18-0: **Flow Networks**

- Directed Graph G
- Each edge weigh is a "capacity"
  - Amount of water/second that can flow through a pipe, for instance
- Single source S, single sink t
- Calculate maximum flow through graph

18-1: **Flow Networks**

- Flow: Function: $V \times V \rightarrow R$
  - Flow from each vertex to every other vertex
  - $f(u, v)$ is the direct flow from u to v
- Properties:
  - $\forall u, v \in V, f(u, v) \leq c(u, v)$
  - $\forall u, v \in V, f(u, v) = -f(v, u)$
  - $\forall u \in V - \{s, t\}, \sum_{v \in V} f(u, v) = 0$
- Total flow, $|f| = \sum_{v \in V} f(s, v) = \sum_{v \in V} f(v, t)$

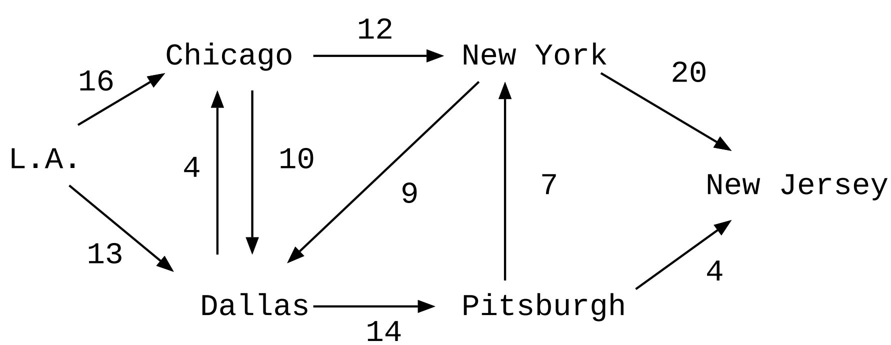
18-2: **Flow Networks**

- Single Source / Single Sink
  - Assume that there is always a single source and a single sink
  - Don't lost any expressive power – always transform a problem with multiple sources and multiple sinks to an equivalent problem with a single source and a single sink
  - How?

18-3: **Flow Networks**

- Example: Shipping product to a warehouse
  - Product produced at a factory, put in crates
  - Crates are shipped to warehouse
  - To cut down costs, use "extra space" in other people's trucks
  - How much product can be produced per day?

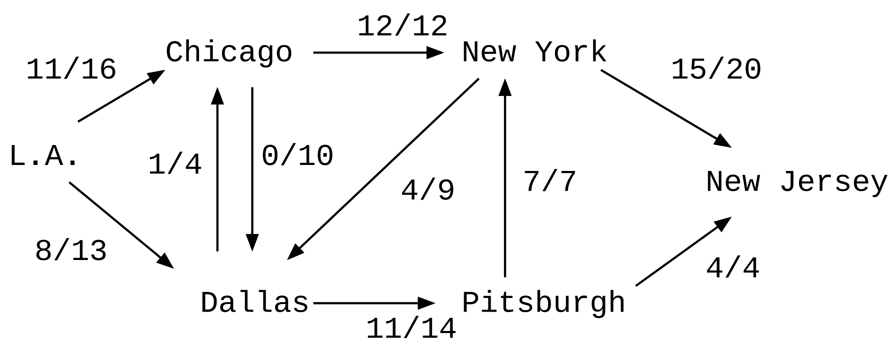
18-4: **Flow Networks**



18-5: **Flow Networks**

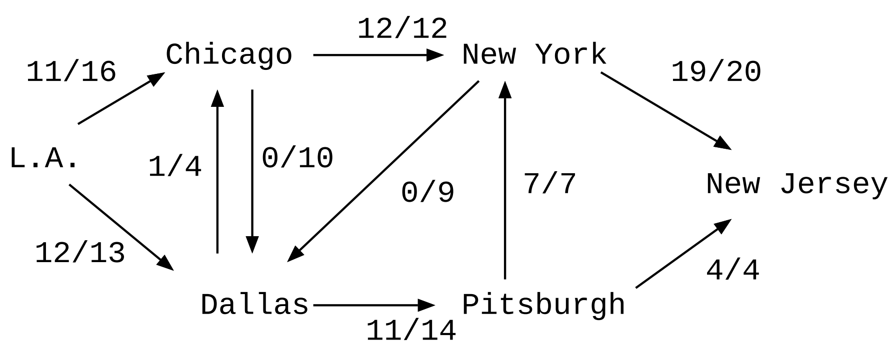
- It would be a little silly to ship 4 crates from Dallas to Chicago, and 7 crates from Chicago to Dallas
  - Could just ship 3 crates from Chicago to Dallas instead
- We will assume that there is only every flow in one direction
  - Flow in the opposite direction "cancels" out

18-6: **Flow Networks**



Is this flow optimal?

18-7: **Flow Networks**



18-8: **Flow Networks**

- Negative flow
  - It is perfectly legal for there to be a negative flow from v to u
  - Negative flow from v to u just means that there is a positive flow from u to v
  - Recall that the total flow over all edge incident to a vertex must be zero, except for source & sink

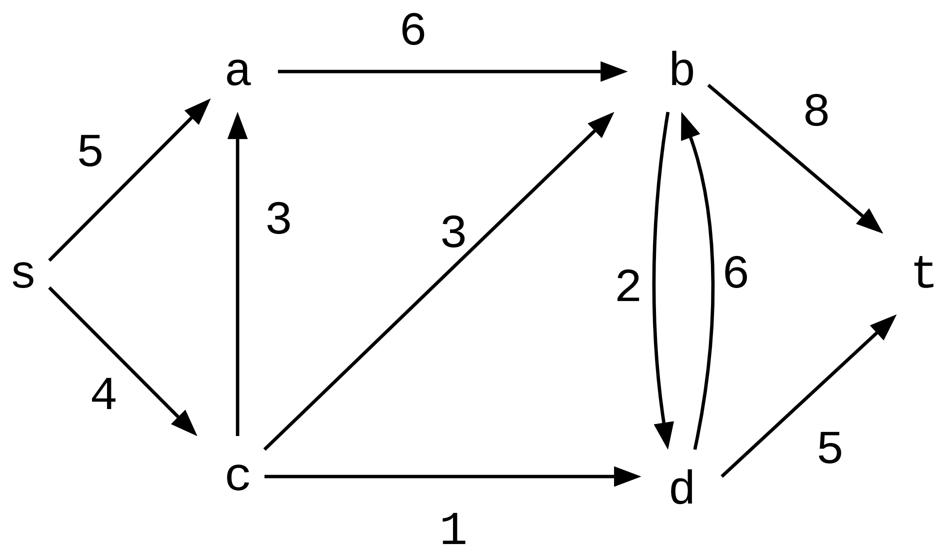
18-9: **Flow Networks**

- Residual capacity
  - $c_f(u, v)$ is the residual capacity of edge (u, v)
  - $c_f(u, v) = c(u, v) - f(u, v)$
  - Note that it is possible for the residual capacity of an edge to be greater than the total capacity
    - Cancelling flow in the opposite direction

