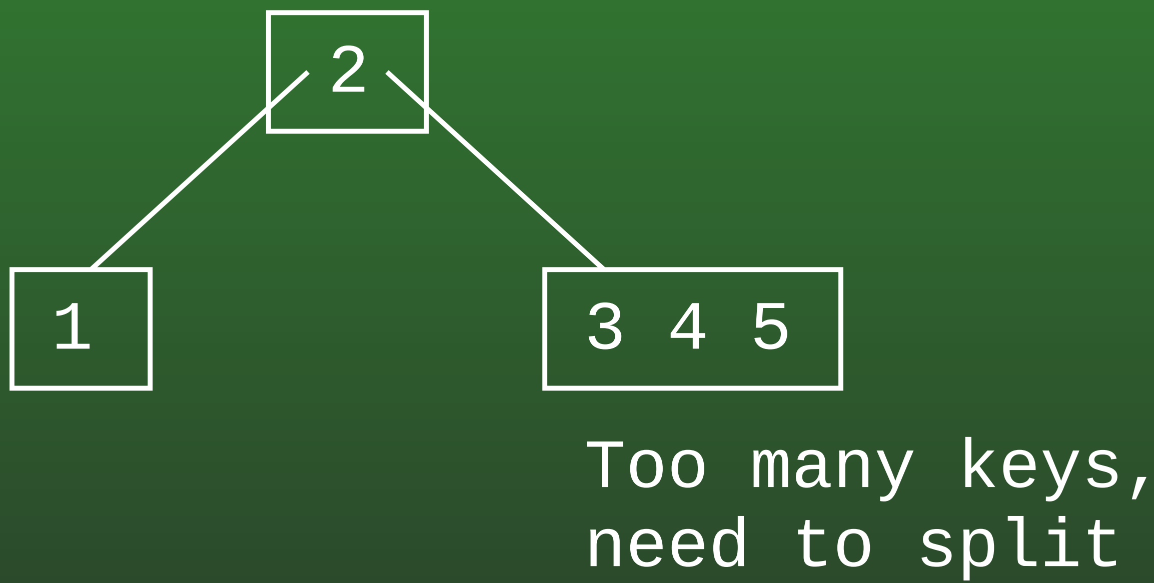
11-18: 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree



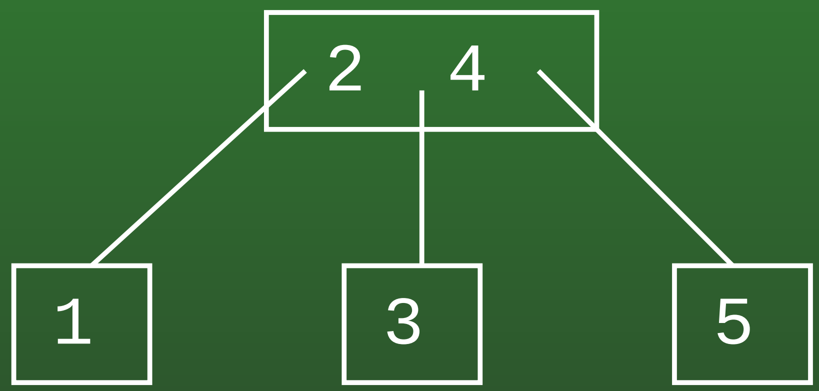
11-19: 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree



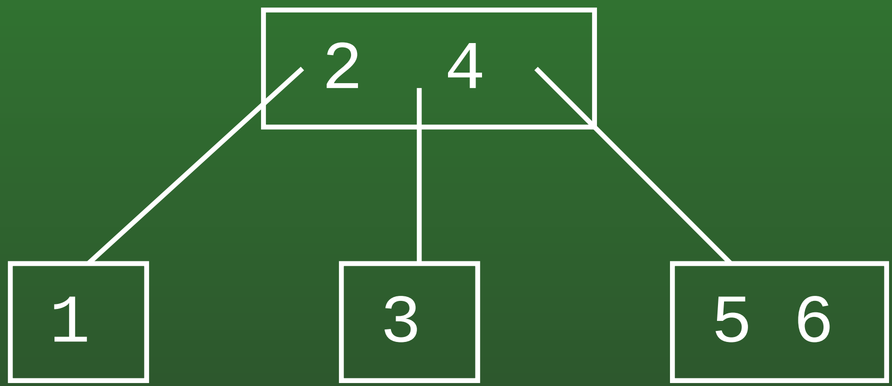
11-20: 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree



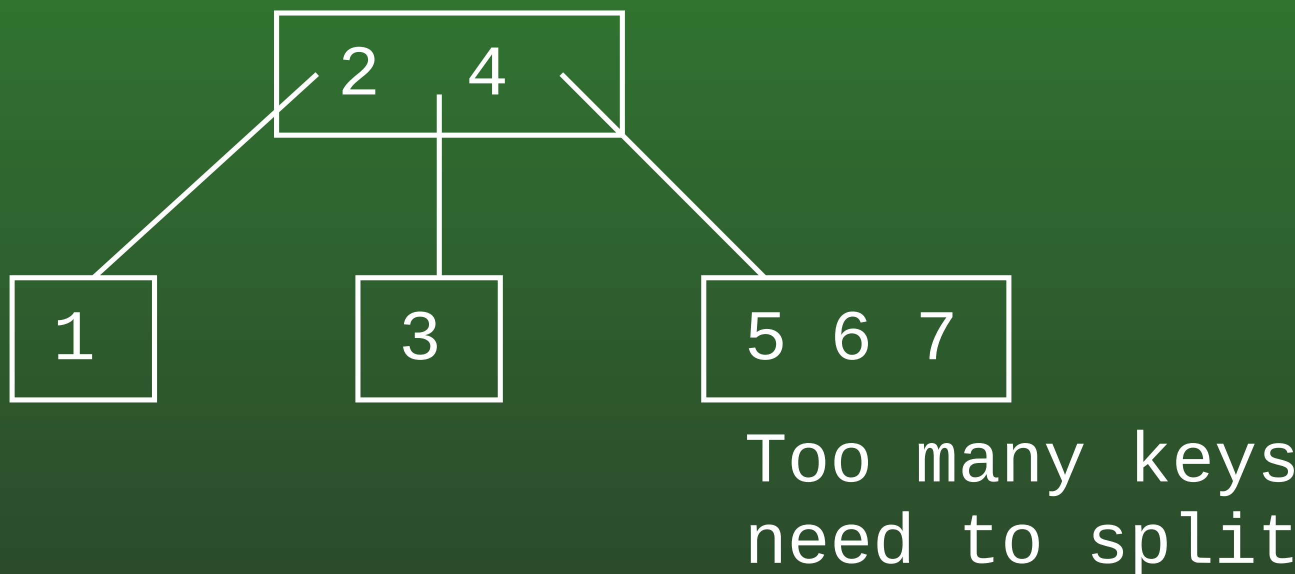
11-21: 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree



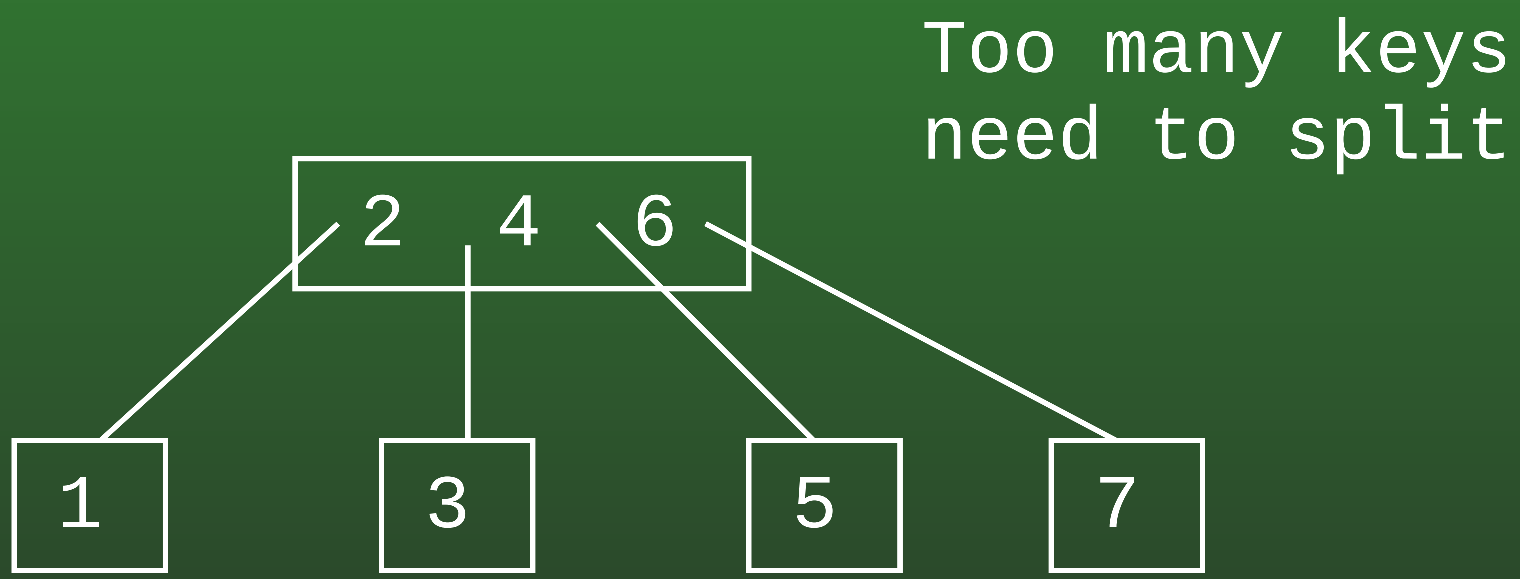
11-22: 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree



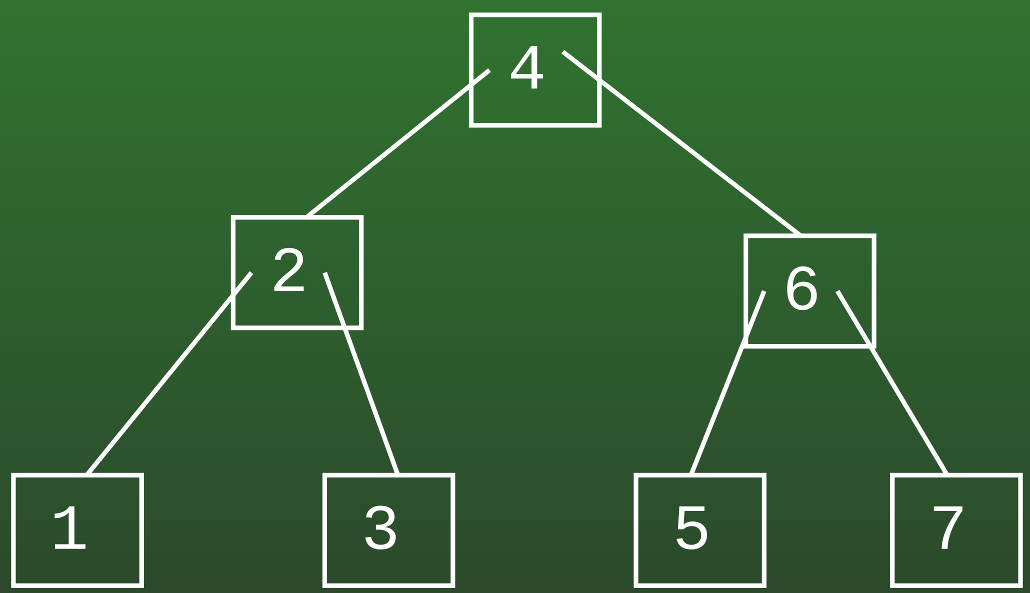
11-23: 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree



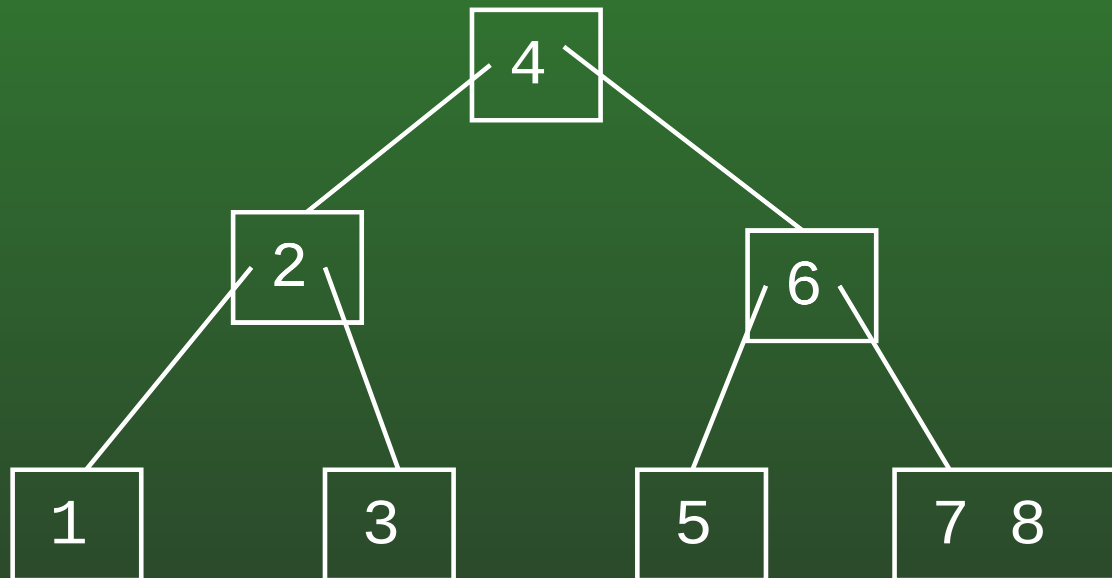
11-24: 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree



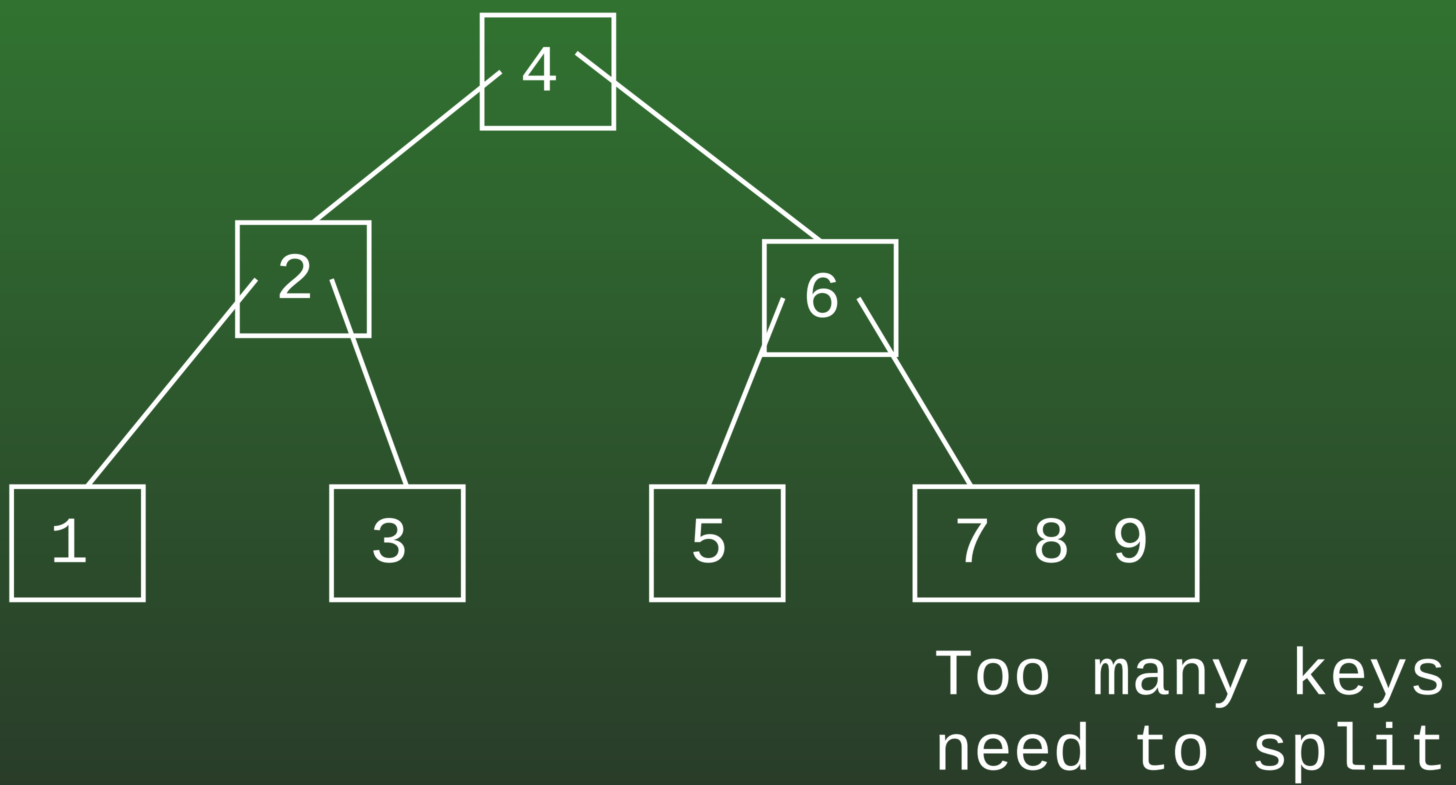
11-25: 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree



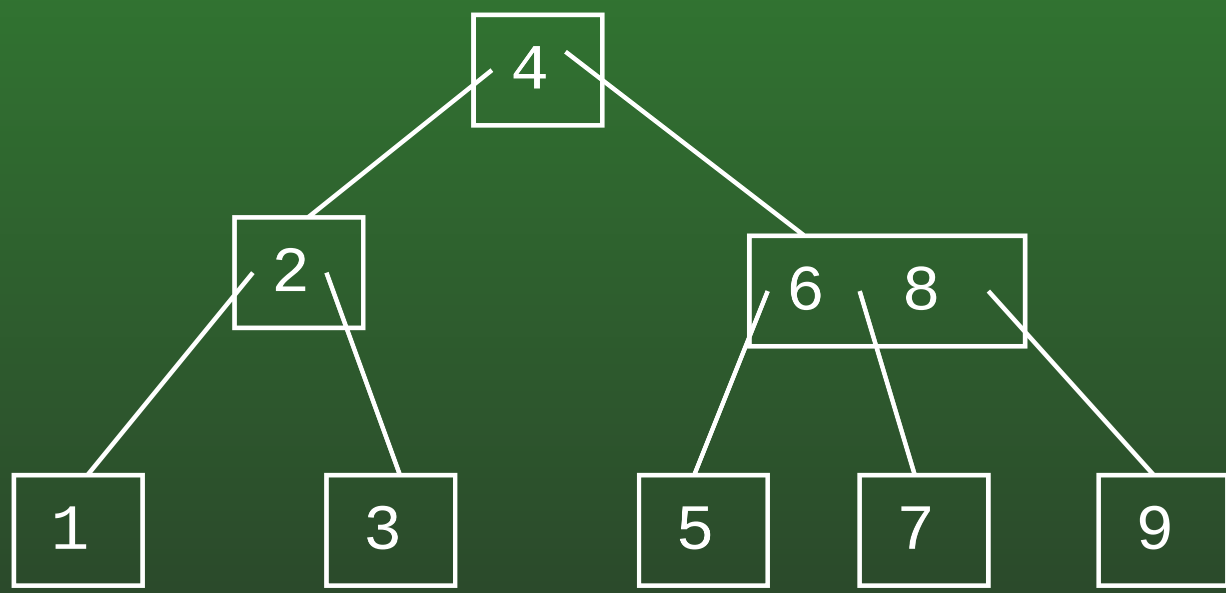
11-26: 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree