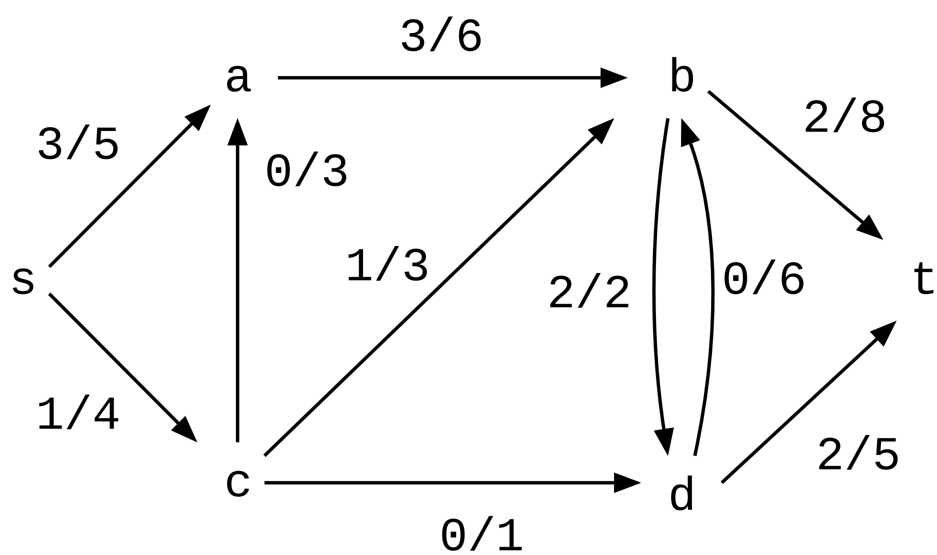
18-10: **Flow Networks**

- Residual Network
  - Given a set of capacities, and a set of current flows, we can create a residual network
  - Residual network can have different edges than the capacity network

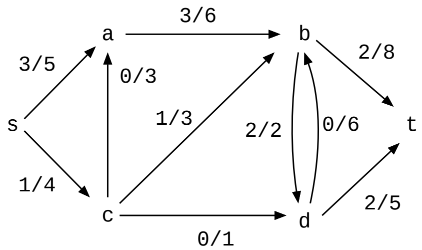
18-11: **Flow Networks**

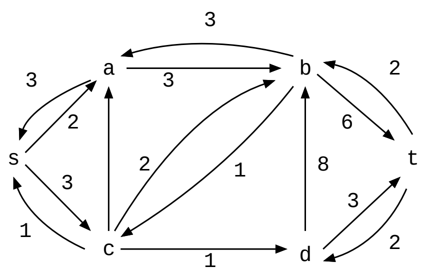


18-12: **Flow Networks**



18-13: **Flow Networks**





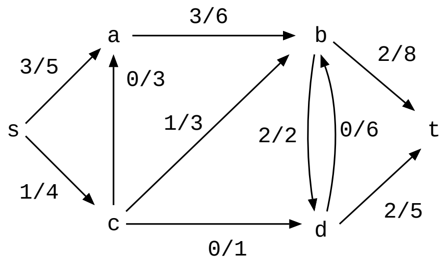
18-14: **Flow Networks**

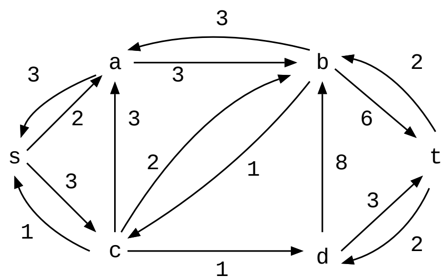
- Given a flow network, with some flows calculated
- Induced residual network
- There is a path from source to sink in the residual network such that:
  - All residual capacities along the path are > 0
- How can we increase the total flow?

18-15: **Augmenting Path**

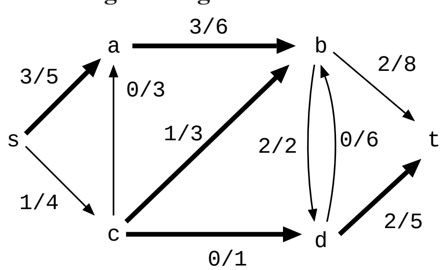
- An *Augmenting path* in a flow network is a path through the network such that all residual capacities along the path > 0
- Given a flow network and an augmenting path, we can increase the total flow by the smallest residual capacity along the path
  - Increase flow along path by smallest residual capacity along the path
  - May involve some flow cancelling

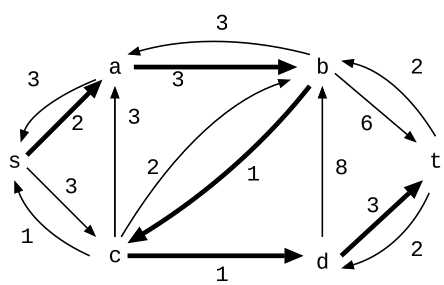
18-16: **Augmenting Path**



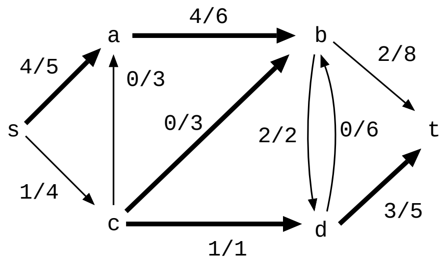


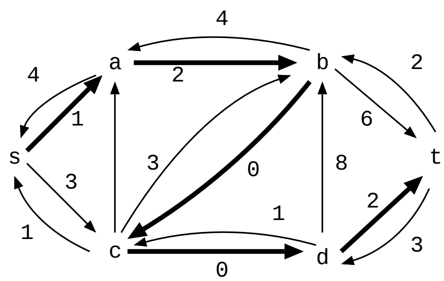
18-17: **Augmenting Path**



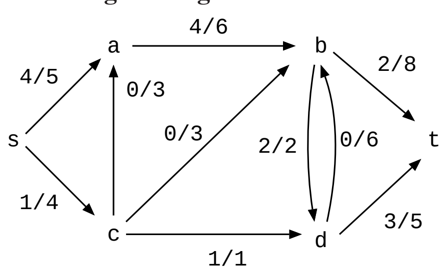


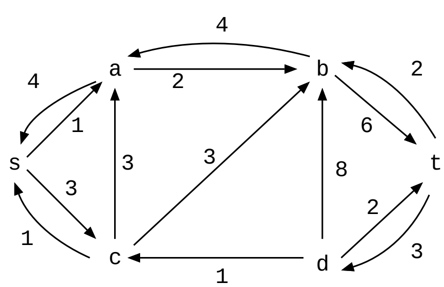
18-18: **Augmenting Path**





18-19: **Augmenting Path**





18-20: **Ford-Fulkerson Method**

Ford-Fulkerson(G, s, t)
    initialize flow f to 0
    while there is an augmenting path p
        augment flow f along p
    return f

18-21: **Ford-Fulkerson Method**

- What is the running time of Ford-Fulkerson Method?
    - Find an augmenting path

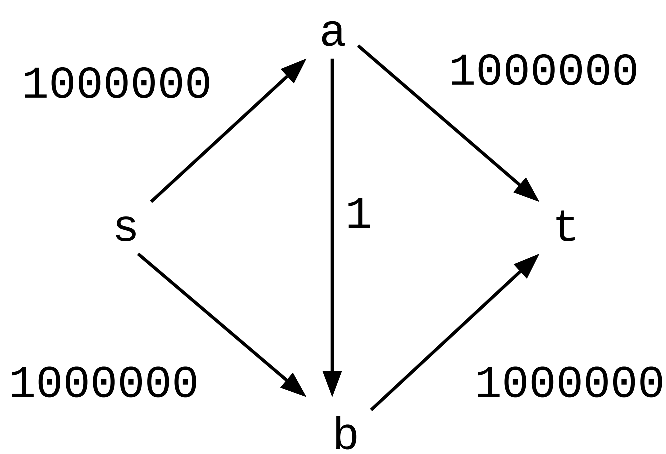
- Update flows / residuals
- Repeat until there are no more augmenting paths