11-27: 2-3 Tree Example

- Inserting elements 1-9 (in order) into a 2-3 tree



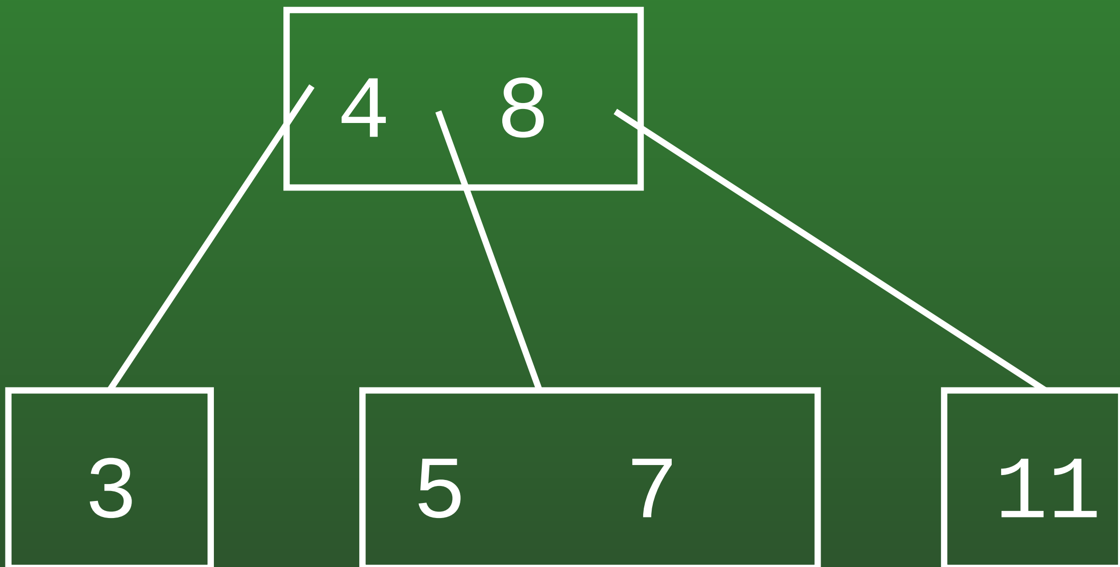
11-28: Deleting from 2-3 Tree

- As with BSTs, we will have 2 cases:
  - Deleting a key from a leaf
  - Deleting a key from an internal node

11-29: Deleting Leaves

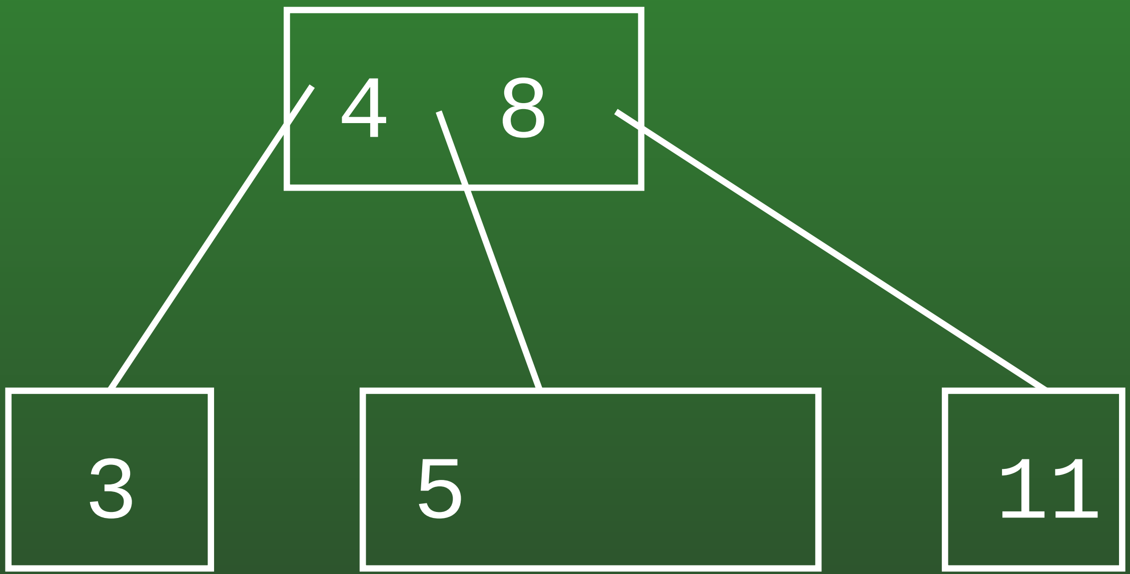
- If leaf contains 2 keys
  - Can safely remove a key

11-30: Deleting Leaves



- Deleting 7

11-31: Deleting Leaves

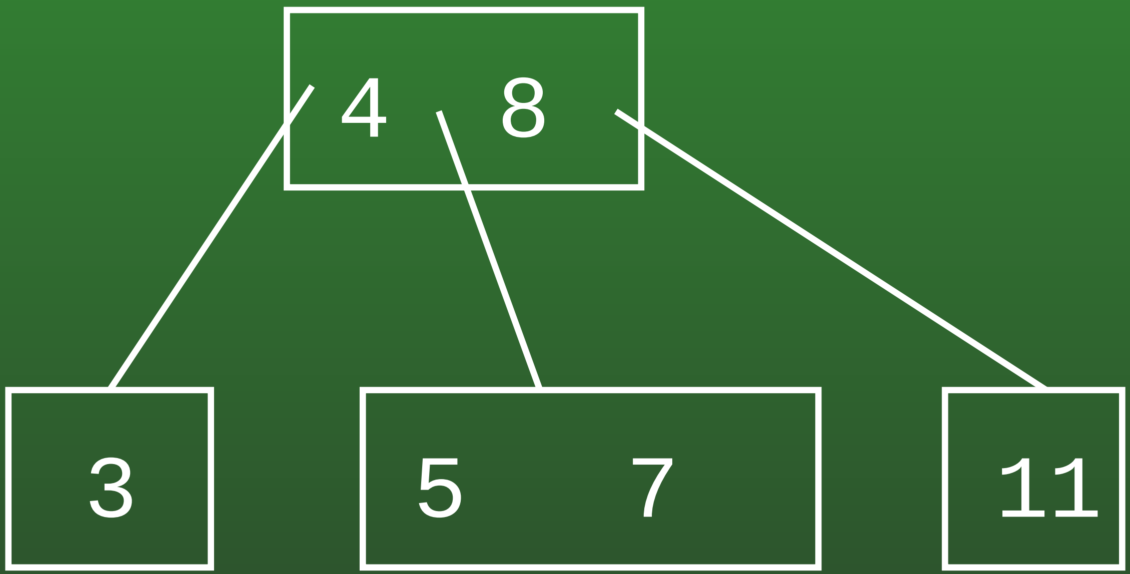


- Deleting 7

11-32: Deleting Leaves

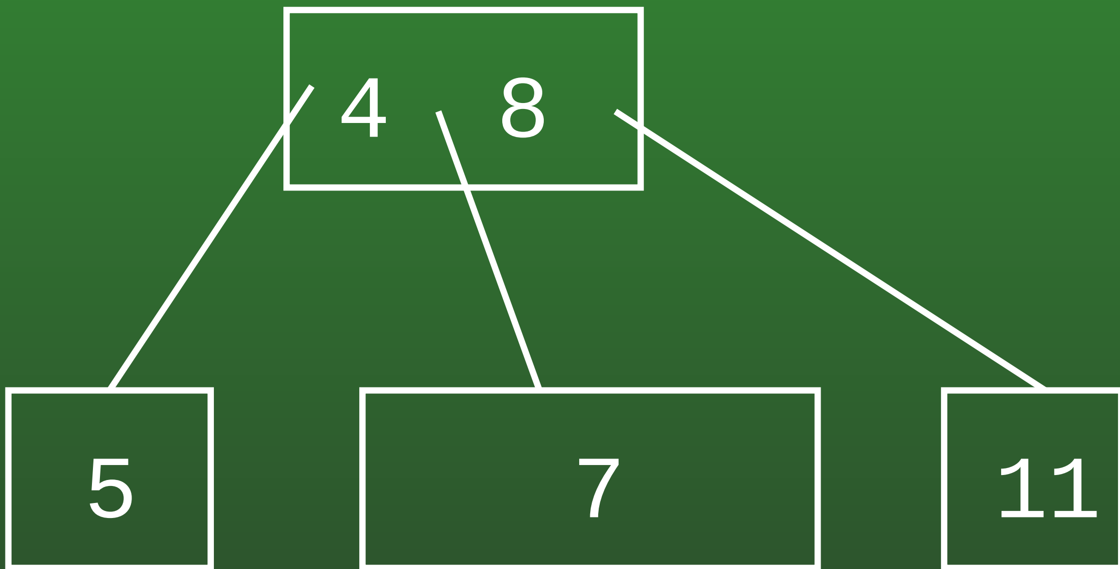
- If leaf contains 1 key
  - Cannot remove key without making leaf empty
  - Try to steal extra key from sibling

11-33: Deleting Leaves



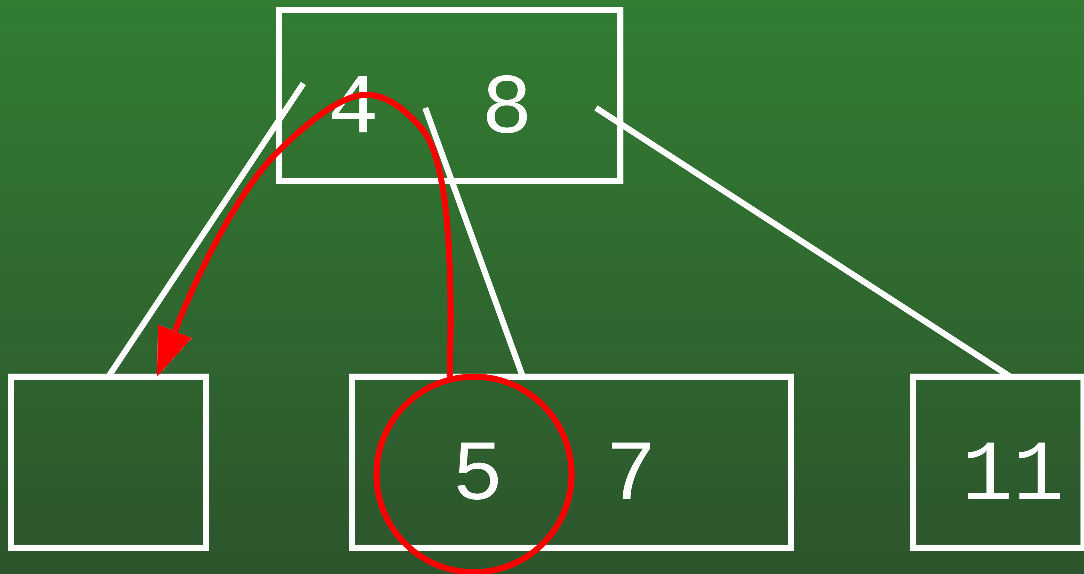
- Deleting 3 – we can steal the 5

11-34: Deleting Leaves



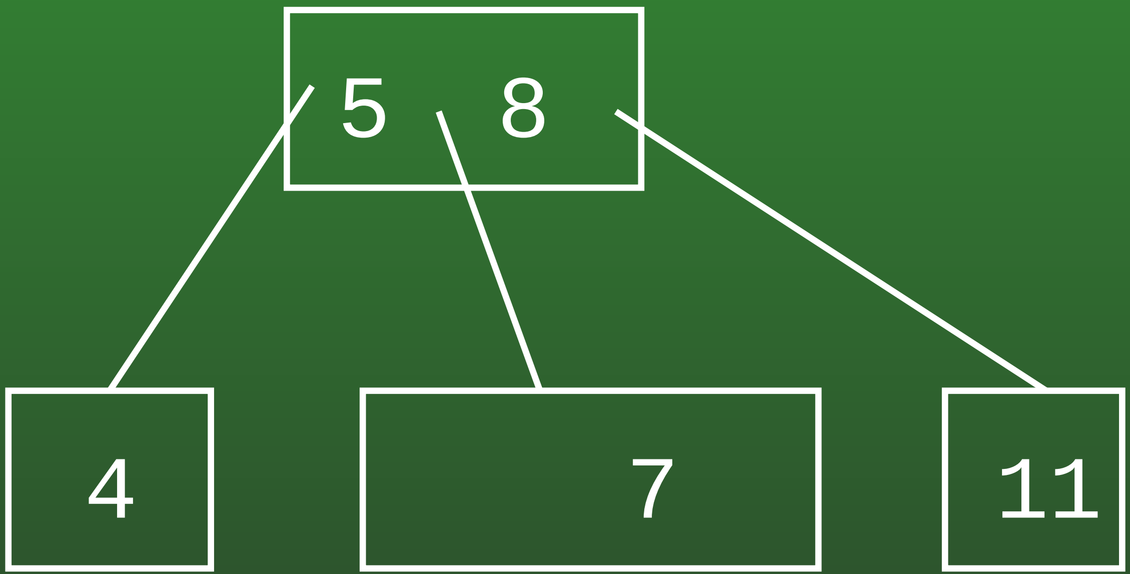
- Not a 2-3 tree. What can we do?

11-35: Deleting Leaves



- Steal key from sibling *through parent*

11-36: Deleting Leaves

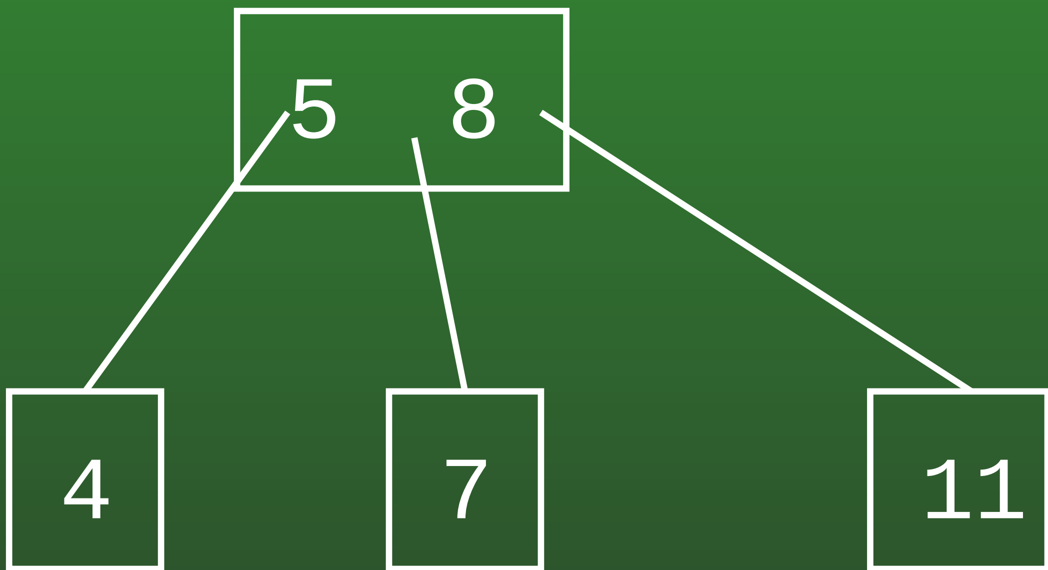


- Steal key from sibling *through parent*

11-37: Deleting Leaves

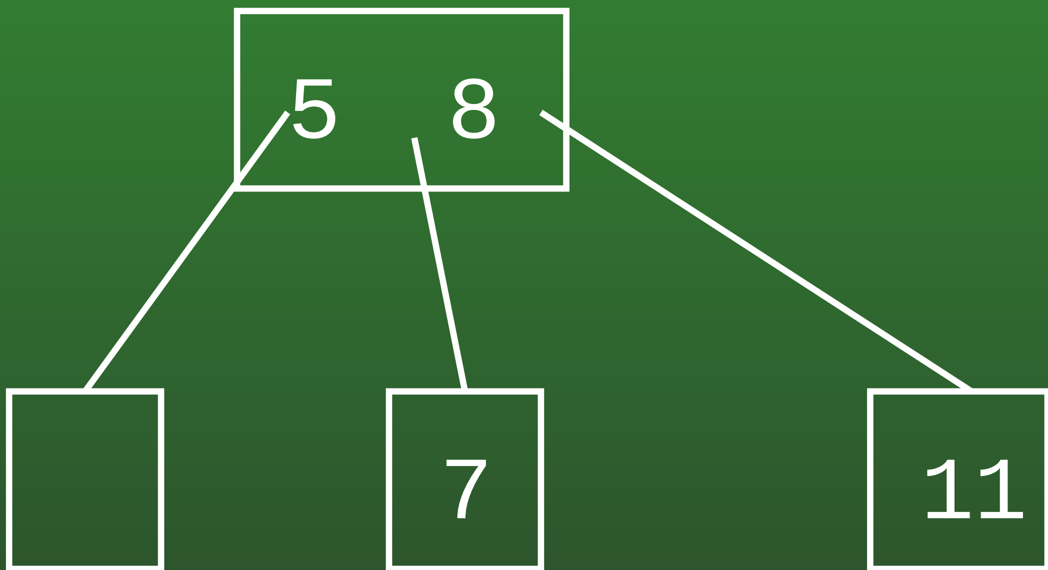
- If leaf contains 1 key, and no sibling contains extra keys
  - Cannot remove key without making leaf empty
  - Cannot steal a key from a sibling
  - Merge with sibling
    - split in reverse

11-38: Merging Nodes



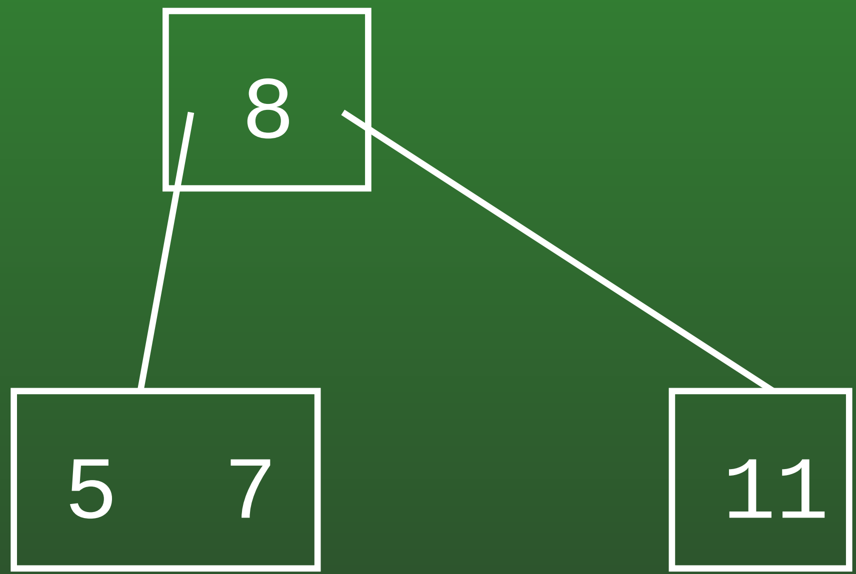
- Removing the 4

11-39: Merging Nodes



- Removing the 4
- Combine 5, 7 into one node

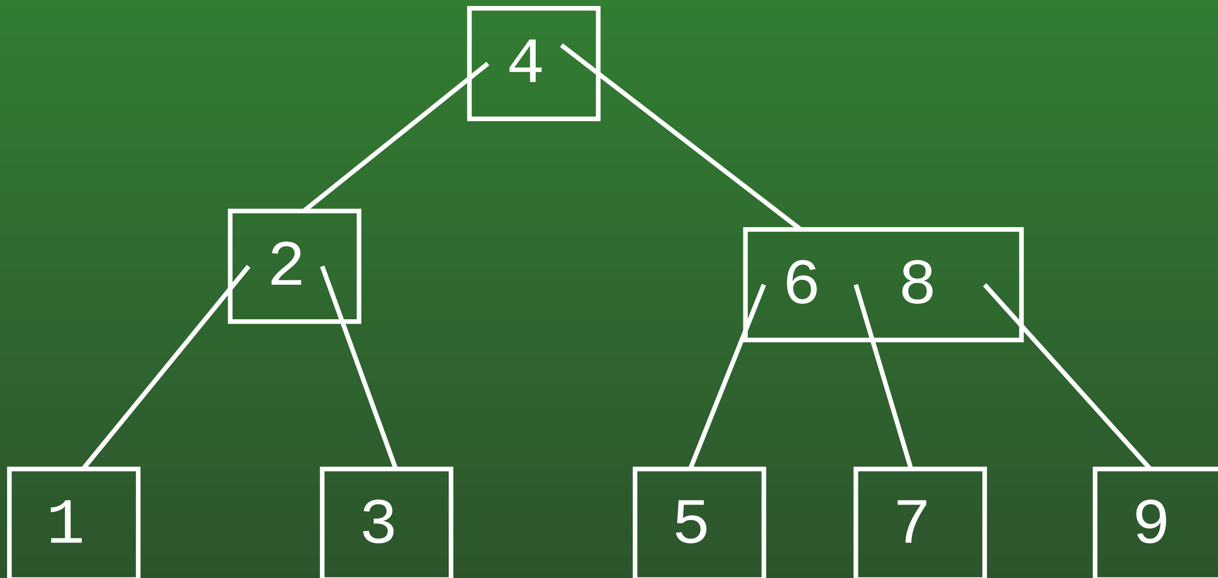
11-40: Merging Nodes



11-41: Merging Nodes

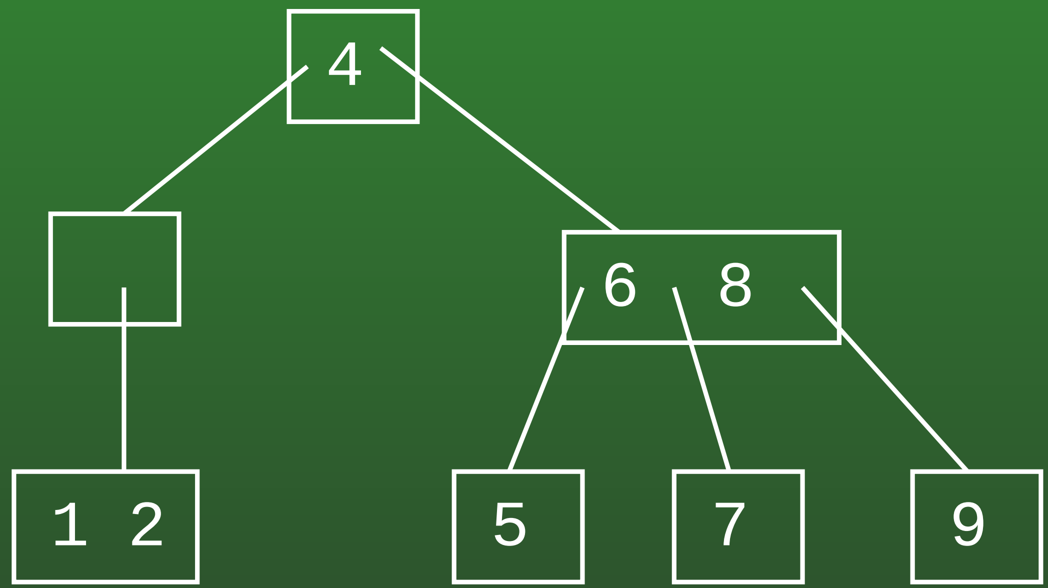
- Merge decreases the number of keys in the parent
  - May cause parent to have too few keys
- Parent can steal a key, or merge again

11-42: Merging Nodes



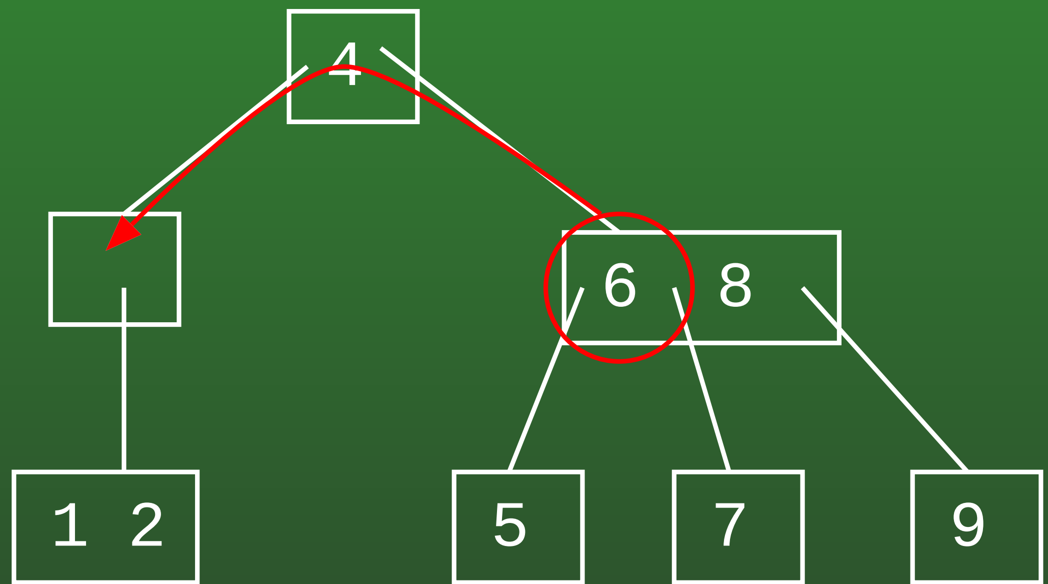
- Deleting the 3 – cause a merge

11-43: Merging Nodes



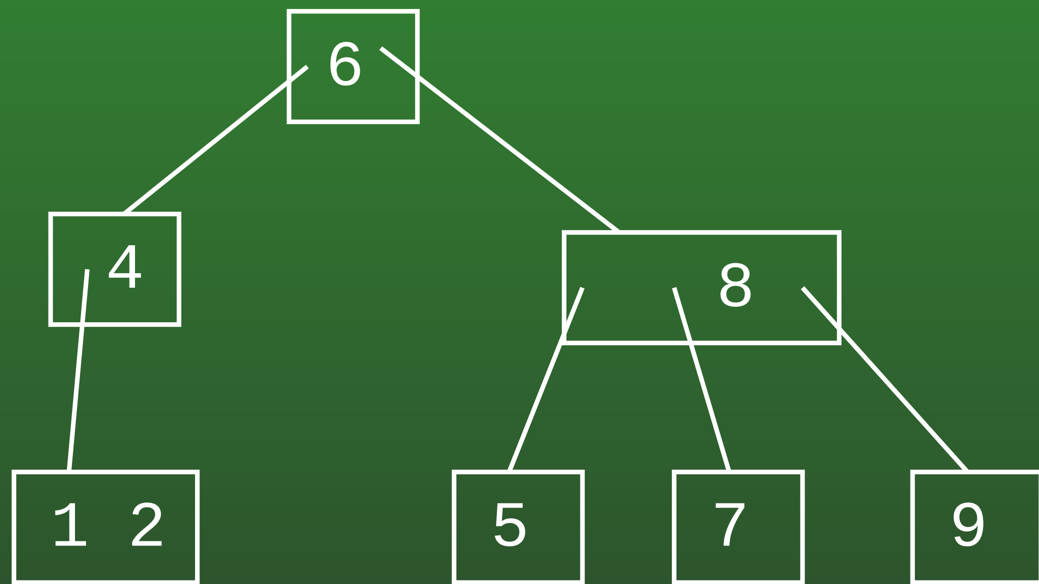
- Deleting the 3 – cause a merge
- Not enough keys in parent

11-44: Merging Nodes



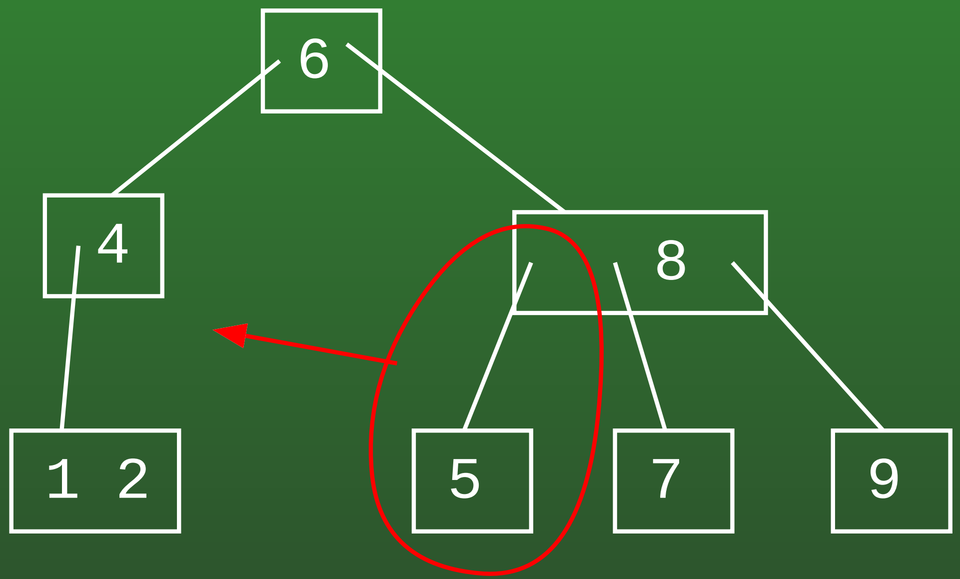
- Steal key from sibling

11-45: Merging Nodes



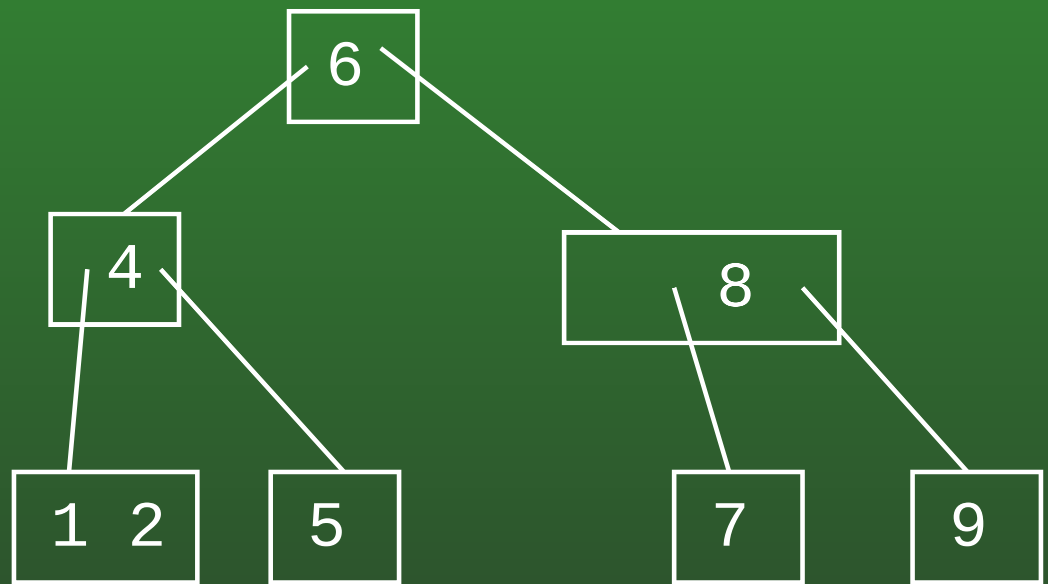
- Steal key from sibling

11-46: Merging Nodes



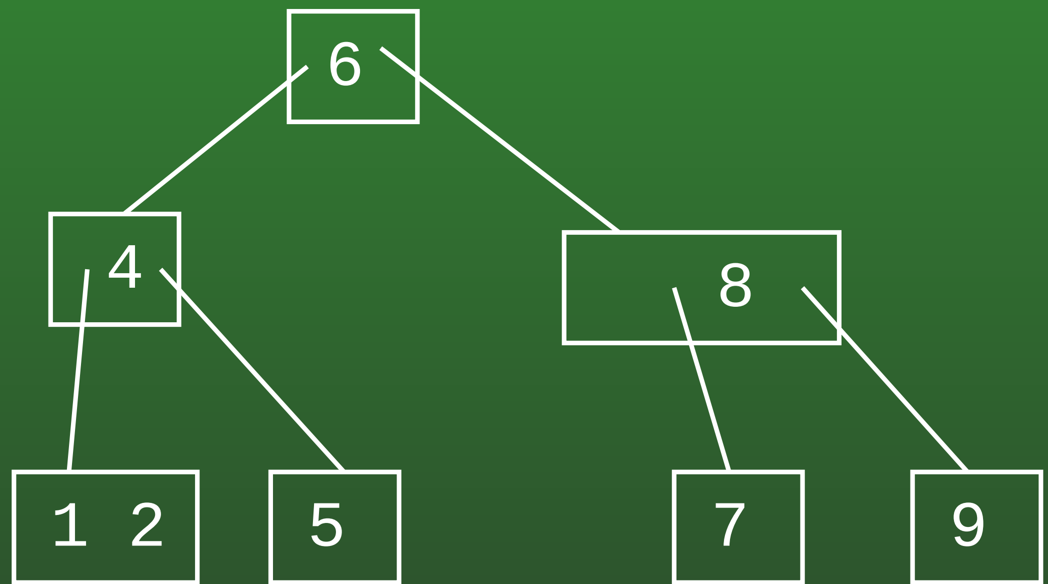
- When we steal a key from an internal node, steal nearest subtree as well

11-47: Merging Nodes



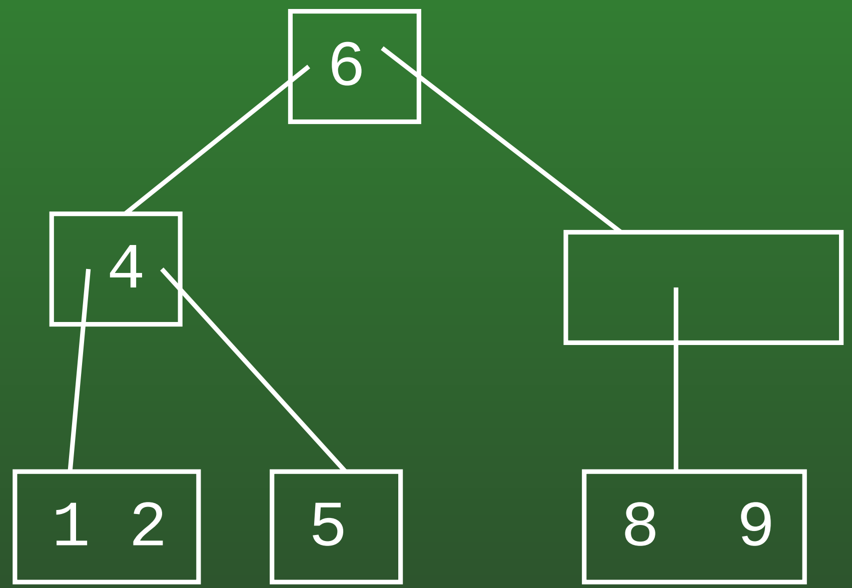
- When we steal a key from an internal node, steal nearest subtree as well

11-48: Merging Nodes



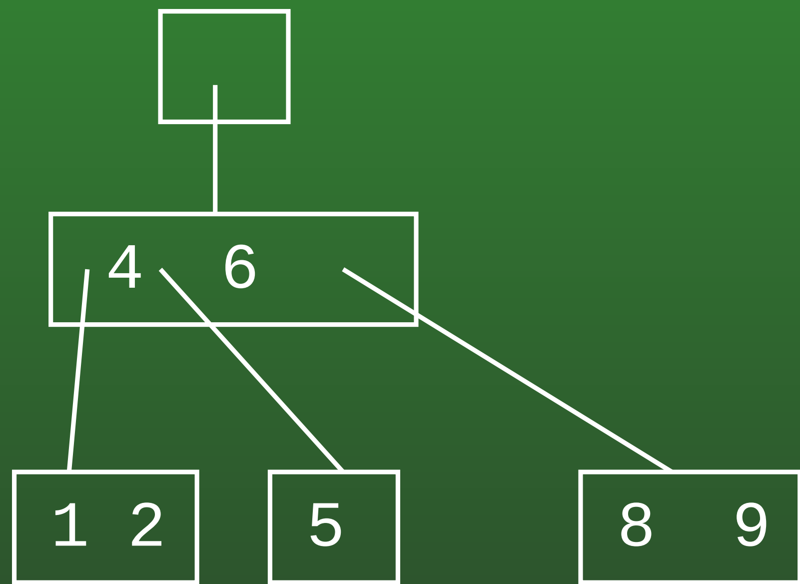
- Deleting the 7 – cause a merge

11-49: Merging Nodes



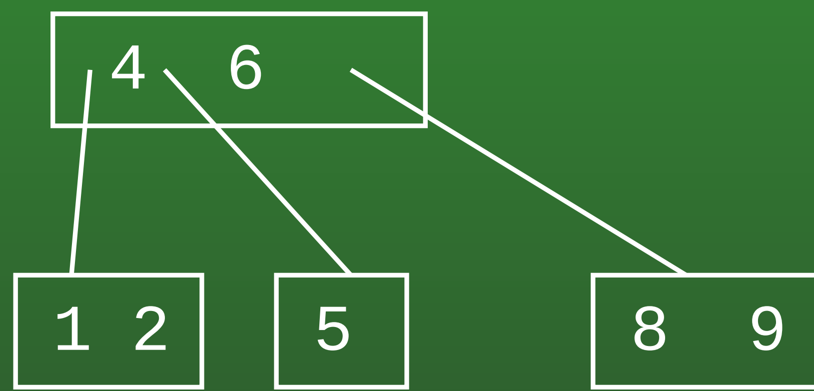
- Parent has too few keys – merge again

Merging Nodes



- Root has no keys – delete

Merging Nodes



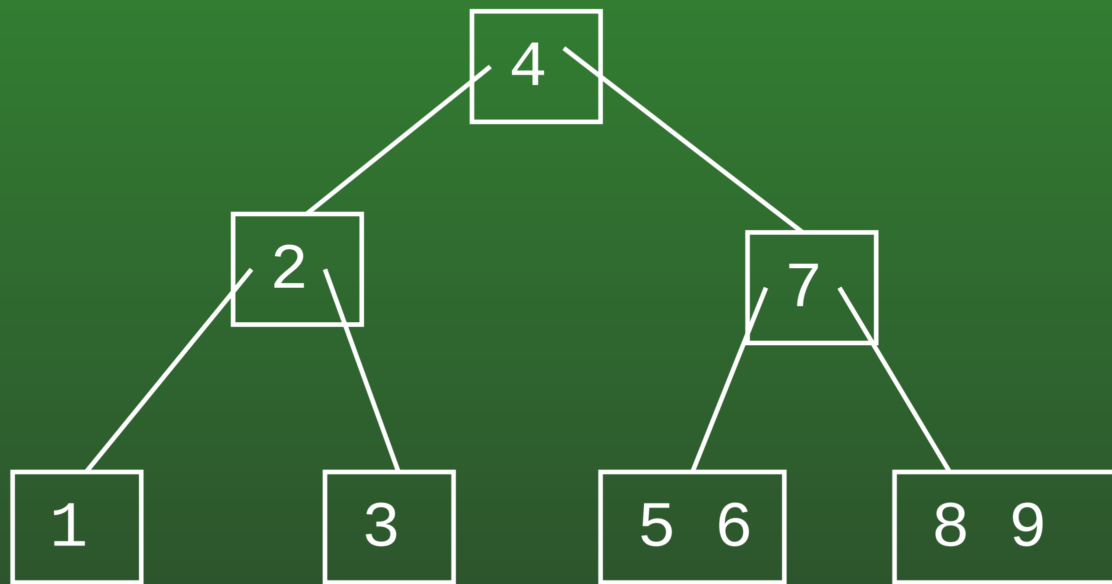
11-52: Deleting Interior Keys

- How can we delete keys from non-leaf nodes?
  - *HINT:* How did we delete non-leaf nodes in standard BSTs?