18-22: **Ford-Fulkerson Method**

- What is the running time of Ford-Fulkerson Method?
  - Find an augmenting path
    - Using DFS, $O(|E|)$
  - Update flows / residuals
    - $O(|E|)$
  - Repeat until there are no more augmenting paths
    - Each iteration could increase the flow by 1, could have $|f|$ iterations!
- Total: $O(|f| * |E|)$

18-23: **Ford-Fulkerson Method**

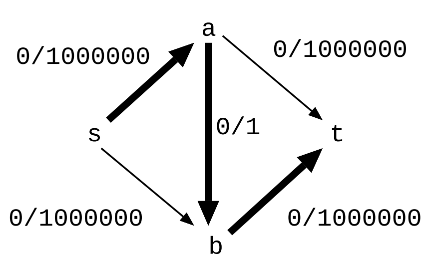
- Could take as many as $|f|$ iterations:



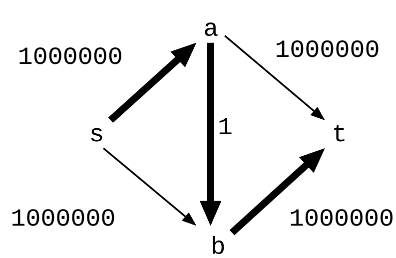
18-24: **Ford-Fulkerson Method**

- Could take as many as $|f|$ iterations:

Flow Network

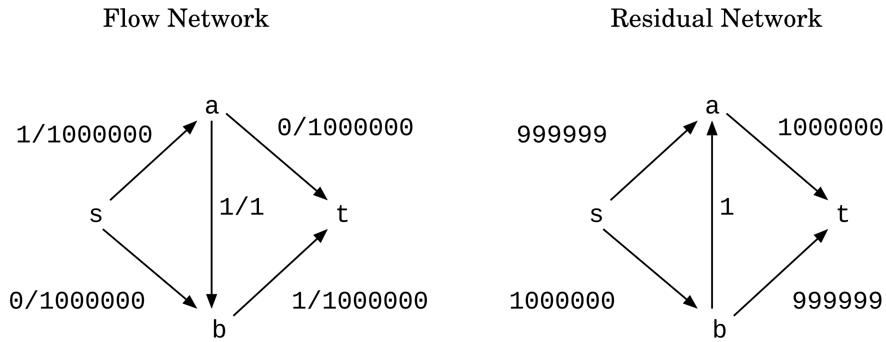


Residual Network



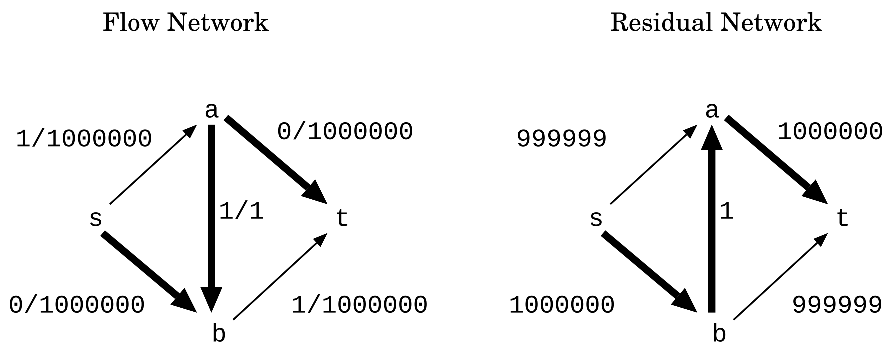
18-25: **Ford-Fulkerson Method**

- Could take as many as $|f|$ iterations:



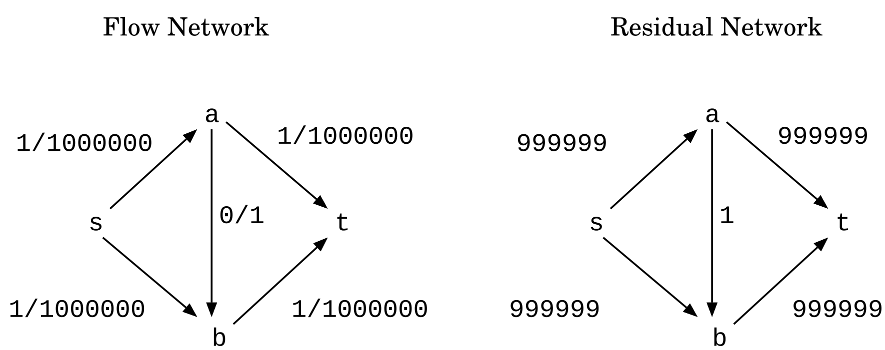
18-26: **Ford-Fulkerson Method**

- Could take as many as $|f|$ iterations:



18-27: **Ford-Fulkerson Method**

- Could take as many as $|f|$ iterations:



18-28: **Ford-Fulkerson Method**

- How can we be smart about choosing the augmenting path, to avoid the previous case?

18-29: **Edmonds-Karp Algorithm**

- How can we be smart about choosing the augmenting path, to avoid the previous case?
  - We can get better performance by always picking the shortest path (path with the fewest edges)

- We can quickly find the shortest path by doing a BFS from the source in the residual network, to find the shortest augmenting path
- If we always choose the shortest augmenting path (i.e., smallest number of edges), total number of iterations is $O(|V| * |E|)$, for a total running time of $O(|V| * |E|^2)$

18-30: **Edmonds-Karp Algorithm**

- If we always pick the shortest augmenting path, no more than $|V| * |E|$ iterations:
  - Lemma #1: Shortest path from source s to any other vertex in residual graph can only increase, not decrease.
    - Residual graph changes over time – edges are added and removed
    - However, shortest path from source to any vertex in the residual graph will only increase over time, never decrease

18-31: **Edmonds-Karp Algorithm**

- Lemma #1: Shortest path from source s to any other vertex in residual graph can only increase, not decrease. Proof by contradiction
  - Assume shortest path from source to some other vertex changes after an augmentation
  - Let f be the flow right before the shortest path decrease, and f' be the flow right after
  - Let v be a vertex such that $\delta_{f'}(s, v) < \delta_f(s, v)$. If there is more than once such v, pick the one with the smallest $\delta_{f'}(s, v)$ value
  - Let $p = s \rightarrow \ldots \rightarrow u \rightarrow v$ be the shortest path from s to v in f'

18-32: **Edmonds-Karp Algorithm**

- Lemma #1: Shortest path from source s to any other vertex in residual graph can only increase, not decrease. Proof by contradiction
  - Edge (u, v) (last edge on path from s to v in $G_{f'}$) must not be in G_f
    - $\delta_{f'}(s, u) \geq \delta_f(s, u)$
      - Because $\delta_{f'}(s, u) < \delta_{f'}(s, v)$, and we picked v to be the vertex with the smallest $\delta_{f'}(s, v)$ value that changed
  - If $(u, v) \in G_f$

$$\begin{aligned} \delta_f(s, v) &\leq \delta_f(s, u) + 1 \\ &\leq \delta_{f'}(s, u) + 1 \\ &\leq \delta_{f'}(s, v) \end{aligned}$$

18-33: **Edmonds-Karp Algorithm**

- Lemma #1: Shortest path from source s to any other vertex in residual graph can only increase, not decrease. Proof by contradiction
  - Edge (u, v) must be in $G_{f'}$ but not in G_f – so the augmenting path must include (v, u)
  - We always choose shortest paths as our augmenting path
  - Shortest path from s to u must include (v, u)

$$\begin{aligned} \delta_f(s, v) &= \delta_f(s, u) - 1 \\ &\leq \delta_{f'}(s, u) - 1 \\ &\leq \delta_{f'}(s, v) - 2 \end{aligned}$$

- Contradiction!

18-34: **Edmonds-Karp Algorithm**

- If we always pick the shortest augmenting path, no more than $|V| * |E|$ iterations:
  - An edge on an augmenting path is critical if it is removed when the flow is augmented (why must there always be at least one critical edge)?
  - Each edge can only be critical at most $|V|/2$ times

18-35: **Edmonds-Karp Algorithm**

- Each edge can only be critical at most $|V|/2$ times
  - When edge (u, v) is critical:
    - $\delta_f(s, v) = \delta_f(s, u) + 1$
  - Critical edge is removed – before it can become critical again, it must be added back by some augmenting path – that path must contain edge (u, v)
  - Let f' be the flow when the edge is added back.

$$\begin{aligned}\delta_{f'}(s,u) &= \delta_{f'}(s,v) + 1 \\ &\geq \delta_f(s,v) + 1 \\ &= \delta_f(s,u) + 1 + 1\end{aligned}$$

  - If an edge (u, v) becomes critical twice, the shortest path from s to u must increase by 2
  - Each edge can only be critical $|V|/2$ times

18-36: **Edmonds-Karp Algorithm**

- Each edge can only be critical at most $|V|/2$ times
  - Let f' be the flow when the edge is added back.

$$\begin{aligned}\delta_{f'}(s,u) &= \delta_{f'}(s,v) + 1 \\ &\geq \delta_f(s,v) + 1 \\ &= \delta_f(s,u) + 1 + 1\end{aligned}$$

  - If an edge (u, v) becomes critical twice, the shortest path from s to u must increase by 2
  - Each edge can only be critical $|V|/2$ times

18-37: **Edmonds-Karp Algorithm**

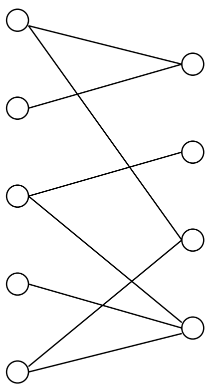
- If we always pick the shortest augmenting path, no more than $|V| * |E|$ iterations:
  - An edge on an augmenting path is critical if it is removed when the flow is augmented (why must there always be at least one critical edge)?
  - Each edge can only be critical at most $|V|/2$ times
  - $|E|$ total edges – no more than $|E| * |V|/2$ iterations

18-38: **Matching Problem**

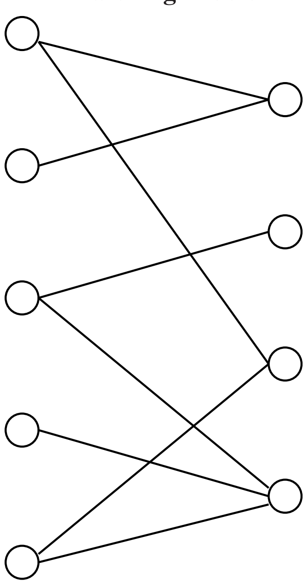
- Given an undirected graph $G = (V, E)$ a matching M is
  - Subset of edges E
  - For any vertex $v \in V$, at most one edge in M is incident to v
- Maximum matching is a matching with largest possible number of edges

18-39: **Matching Problem**

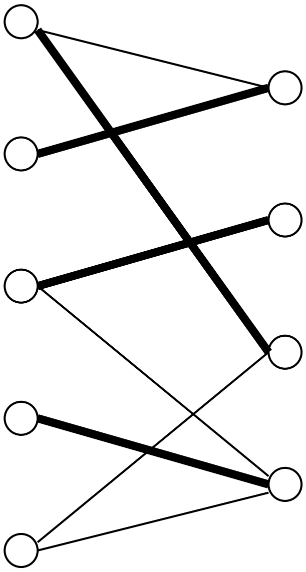
- Bipartite graph
  - Vertices can be divided into two groups, S_1 and S_2
  - Each edge connects a vertex in S_1 with a vertex in S_2



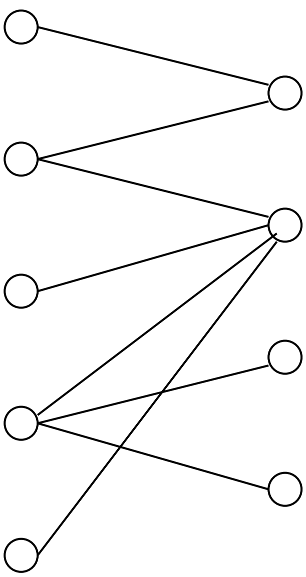
18-40: **Matching Problem**



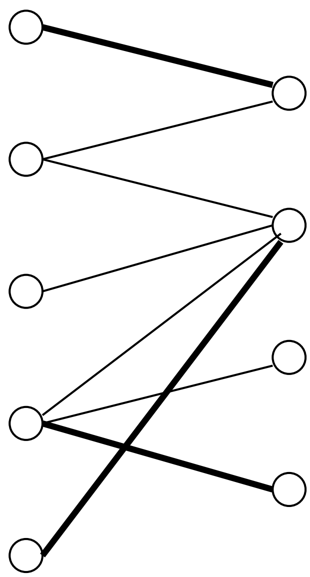
18-41: **Matching Problem**



18-42: **Matching Problem**

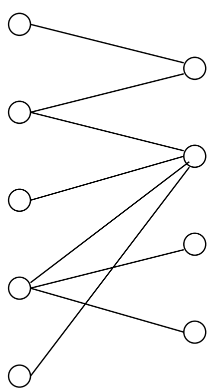


18-43: **Matching Problem**



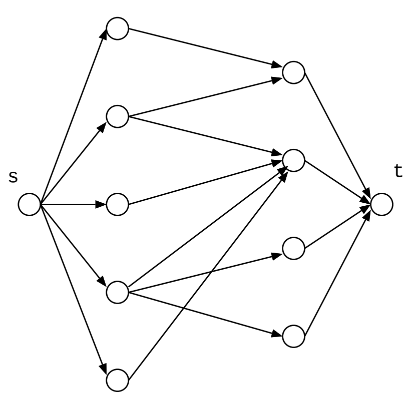
18-44: **Matching Problem**

- Finding a matching in a bipartite graph can be considered a maximum flow problem. How?



18-45: **Matching Problem**

- Finding a matching in a bipartite graph can be considered a maximum flow problem. How?



18-46: **Push-Relabel Algorithms**

- New algorithm for calculating maximum flow
- Basic idea:
    - Allow vertices to be “overfull” (have more inflow than outflow)
    - Push full capacity out of edges from source
    - Push overflow at each vertex forward to the sink
    - Push excess flow back to source

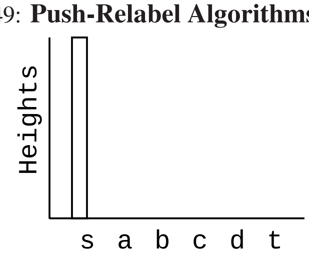
18-47: **Push-Relabel Algorithms**

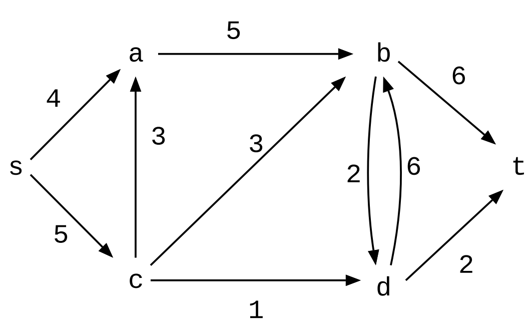
- Think of graph as a bunch of water containers connected by pipes.
- We will raise and lower the vertices, and allow water to flow between them
    - Water can only flow from higher vertex to a lower vertex
- Initially, source is at height $|V|$, all other vertices are at height 1
- Full capacity of each pipe out of the source flows to each vertex adjacent to the source