11-53: Deleting Interior Keys

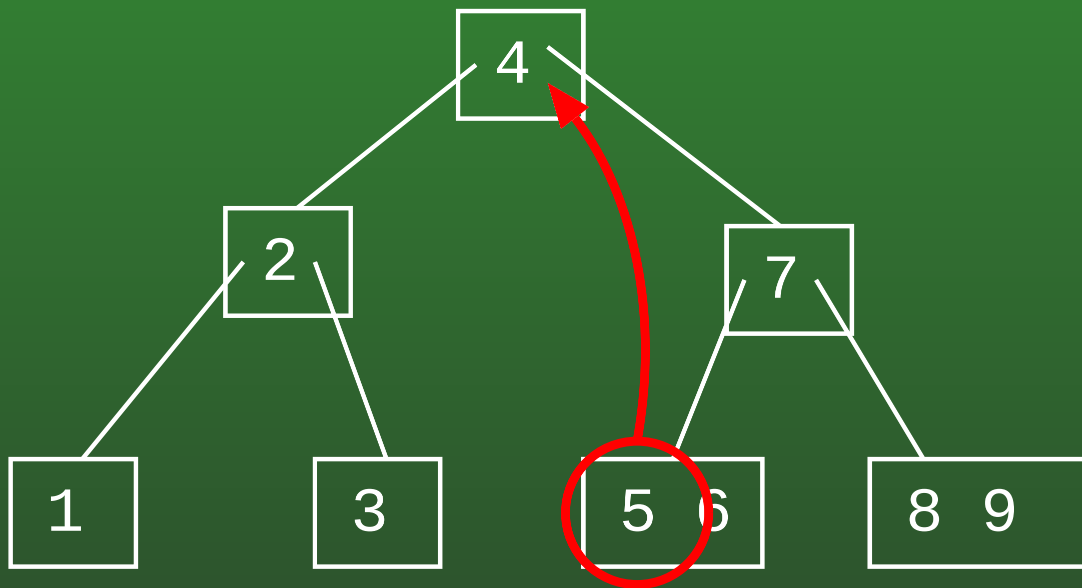
- How can we delete keys from non-leaf nodes?
  - Replace key with smallest element subtree to right of key
  - Recursivly delete smallest element from subtree to right of key
- (can also use largest element in subtree to left of key)

11-54: Deleting Interior Keys



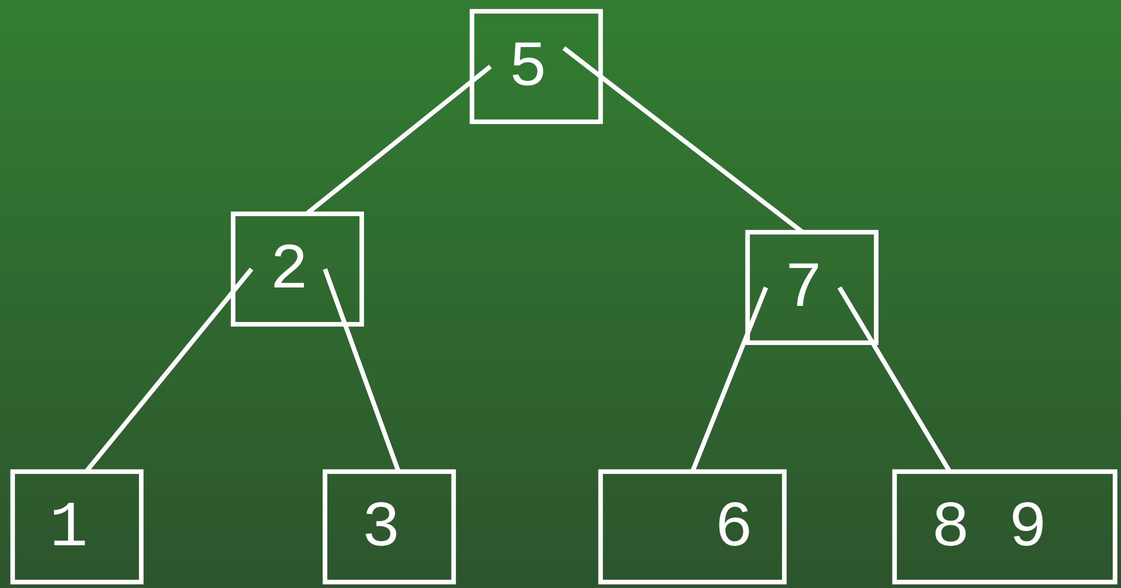
- Deleting the 4

Deleting Interior Keys

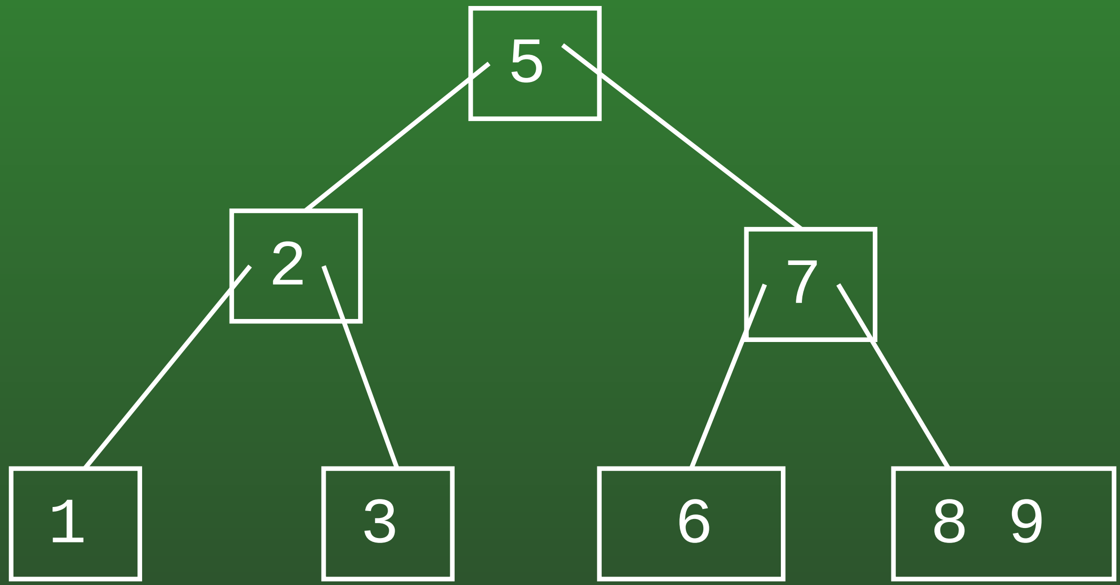


- Deleting the 4
- Replace 4 with smallest element in tree to right of 4

11-56: Deleting Interior Keys

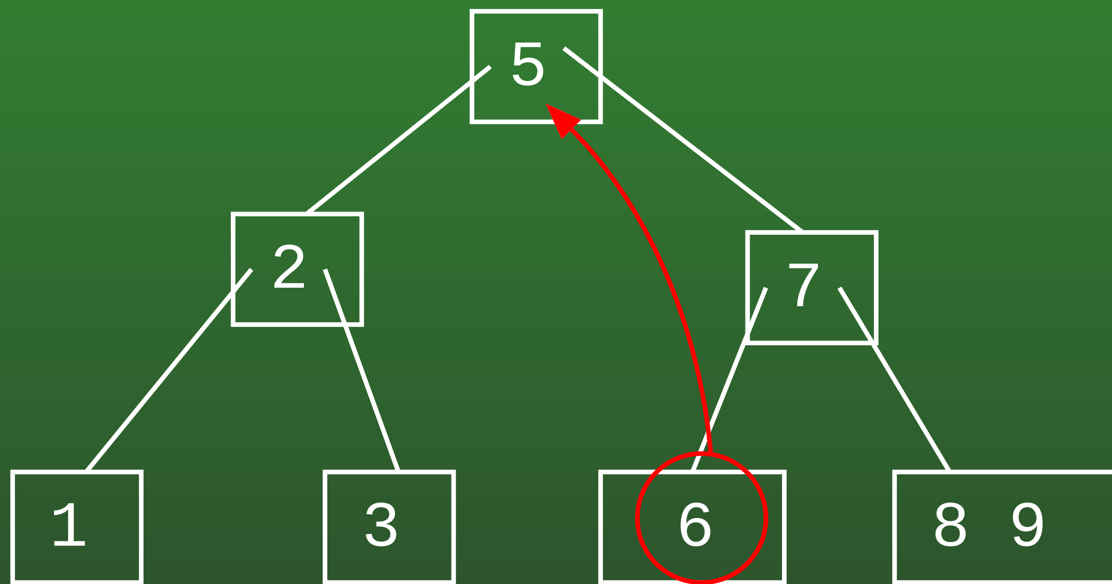


Deleting Interior Keys



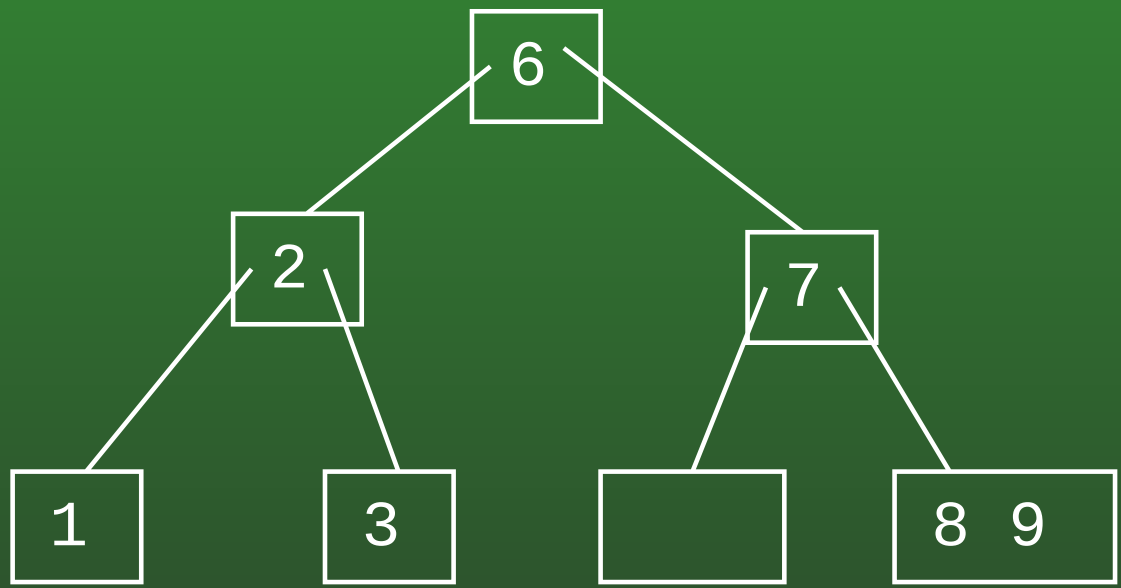
- Deleting the 5

11-58: Deleting Interior Keys



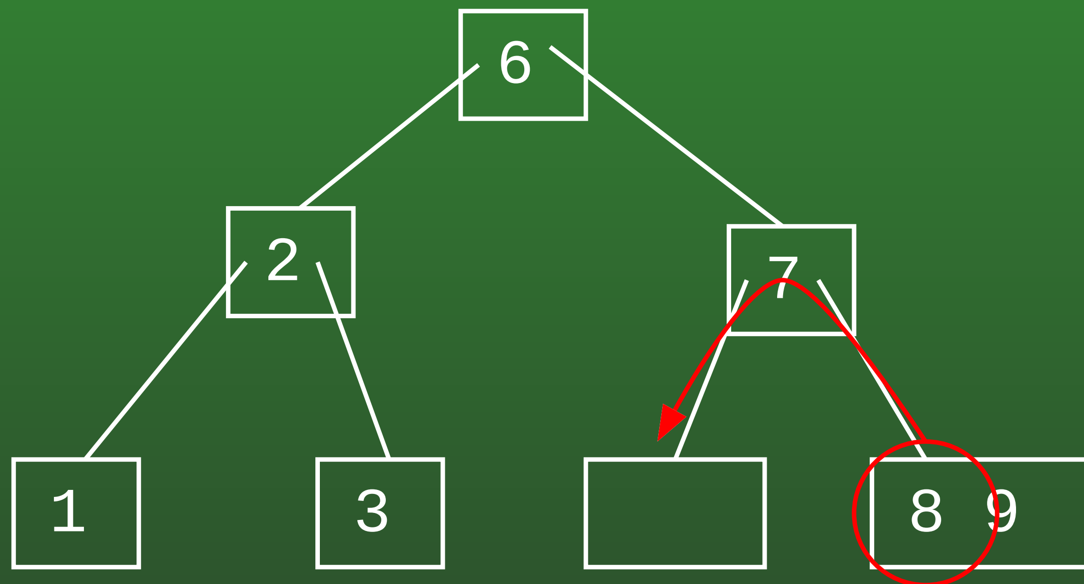
- Deleting the 5
- Replace the 5 with the smallest element in tree to right of 5

11-59: Deleting Interior Keys



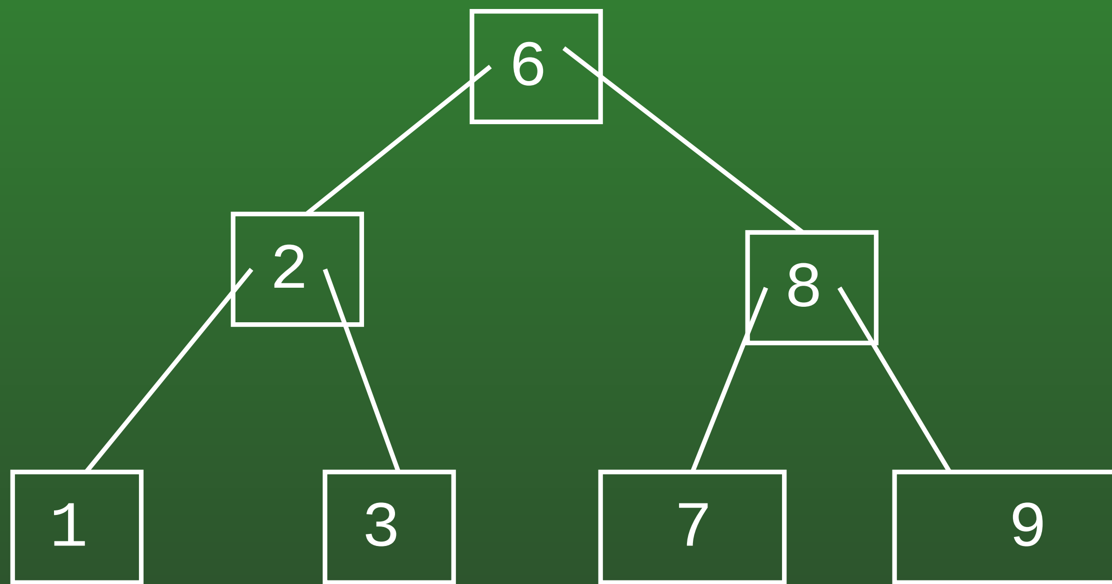
- Deleting the 5
- Replace the 5 with the smallest element in tree to right of 5
- Node with two few keys