18-48: **Push-Relabel Algorithms**

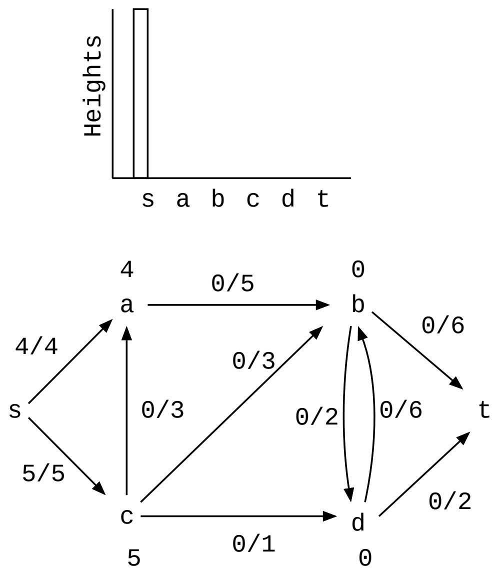
- Full capacity of each pipe out of the source flows back to each vertex adjacent to the source
    - This causes some vertices to be overfull – inflow greater than outflow
- Raise some vertex whose inflow is greater than outflow, to allow water to flow to different vertices
- Repeat until all vertices (other than the sink, which stays at level 0) are at the same level as the source
- If there are still overfull vertices, continue to raise them so that the extra flow spills back into the source

18-49: **Push-Relabel Algorithms**

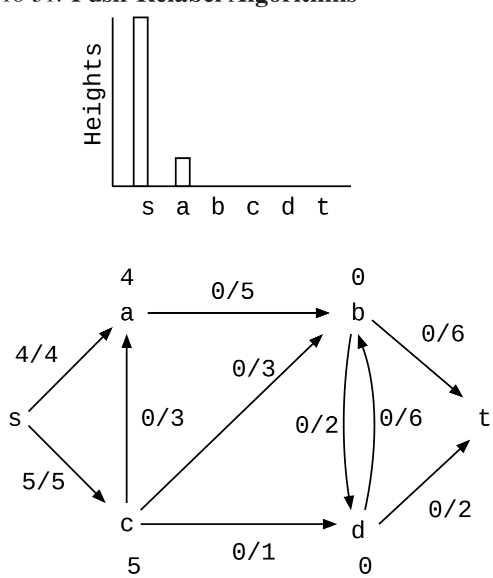




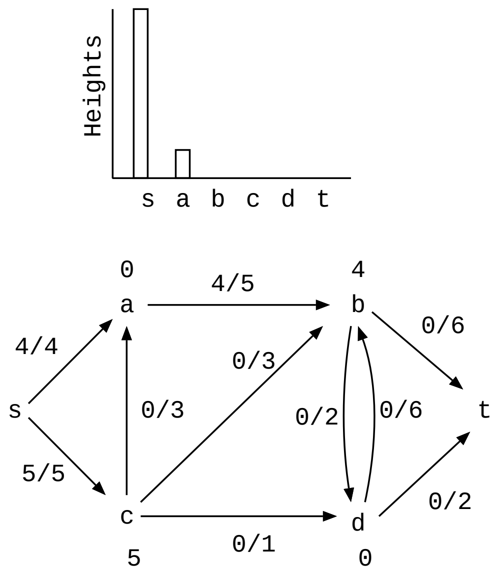
18-50: **Push-Relabel Algorithms**



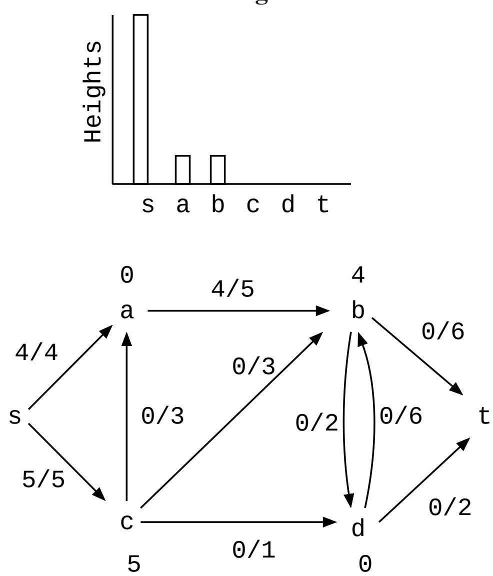
18-51: **Push-Relabel Algorithms**



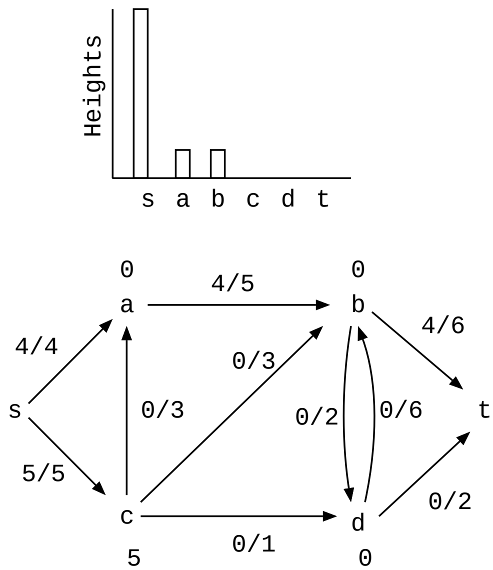
18-52: **Push-Relabel Algorithms**



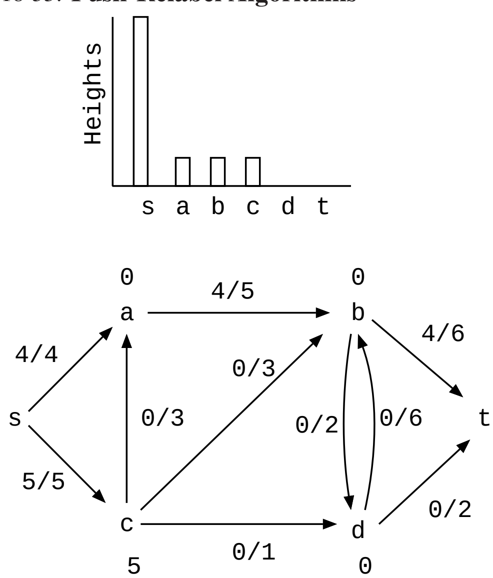
18-53: **Push-Relabel Algorithms**



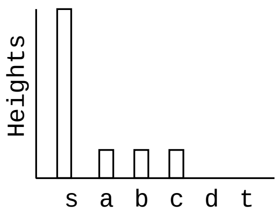
18-54: **Push-Relabel Algorithms**

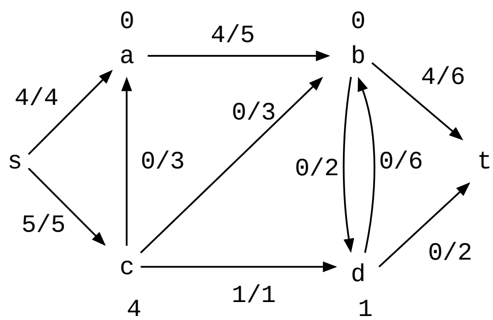


18-55: **Push-Relabel Algorithms**

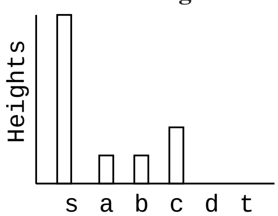


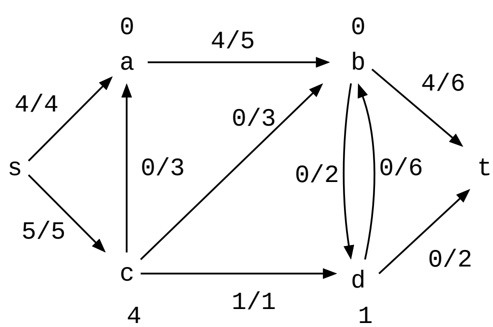
18-56: **Push-Relabel Algorithms**



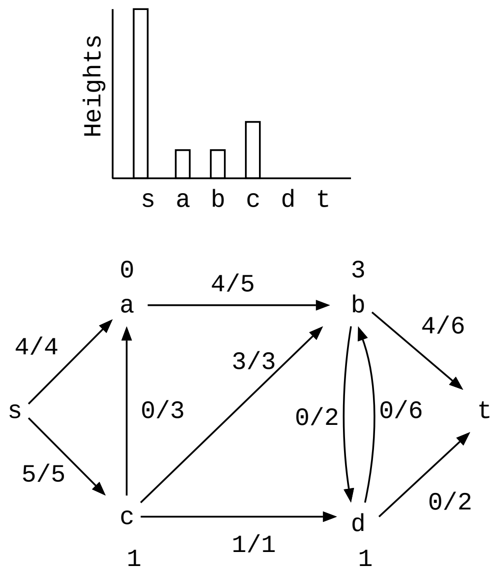


18-57: **Push-Relabel Algorithms**

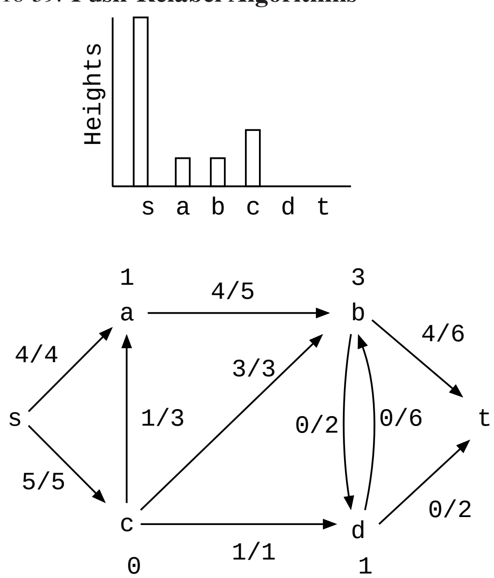