11-60: Deleting Interior Keys

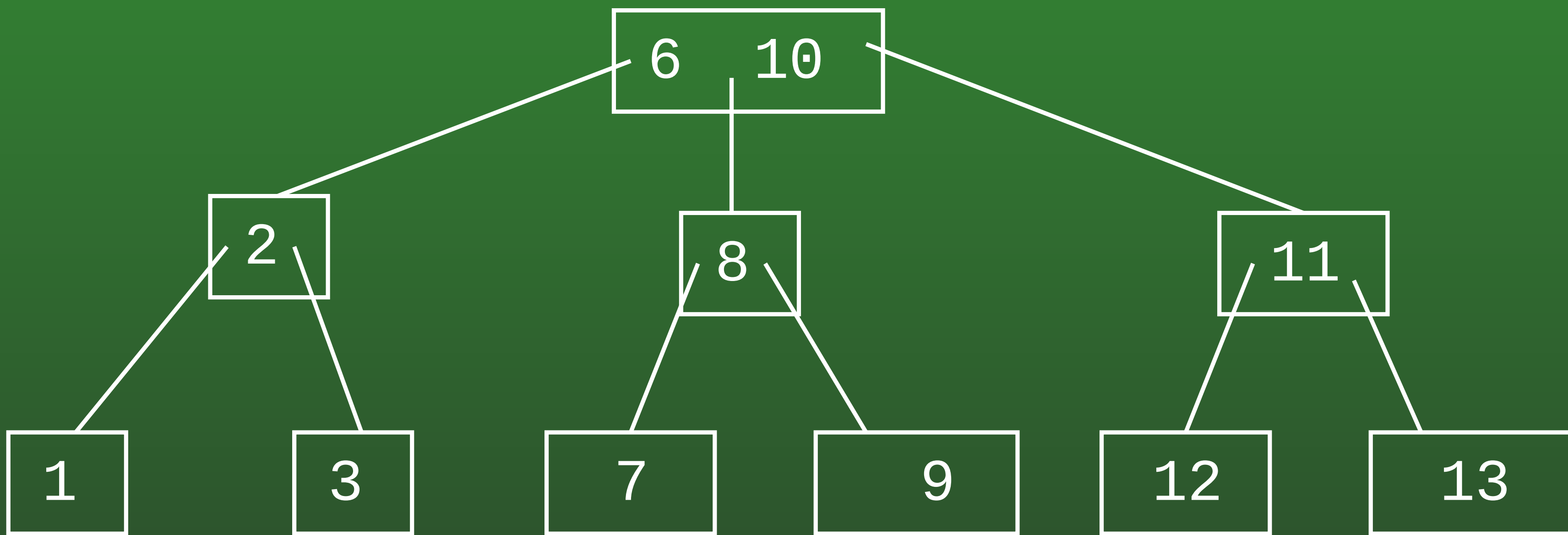


- Node with two few keys
- Steal a key from a sibling

11-61: Deleting Interior Keys

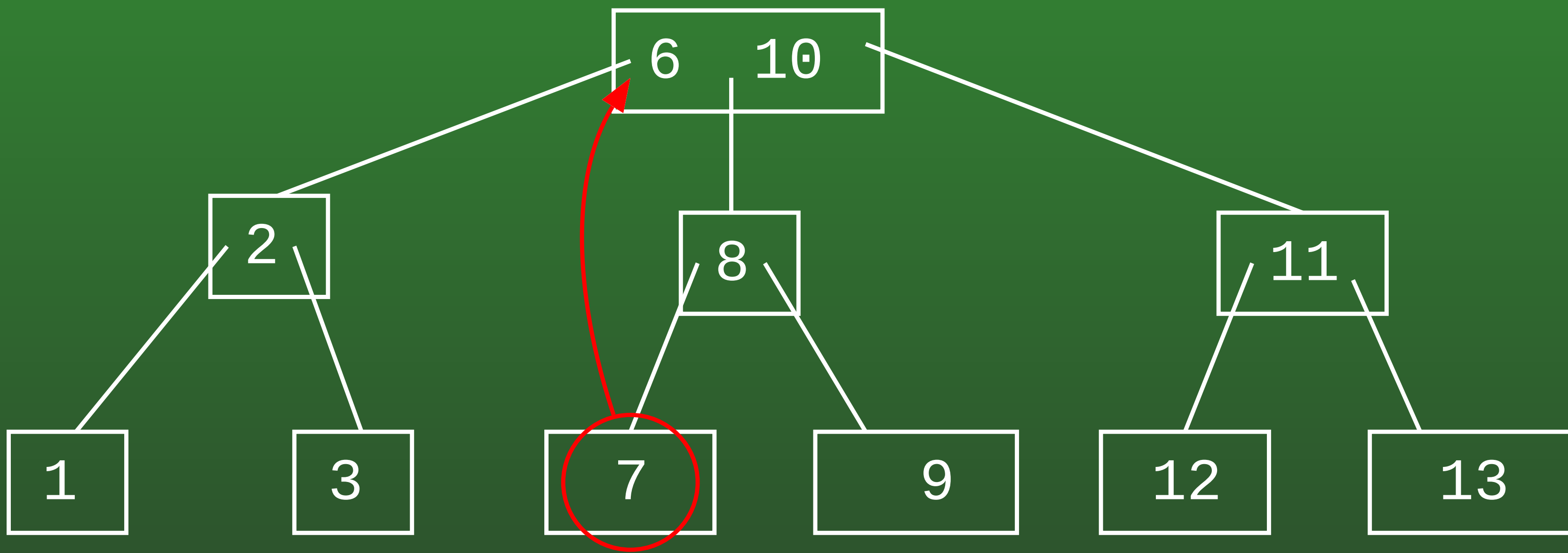


11-62: Deleting Interior Keys



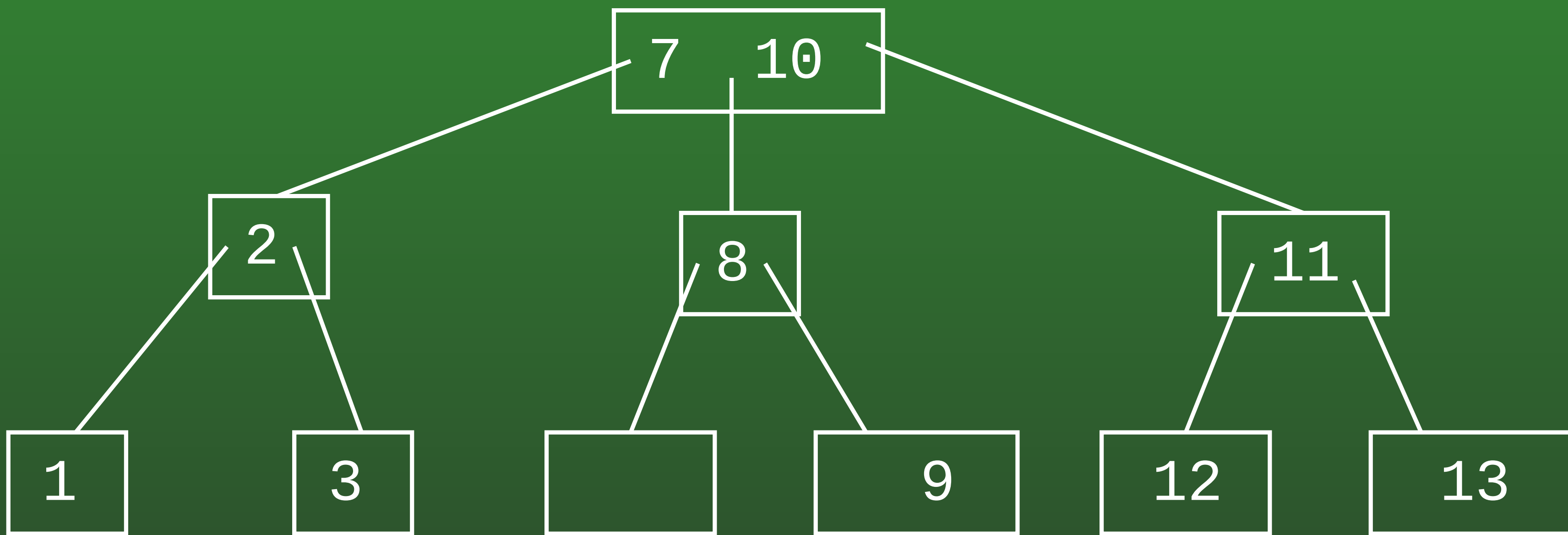
- Removing the 6

11-63: Deleting Interior Keys



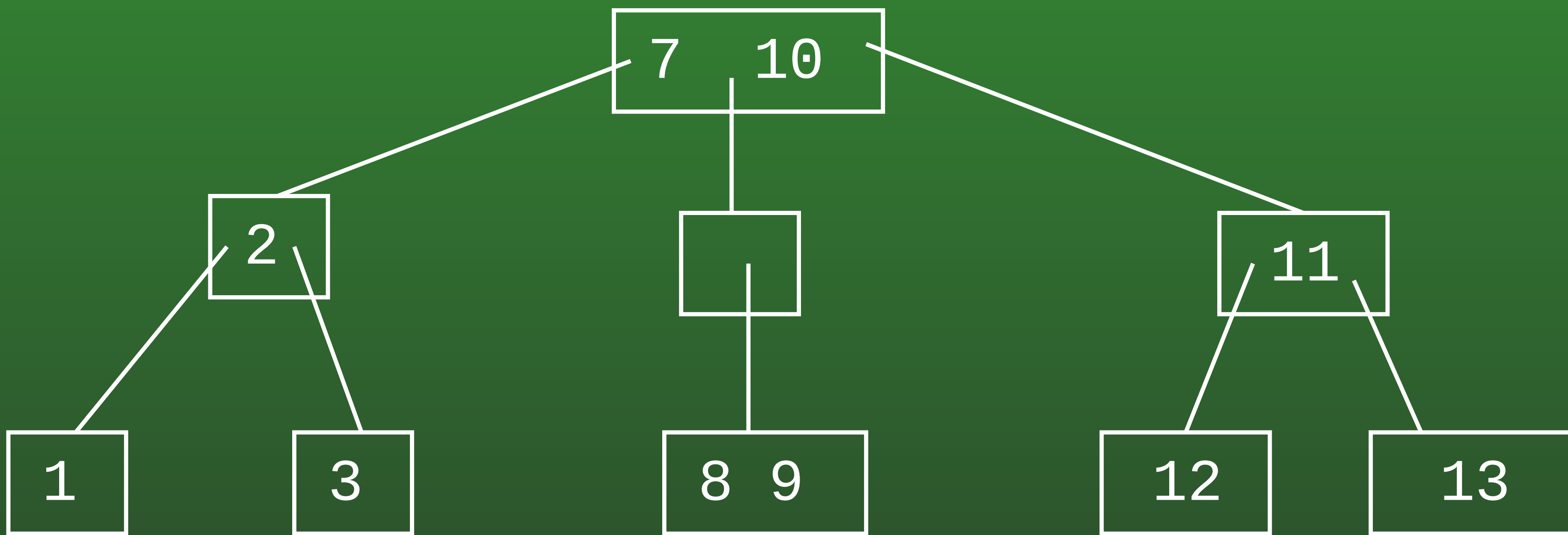
- Removing the 6
- Replace the 6 with the smallest element in the tree to the right of the 6

11-64: Deleting Interior Keys



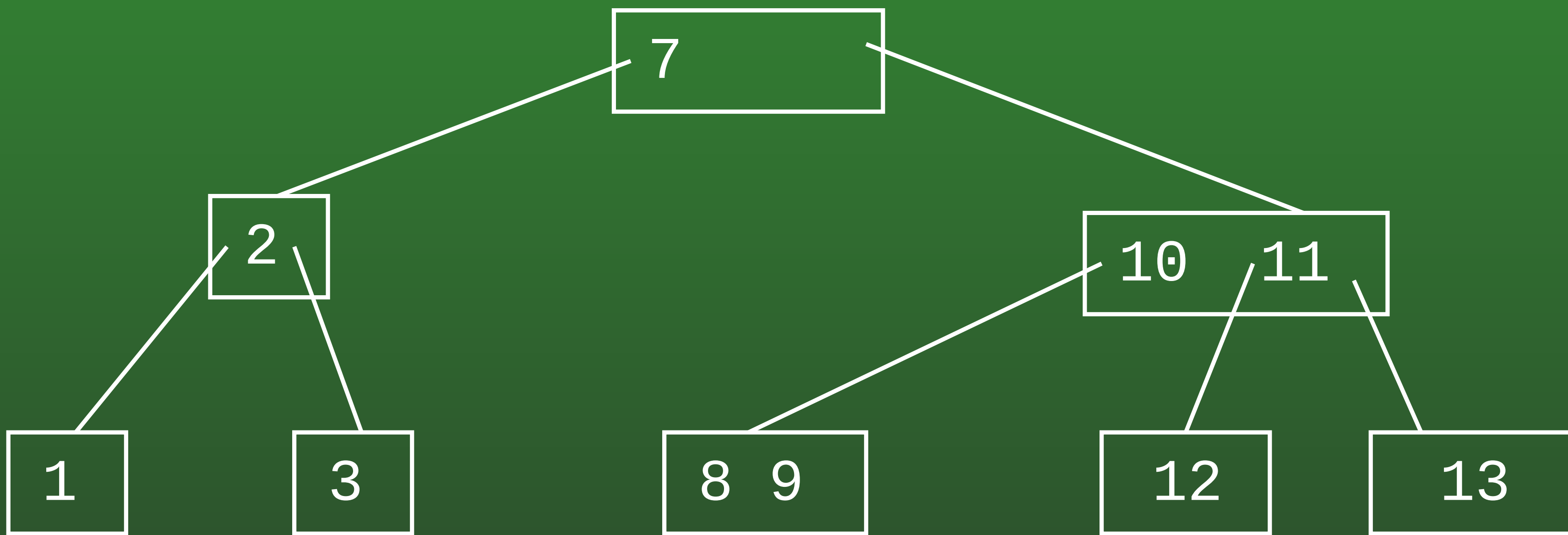
- Node with too few keys
  - Can't steal key from sibling
  - Merge with sibling

Deleting Interior Keys



- Node with too few keys
  - Can’t steal key from sibling
  - Merge with sibling
  - (arbitrarily pick right sibling to merge with)

11-66: Deleting Interior Keys



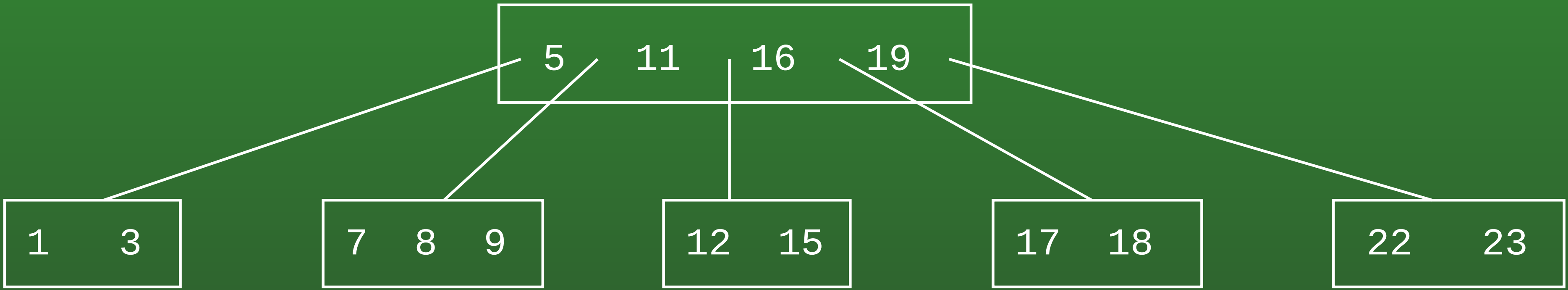
11-67: Generalizing 2-3 Trees

- In 2-3 Trees:
  - Each node has 1 or 2 keys
  - Each interior node has 2 or 3 children
- We can generalize 2-3 trees to allow more keys / node

B-Trees

- A B-Tree of maximum degree k:
  - All interior nodes have $\lceil k/2 \rceil \ldots k$ children
  - All nodes have $\lceil k/2 \rceil - 1 \ldots k - 1$ keys
- 2-3 Tree is a B-Tree of maximum degree 3

B-Trees

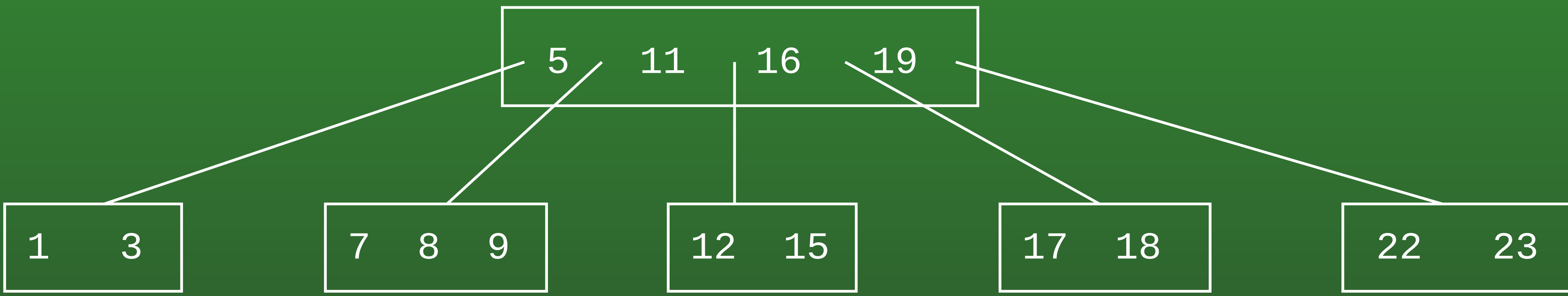


- B-Tree with maximum degree 5
  - Interior nodes have 3 – 5 children
  - All nodes have 2-4 keys

11-70: B-Trees

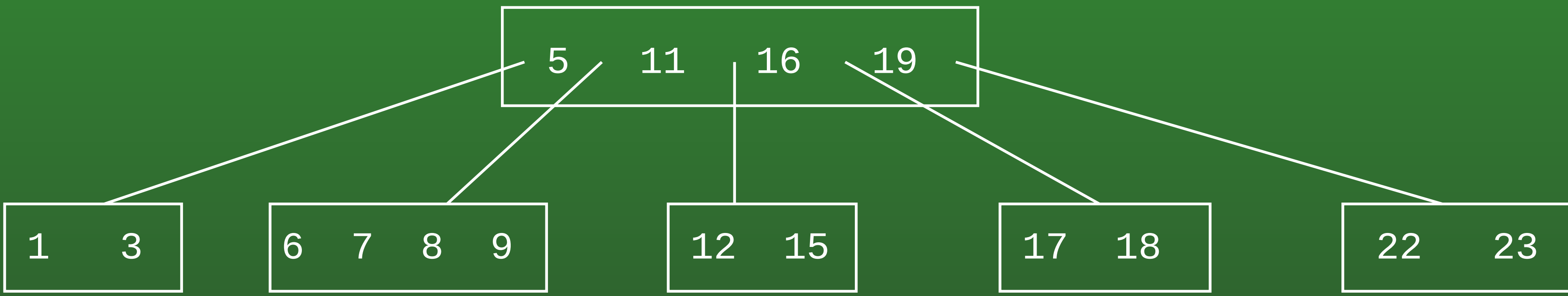
- Inserting into a B-Tree
  - Find the leaf where the element would go
  - If the leaf is not full, insert the element into the leaf
  - Otherwise, split the leaf (which may cause further splits up the tree), and insert the element

B-Trees

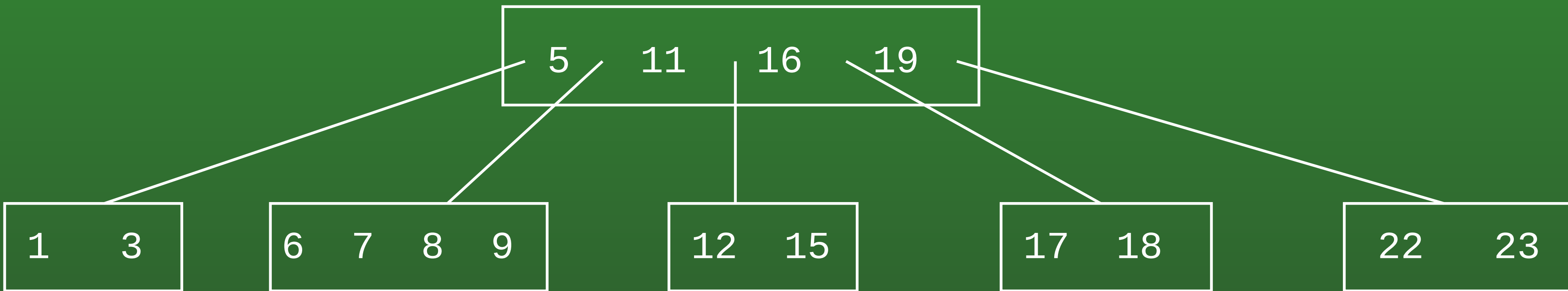


- Inserting a 6 ..

B-Trees

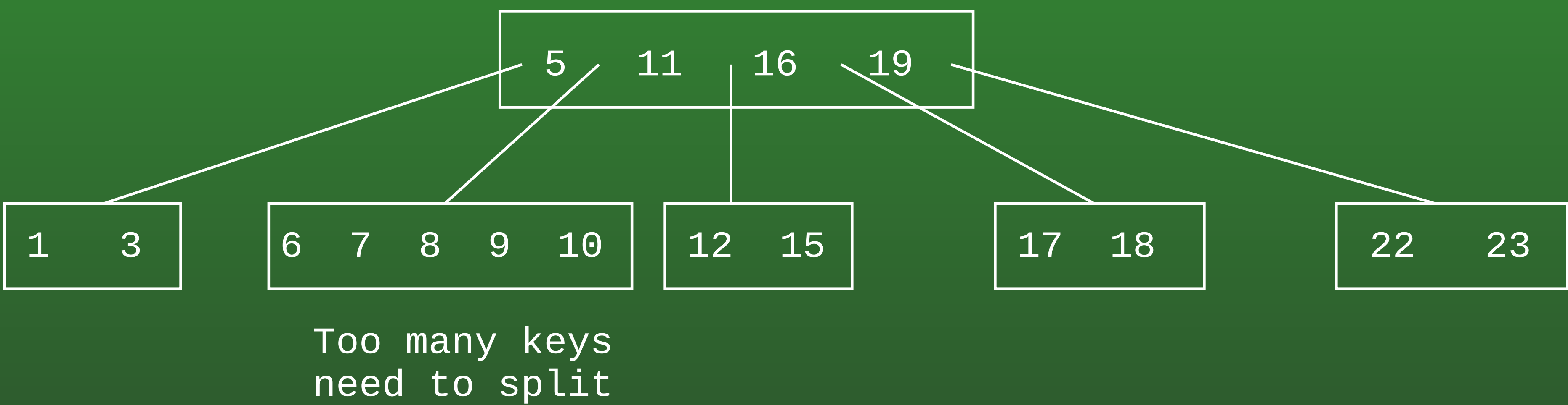


B-Trees



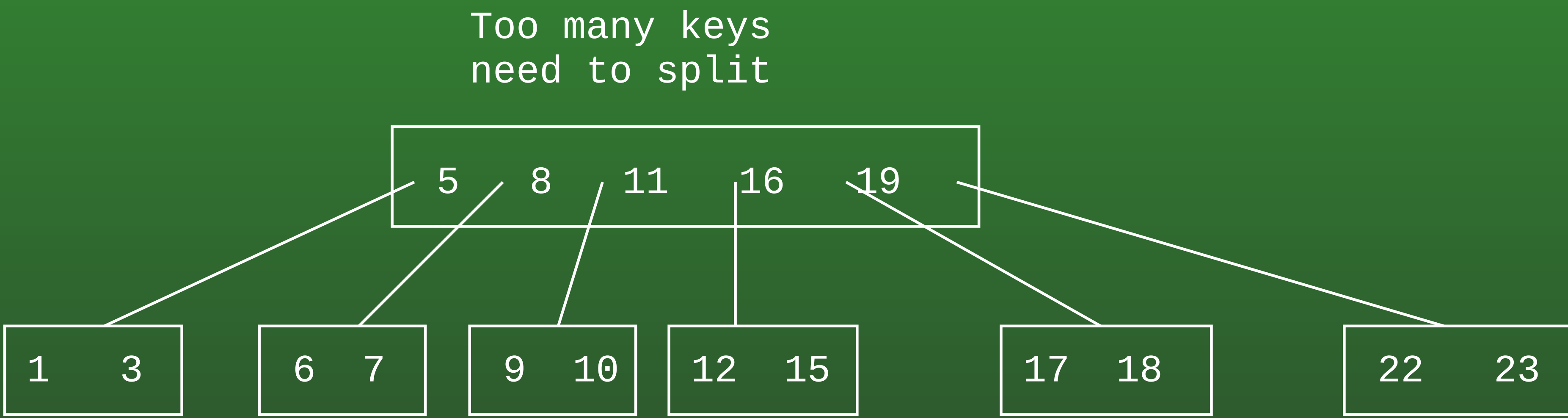
- Inserting a 10 ..