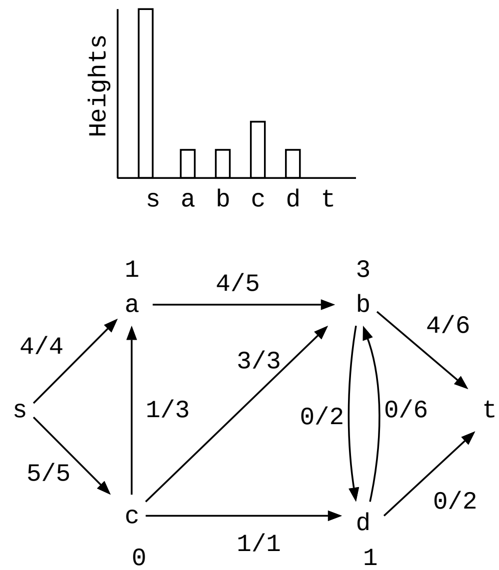
18-58: **Push-Relabel Algorithms**



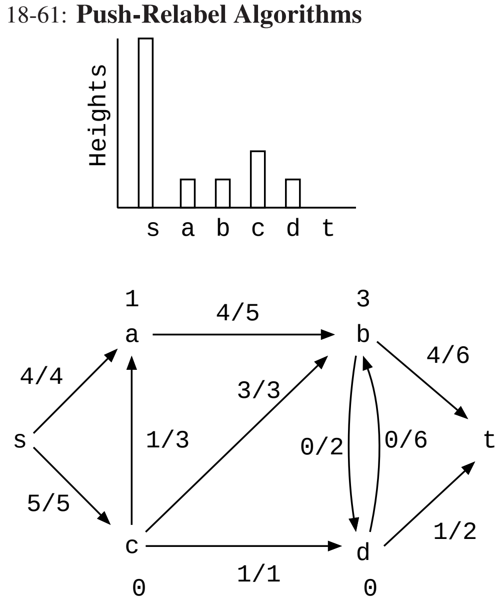
18-59: **Push-Relabel Algorithms**



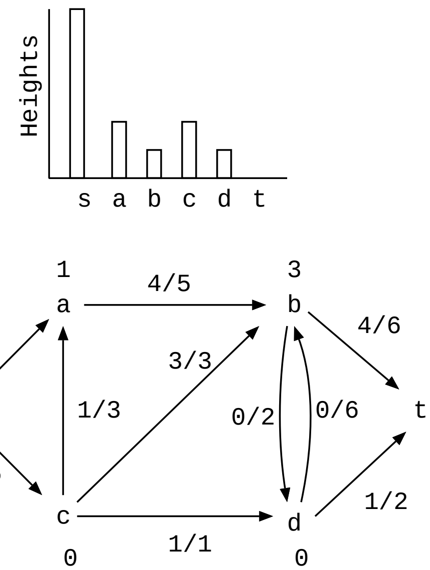
18-60: **Push-Relabel Algorithms**



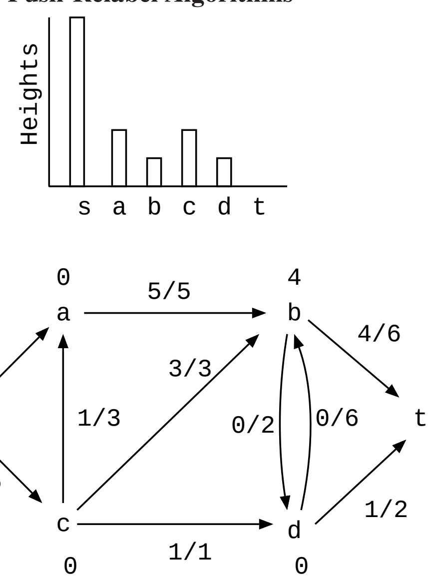
18-61: **Push-Relabel Algorithms**



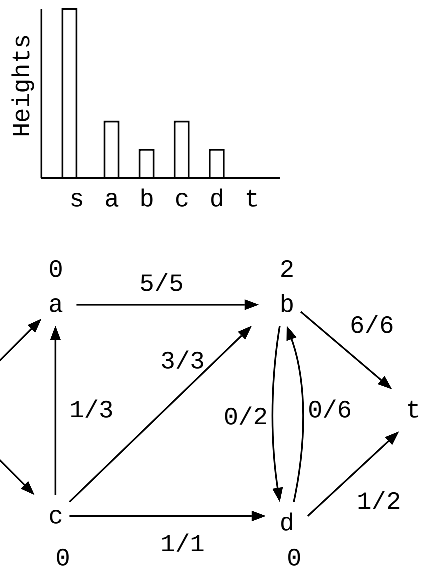
18-62: **Push-Relabel Algorithms**



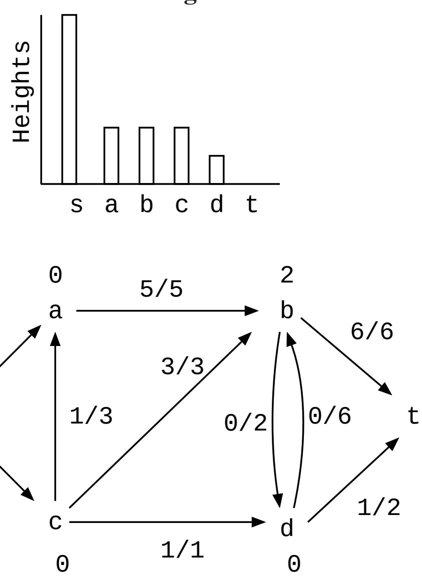
18-63: **Push-Relabel Algorithms**



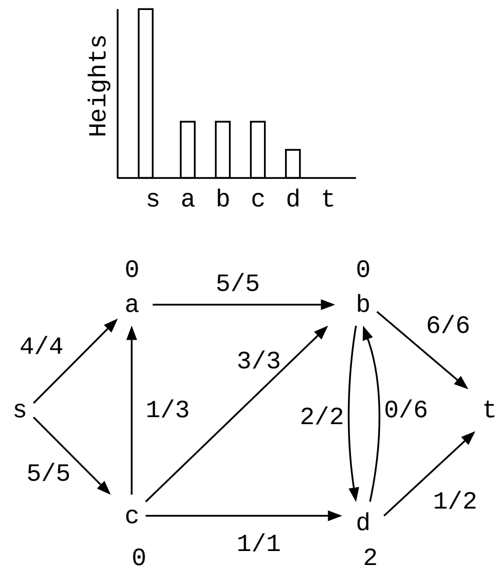
18-64: **Push-Relabel Algorithms**



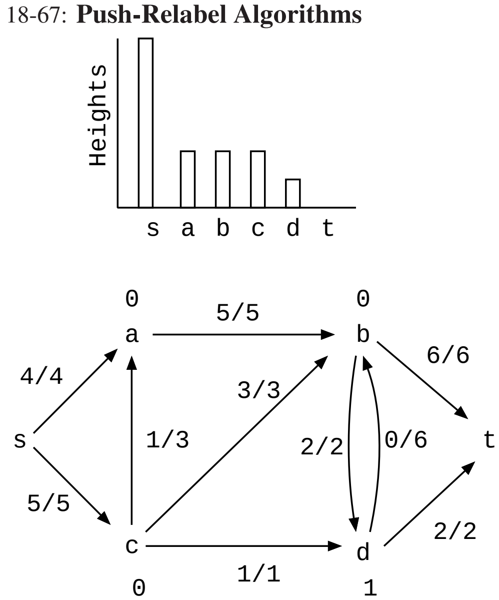
18-65: **Push-Relabel Algorithms**



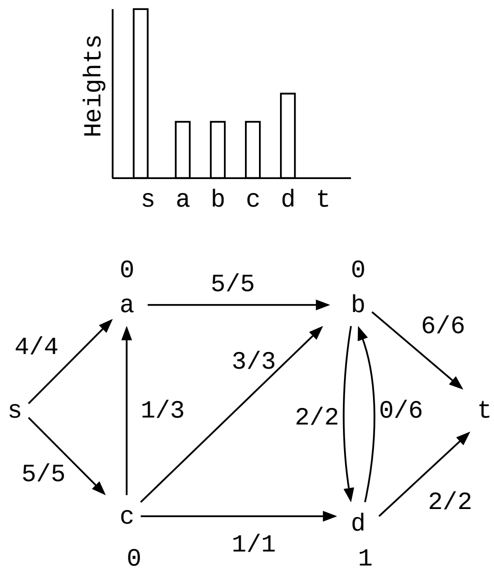
18-66: **Push-Relabel Algorithms**



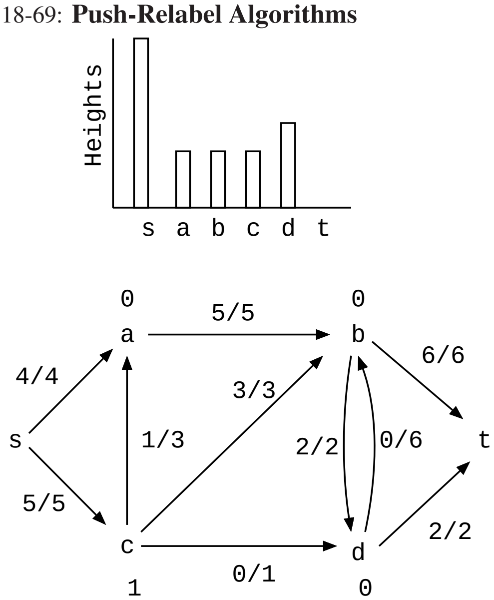
18-67: **Push-Relabel Algorithms**



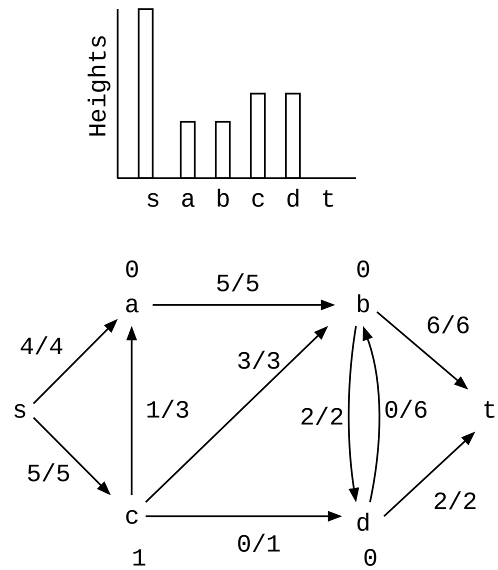