11-74: B-Trees



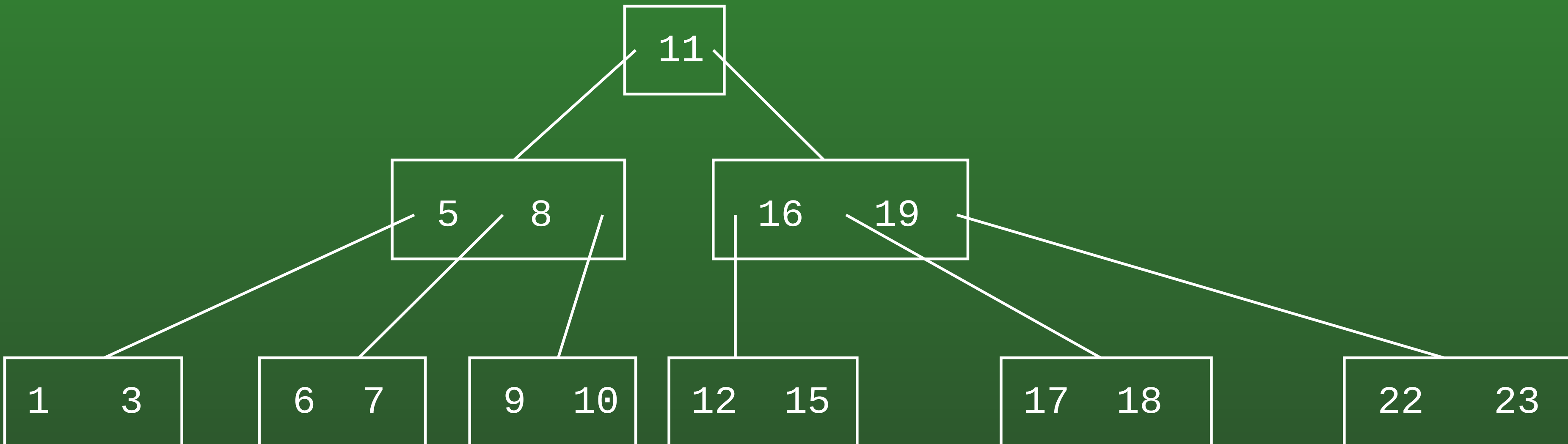
- Promote 8 to parent (between 5 and 11)
- Make nodes out of (6, 7) and (9, 10)

11-75: B-Trees



- Promote 11 to parent (new root)
- Make nodes out of (5, 8) and (6, 19)

B-Trees



- Note that the root only has 1 key, 2 children
- All nodes in B-Trees with maximum degree 5 should have at least 2 keys
- The root is an exception – it may have as few as one key and two children for any maximum degree

B-Trees

- B-Tree of maximum degree k
    - Generalized BST
    - All leaves are at the same depth
    - All nodes (other than the root) have $\lceil k/2 \rceil - 1 \ldots k - 1$ keys
    - All interior nodes (other than the root) have $\lceil k/2 \rceil \ldots k$ children

11-78: B-Trees

- B-Tree of maximum degree k
  - Generalized BST
  - All leaves are at the same depth
  - All nodes (other than the root) have $\lceil k/2 \rceil - 1 \ldots k-1$ keys
  - All interior nodes (other than the root) have $\lceil k/2 \rceil \ldots k$ children
- Why do we need to make exceptions for the root?

11-79: B-Trees

- Why do we need to make exceptions for the root?
  - Consider a B-Tree of maximum degree 5 with only one element

B-Trees

- Why do we need to make exceptions for the root?
  - Consider a B-Tree of maximum degree 5 with only one element
  - Consider a B-Tree of maximum degree 5 with 5 elements

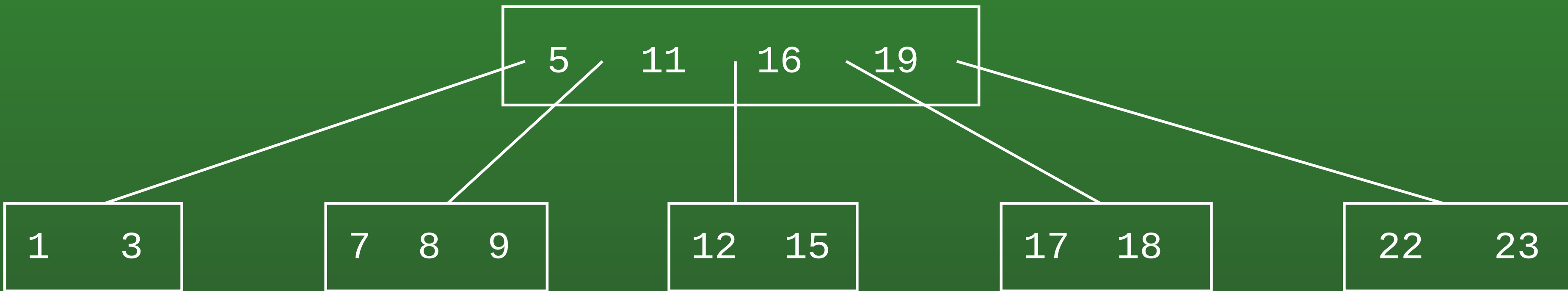
11-81: B-Trees

- Why do we need to make exceptions for the root?
  - Consider a B-Tree of maximum degree 5 with only one element
  - Consider a B-Tree of maximum degree 5 with 5 elements
  - Even when a B-Tree *could* be created for a specific number of elements, creating an exception for the root allows our split/merge algorithm to work correctly.

B-Trees

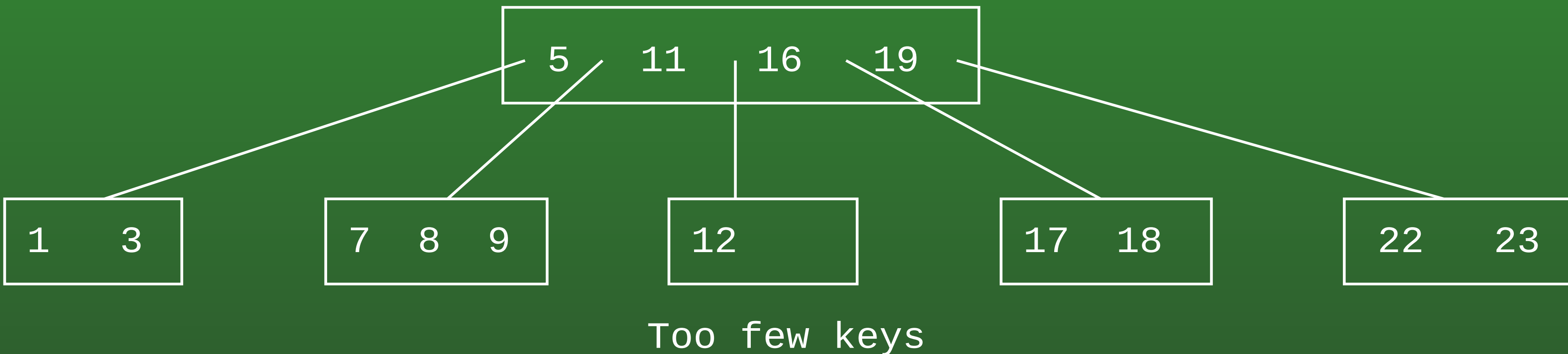
- Deleting from a B-Tree (Key is in a leaf)
  - Remove key from leaf
  - Steal / Split as necessary
  - May need to split up tree as far as root

11-83: B-Trees

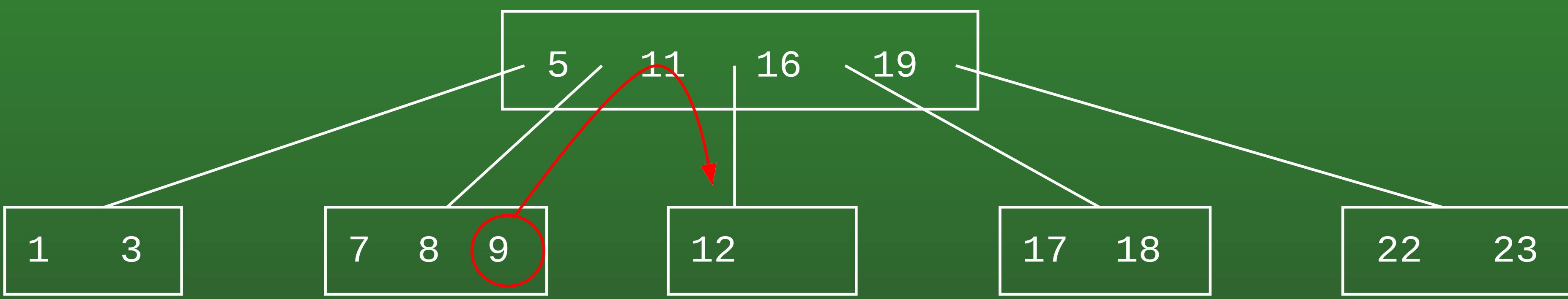


- Deleting the 15

11-84: B-Trees

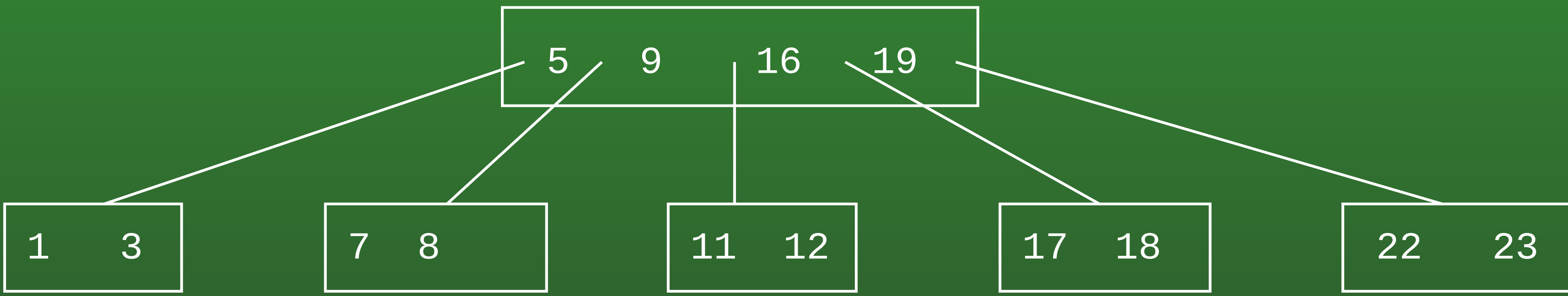


11-85: B-Trees

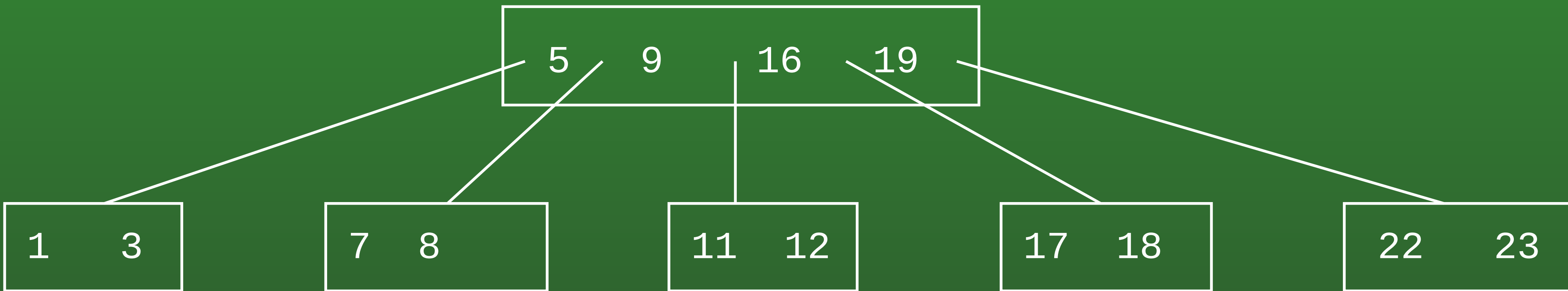


- Steal a key from sibling

B-Trees

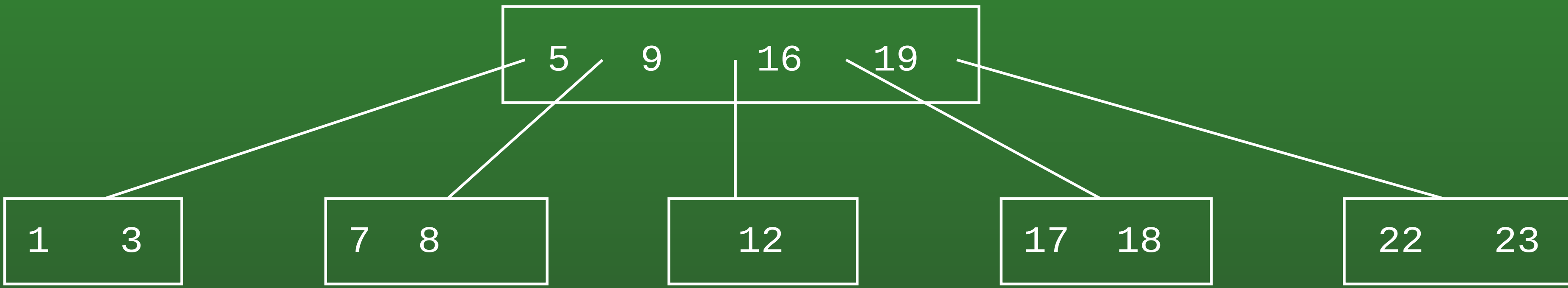


11-87: B-Trees



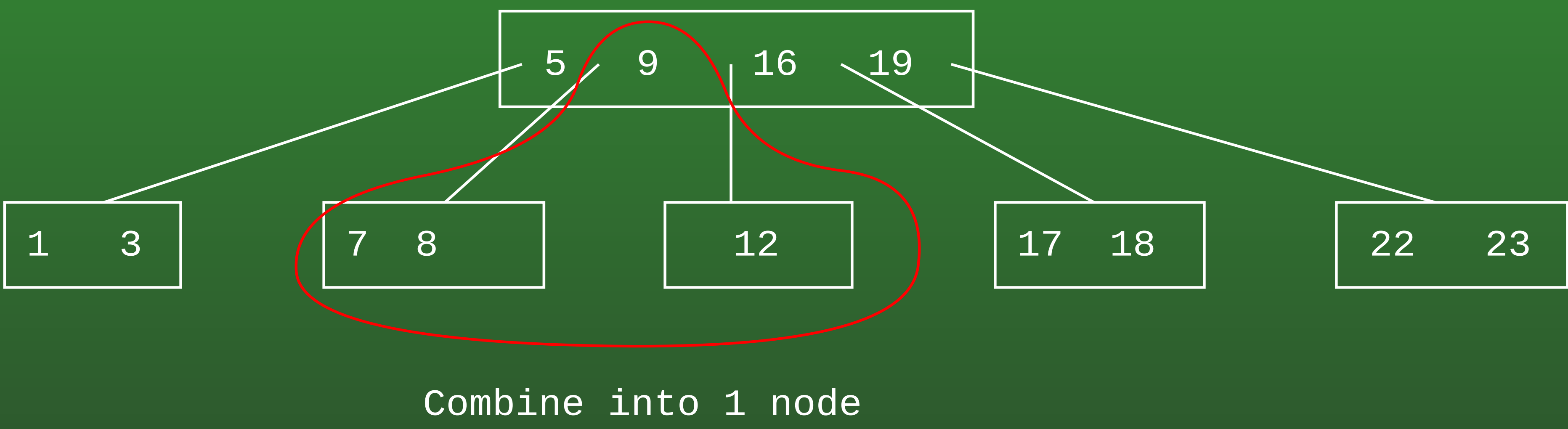
- Delete the 11

B-Trees



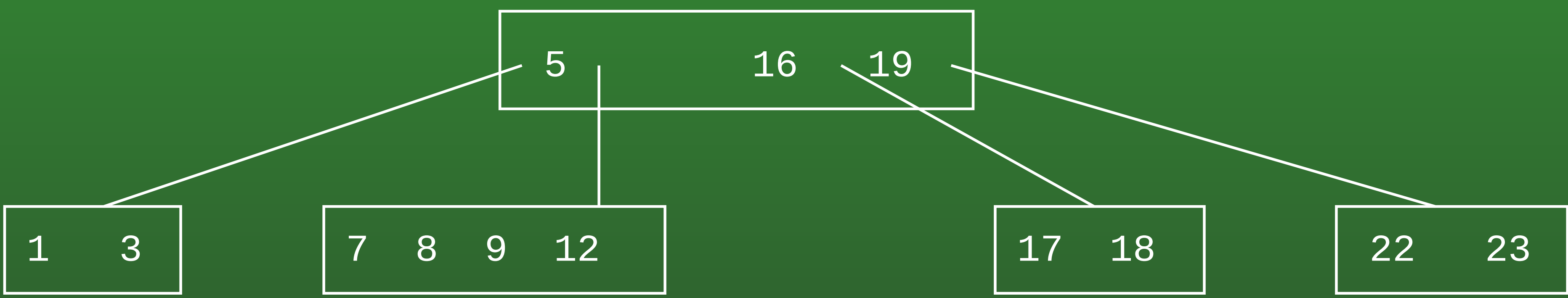
Too few keys

B-Trees



- Merge with a sibling (pick the left sibling arbitrarily)

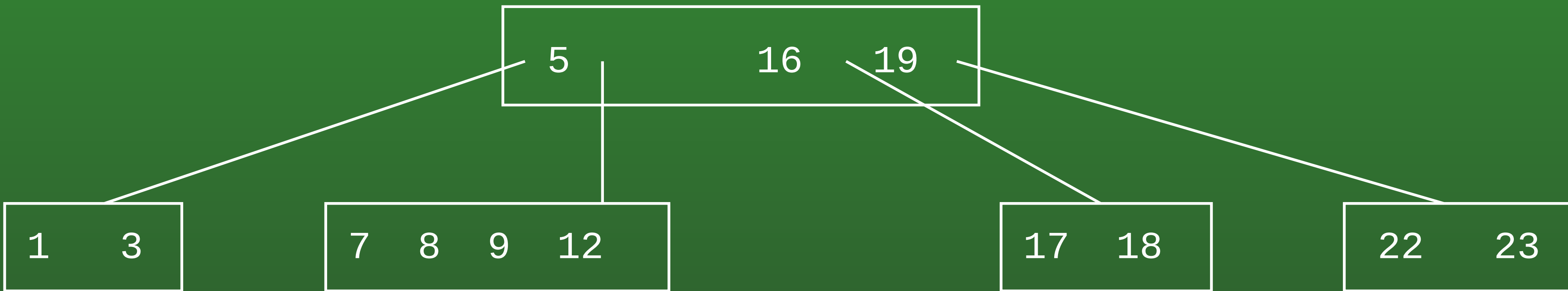
B-Trees



B-Trees

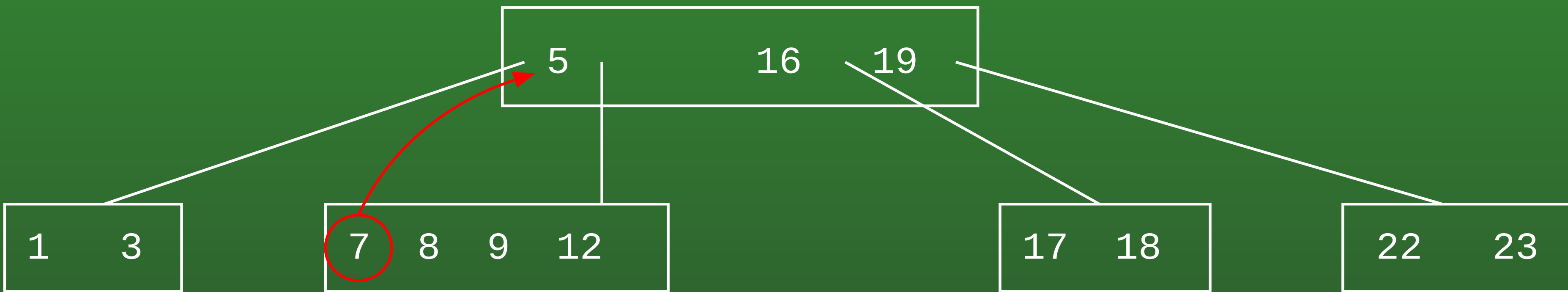
- Deleting from a B-Tree (Key in internal node)
  - Replace key with largest key in right subtree
  - Remove largest key from right subtree
  - (May force steal / merge)

B-Trees



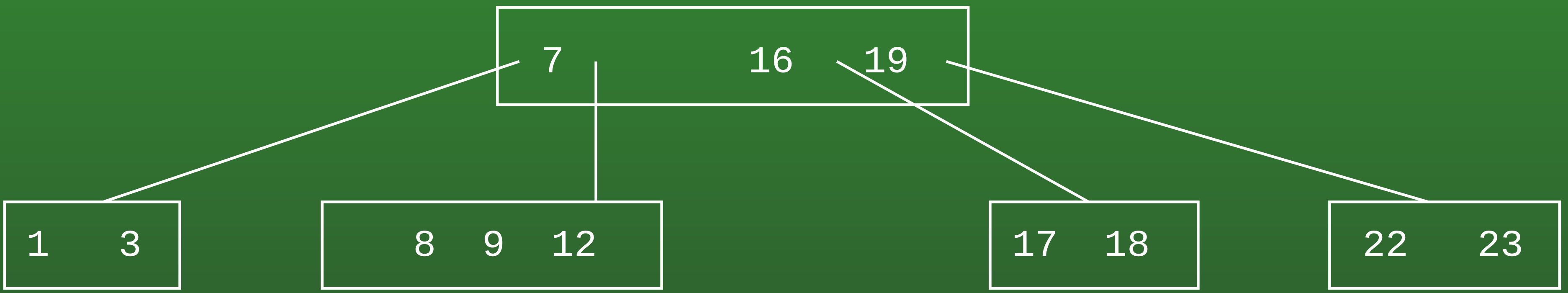
- Remove the 5

B-Trees

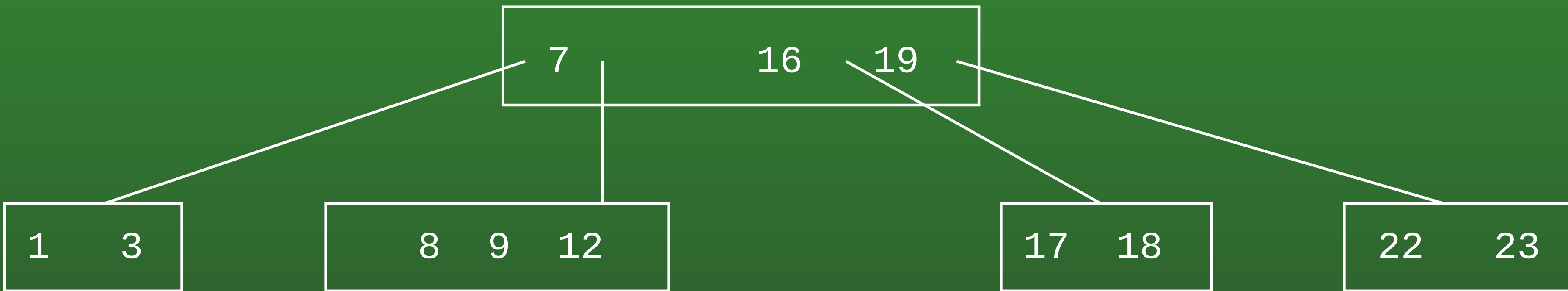


- Remove the 5

B-Trees

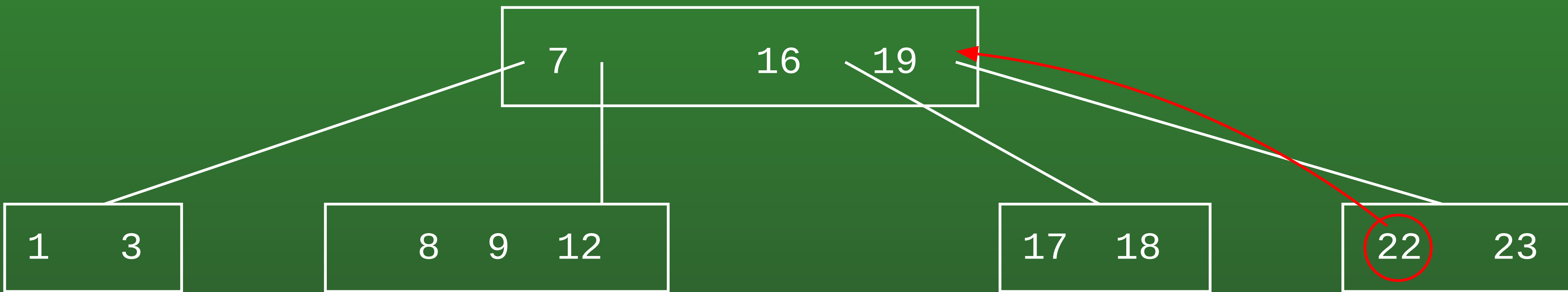


B-Trees



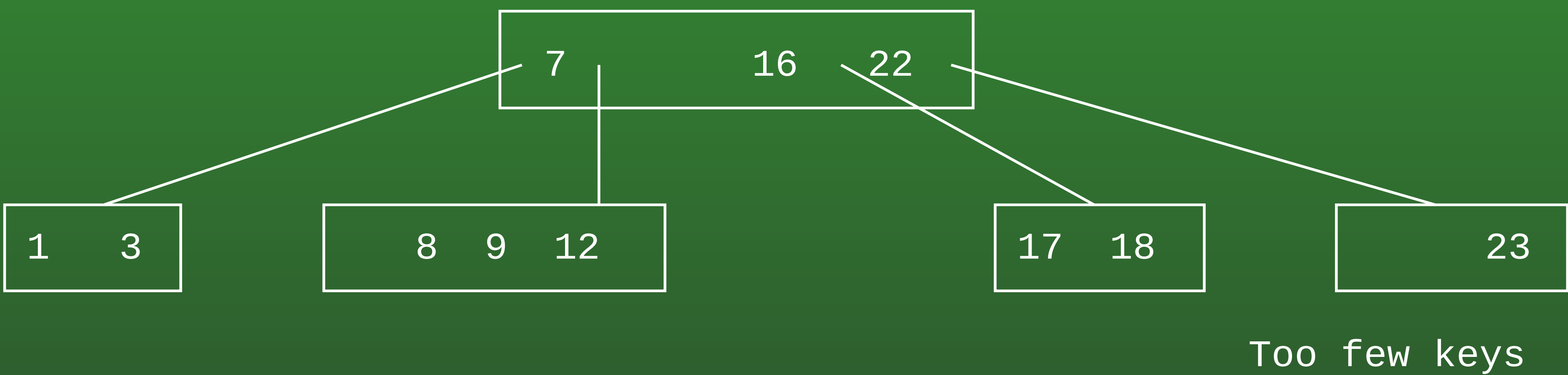
- Remove the 19

B-Trees

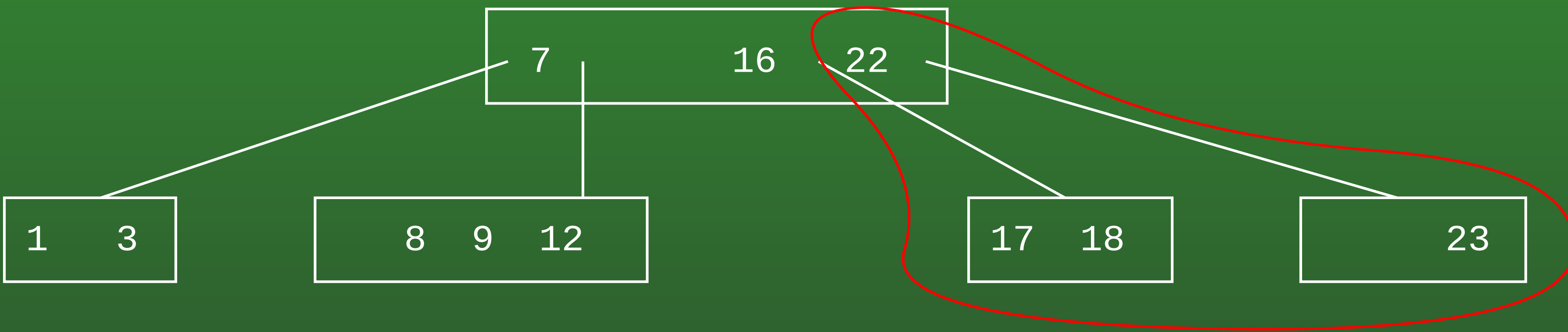


- Remove the 19

B-Trees

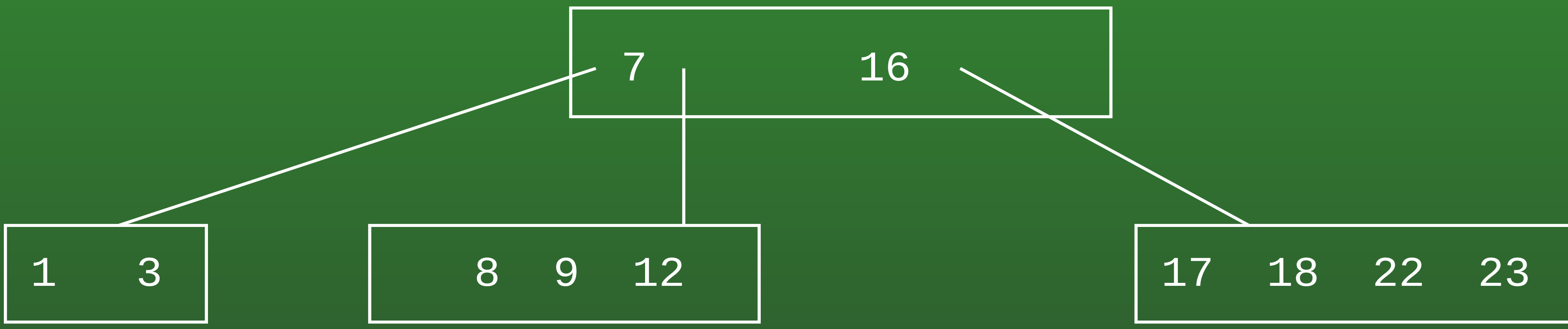


B-Trees



- Merge with left sibling

B-Trees



11-100: B-Trees

- Almost all databases that are large enough to require storage on disk use B-Trees
- Disk accesses are *very* slow
  - Accessing a byte from disk is 10,000 – 100,000 times as slow as accessing from main memory
  - Recently, this gap has been getting even bigger
- Compared to disk accesses, all other operations are essentially free
- Most efficient algorithm minimizes disk accesses as much as possible

11-101: B-Trees

- Disk accesses are slow – want to minimize them
- Single disk read will read an entire sector of the disk
- Pick a maximum degree k such that a node of the B-Tree takes up exactly one disk block
  - Typically on the order of 100 children / node

11-102: B-Trees

- With a maximum degree around 100, B-Trees are very shallow
- Very few disk reads are required to access any piece of data
- Can improve matters even more by keeping the first few levels of the tree in main memory
  - For large databases, we can’t store the entire tree in main memory – but we can limit the number of disk accesses for each operation to be very small

11-103: B-Trees

- If the maximum degree of a B-Tree is odd (2-3 tree, 3-4-5 tree), then we can only split a node when it gets “over-full”
  - Examples for 2-3 trees on board
- If the maximum degree of a B-Tree is even (2-3-4 tree, 3-4-5-6, etc.):
  - We can split a node before it is “over-full”
  - We can merge nodes before they are “under-full”

11-104: B-Trees

- Preemptive Splitting
  - If the maximum degree is even, we can implement an insert with a single pass down the tree (instead of a pass down, and then a pass up to clean up)
  - When inserting into any subtree tree, if the root of that tree is full, split the root before inserting
    - Every time we want to do a split, we know our parent is not full.

(examples, use visualization)

B-Trees

- Preemptive Combining – Deleting from Leaves
  - If the maximum degree is even, we can implement a delete with a single pass down the tree (instead of a pass down, and then a pass up to clean up)
  - When deleting from any node (other than the root), combine / steal as necessary so that the node has more then the minimum # of keys
  - When you get to a leaf, you are guaranteed that there will be an extra key in the leaf

(examples, deleting from leaves)

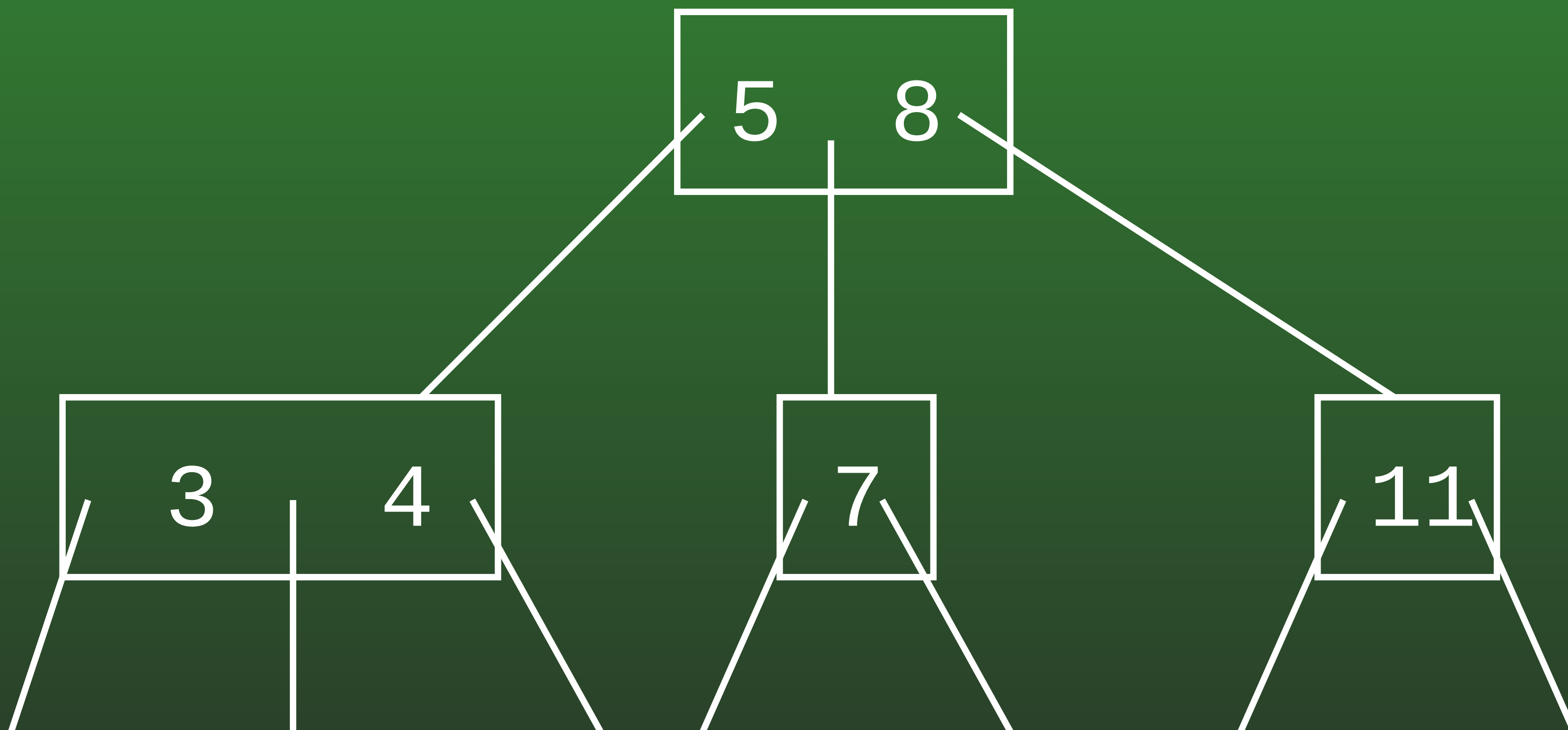
B-Trees

- Preemptive Combining
  - Deleting k from a non-leaf:
    - If the subtree left of k has $>$ minimum number of elements, replace k with largest element in the left subtree, splitting as you go down
    - If the subtree right of k has $>$ minimum number of elements, replace k with smallest element in the right subtree, splitting as you go down

(examples)

B-Trees

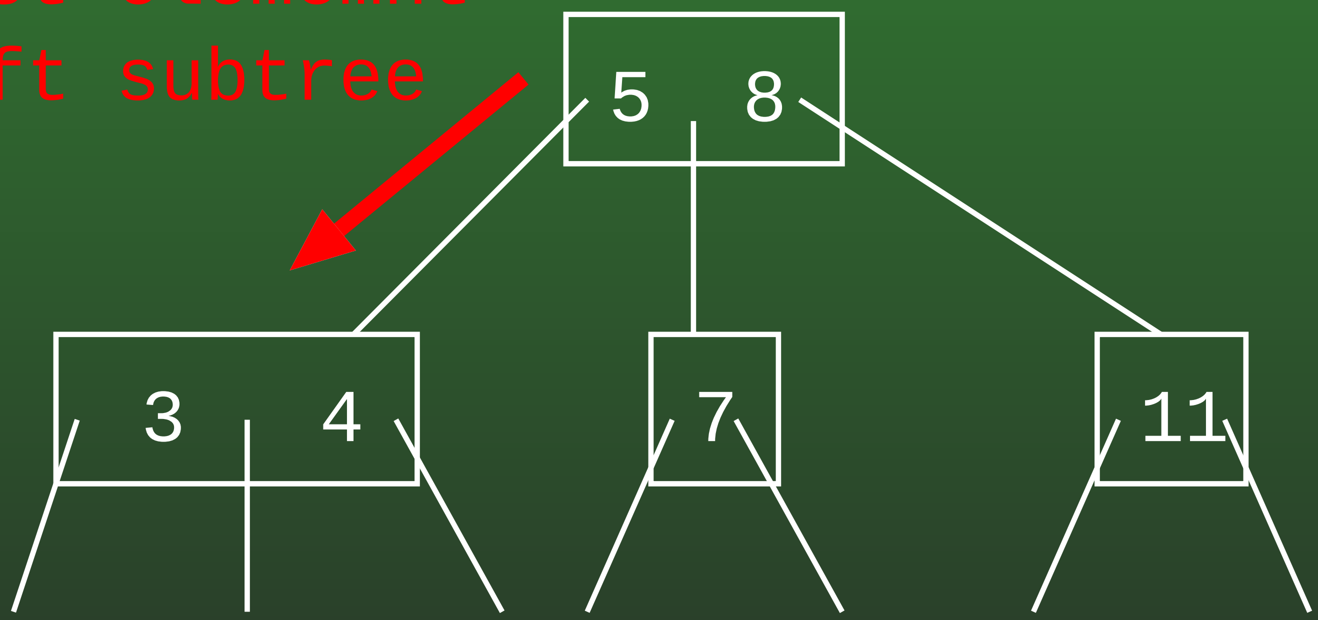
Deleting 5:



B-Trees

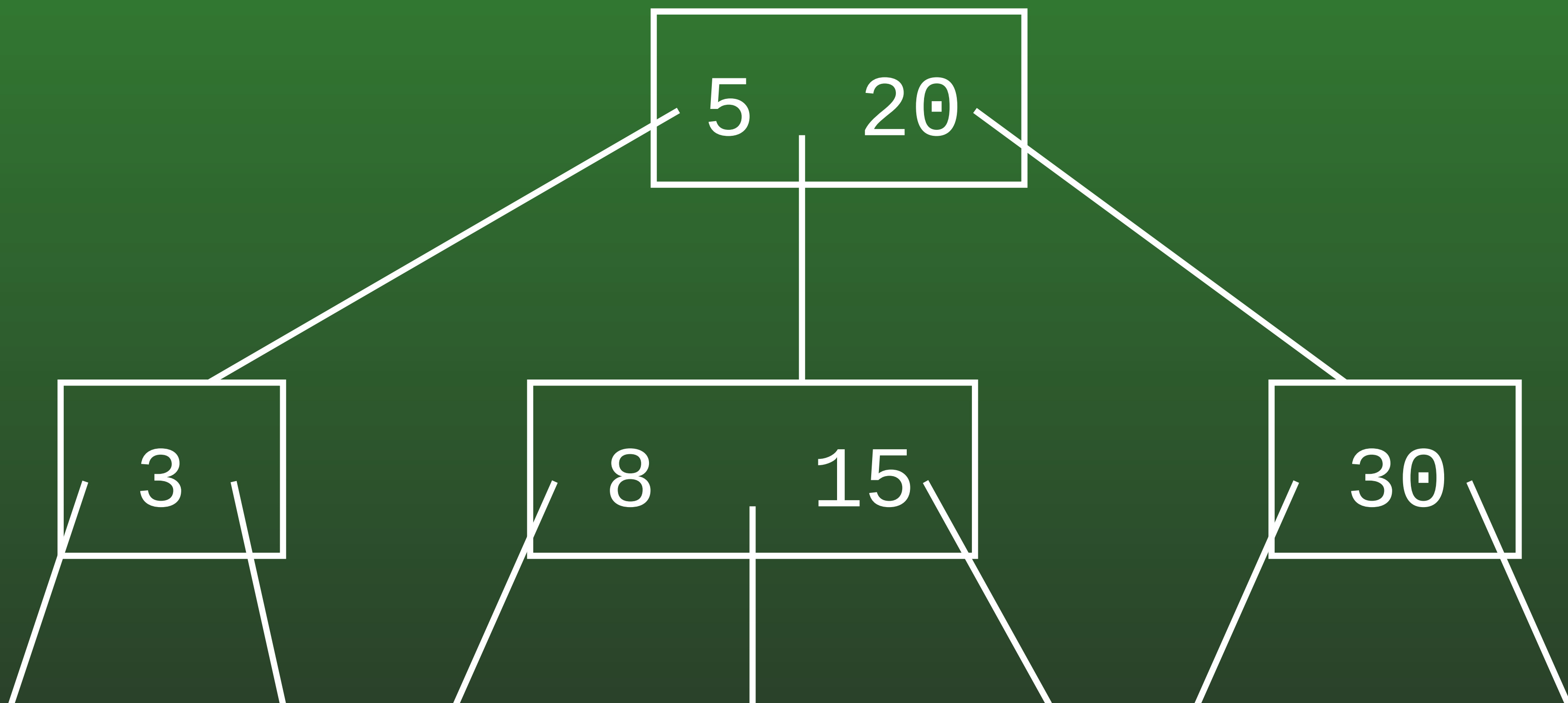
Deleting 5:

Replace 5 with
largest elememnt
in left subtree



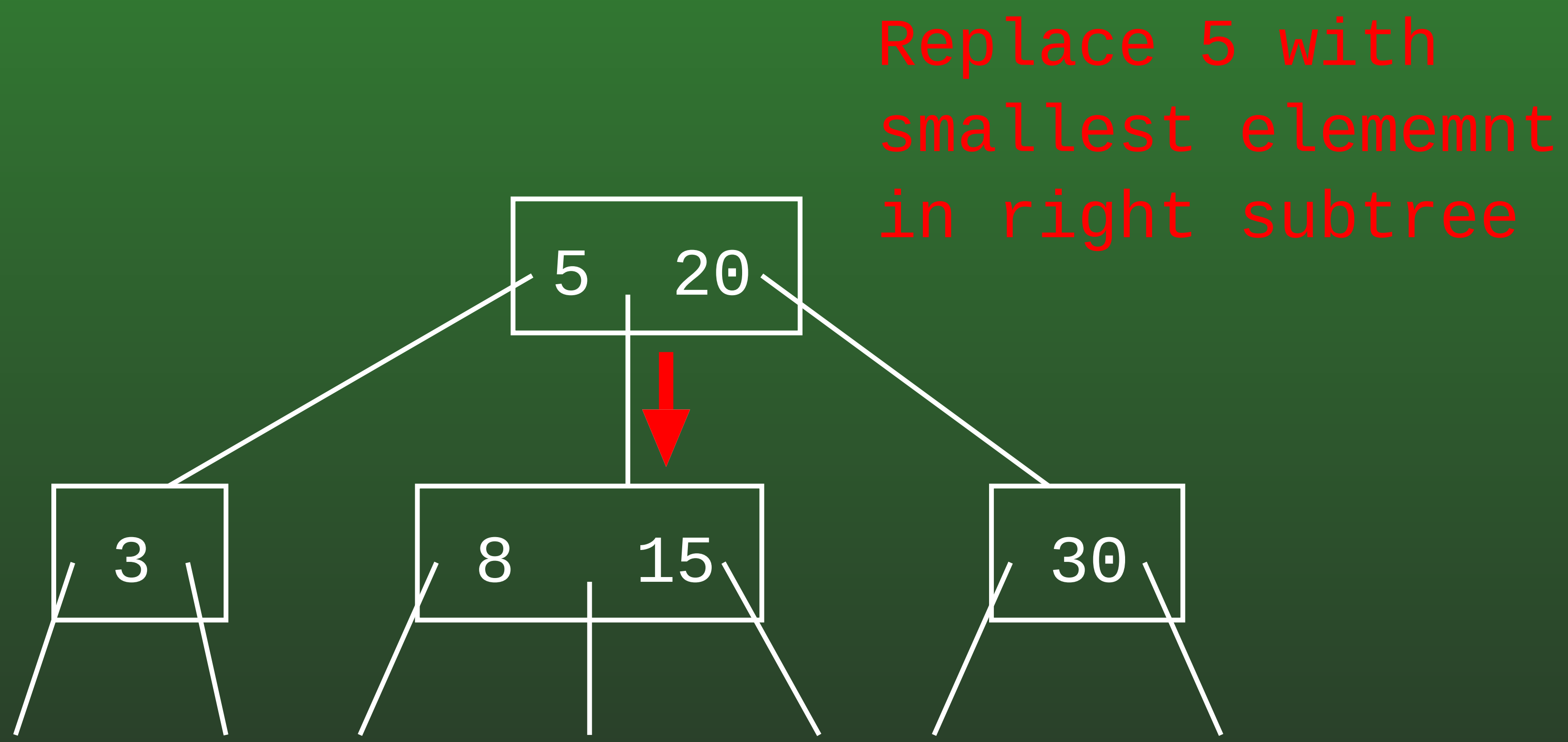
B-Trees

Deleting 5:



11-110: B-Trees

Deleting 5:

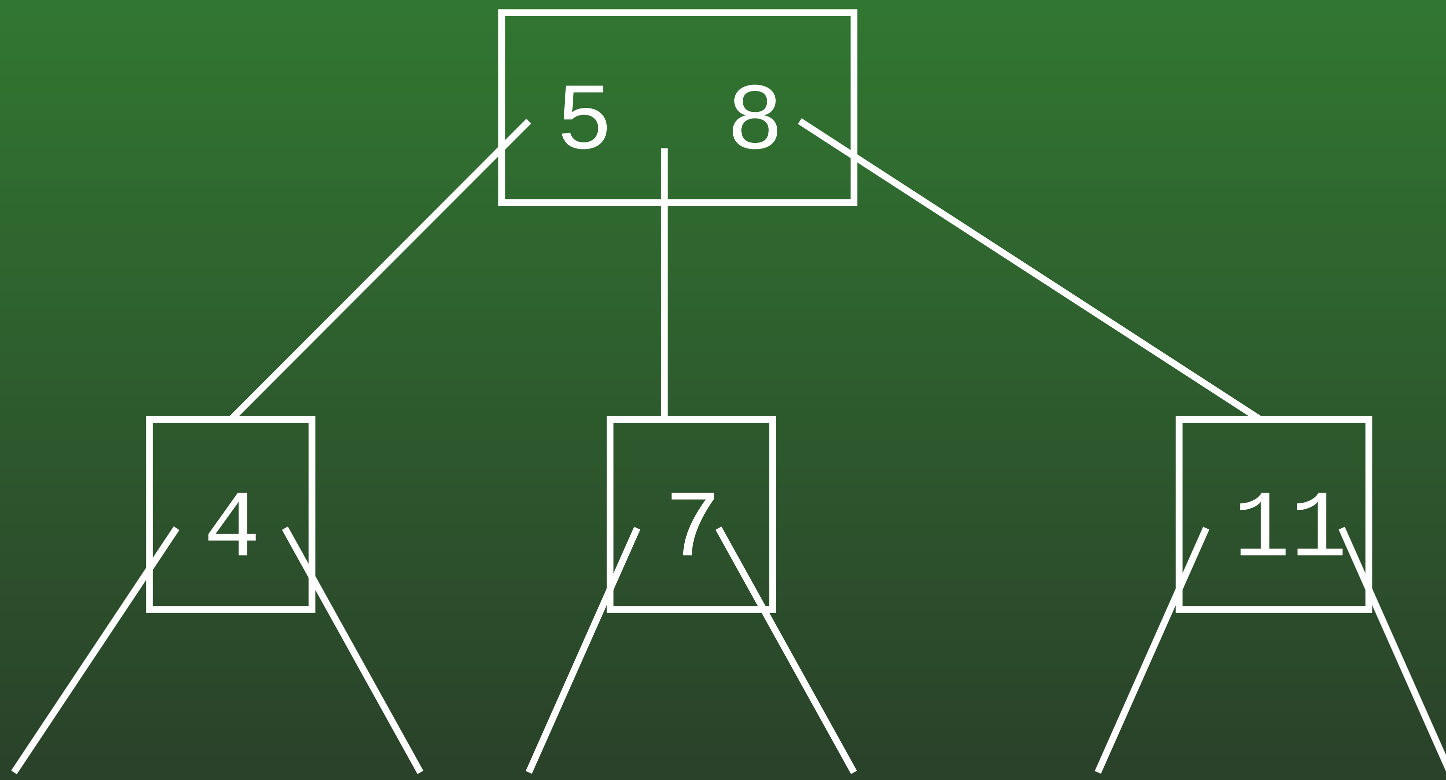


B-Trees

- Preemptive Combining
  - Deleting k from a non-leaf:
    - If the subtrees to the left & right of k subtrees both have the minimum # of elements, combine around k
    - Recursively remove k from this new node

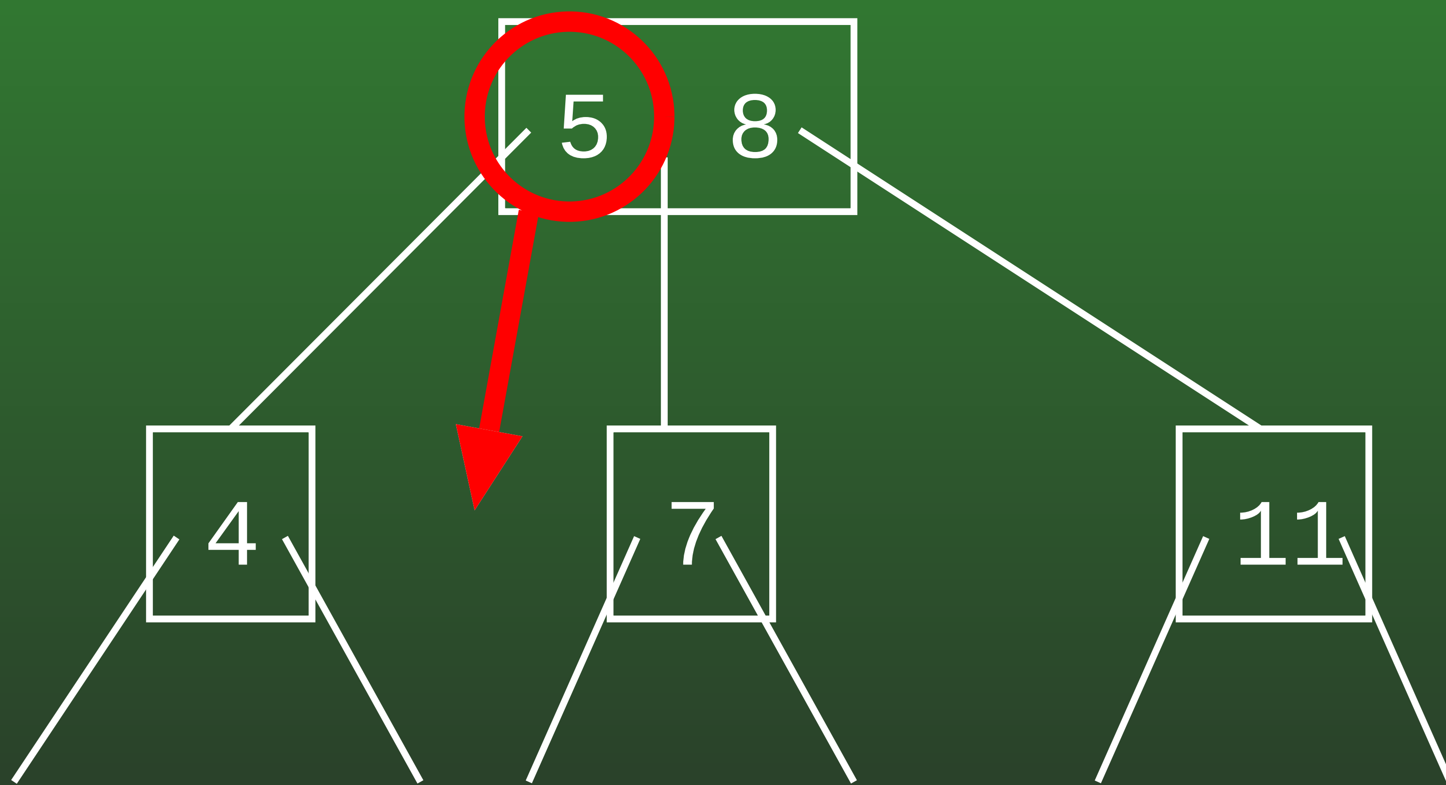
B-Trees

Deleting 5:



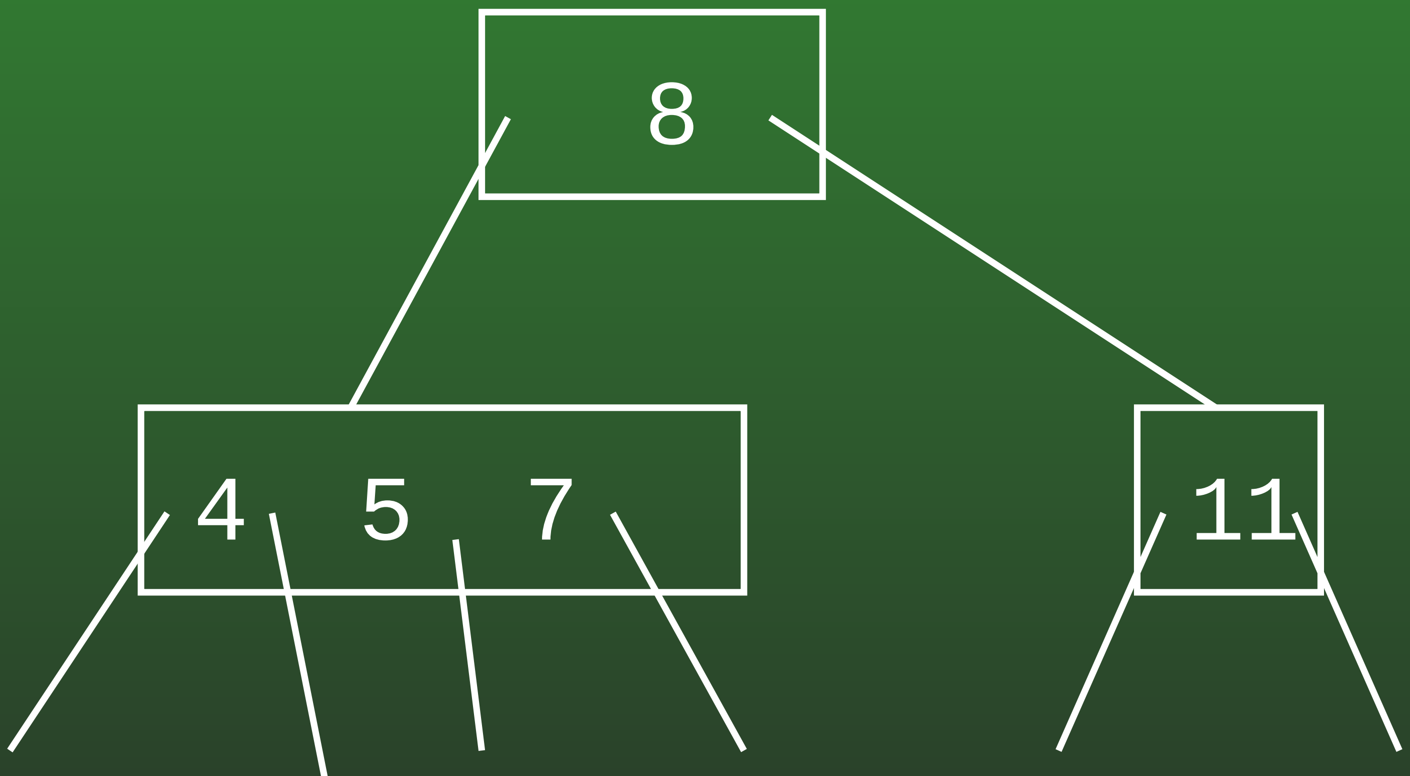
B-Trees

Merge around 5:



B-Trees

Delete 5 from new node:



11-115: B-Trees

- Preemptive Combining
  - Deleting k from a non-leaf:
    - If the subtrees to the left & right of k subtrees both have the minimum # of elements, combine around k
    - Recursively remove k from this new node
- Why do we need this case? Why can't we just replace key with largest value in left subtree, or smallest value in right subtree?

B-Trees

- Preemptive Combining
  - Deleting k from a non-leaf:
    - If the subtrees to the left & right of k subtrees both have the minimum # of elements, combine around k
    - Recursively remove k from this new node
- Why do we need this case? Why can't we just replace key with largest value in left subtree?
  - Immediately cause a merge, anyway
  - Harder to determine which location to copy largest element into

B-Trees

- Preemptive split/merge vs. "standard" split/merge
  - Advantages of the "standard" method?
  - Advantages of the "preemptive" method?
- Textbook uses "preemptive" method
  - Defines "minimum degree k" (with maximum degree = $2k$) instead of "maximum degree k" (with minimum degree = $\lceil \frac{k}{2} \rceil$)