18-68: **Push-Relabel Algorithms**



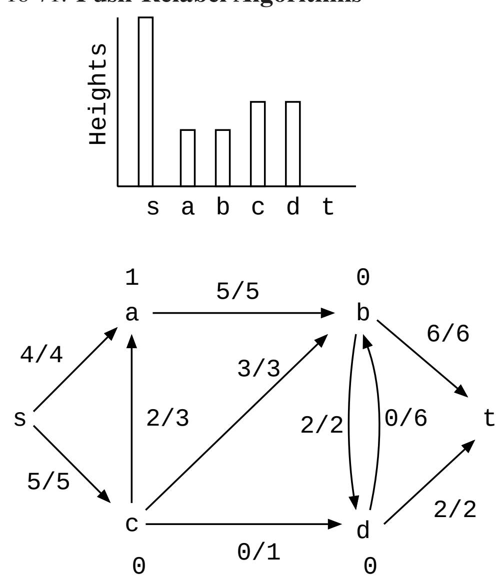
18-69: **Push-Relabel Algorithms**



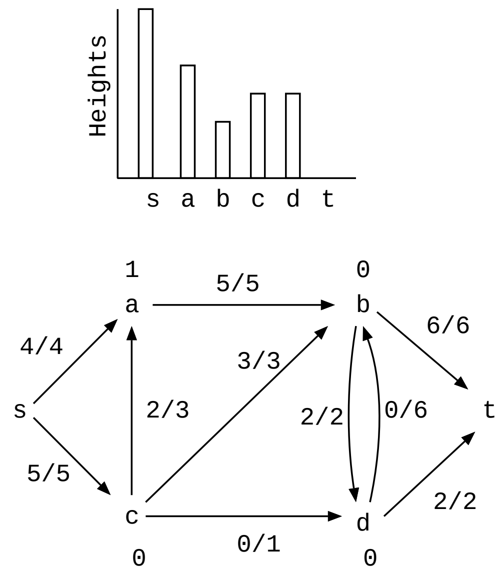
18-70: **Push-Relabel Algorithms**



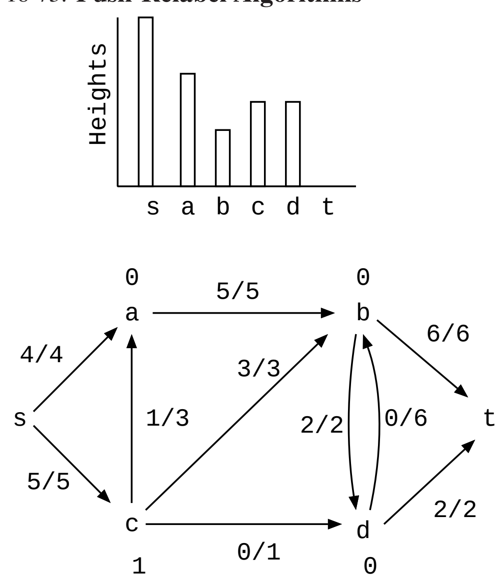
18-71: **Push-Relabel Algorithms**



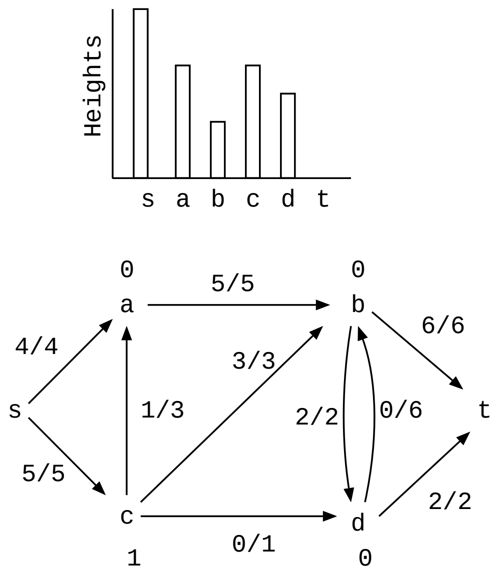
18-72: **Push-Relabel Algorithms**



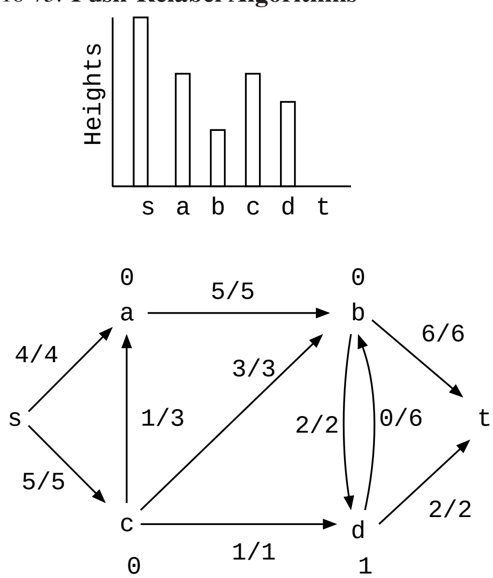
18-73: **Push-Relabel Algorithms**



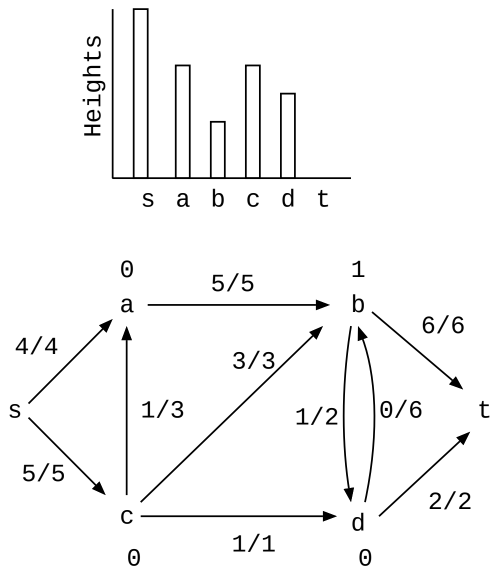
18-74: **Push-Relabel Algorithms**



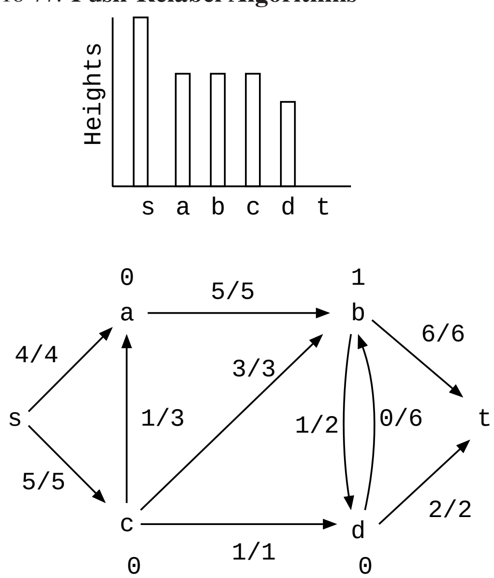
18-75: **Push-Relabel Algorithms**



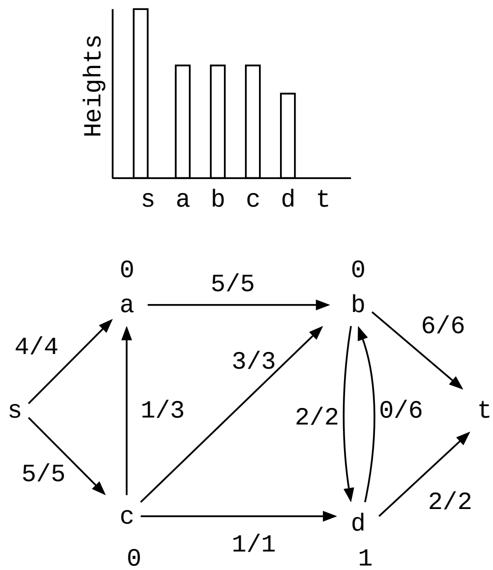
18-76: **Push-Relabel Algorithms**



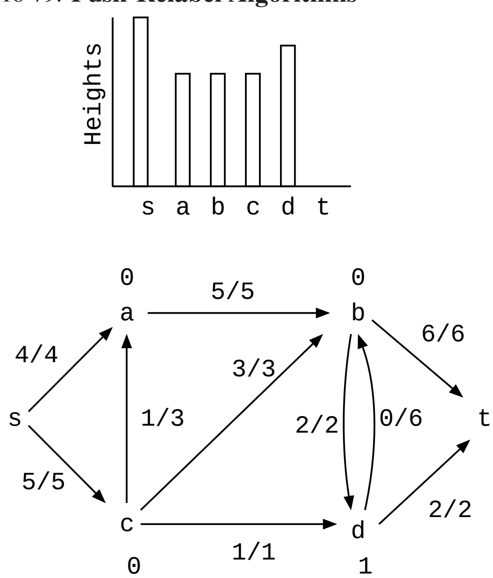
18-77: **Push-Relabel Algorithms**



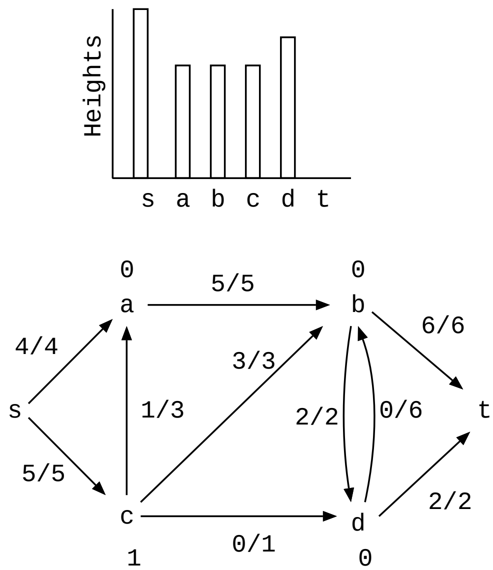
18-78: **Push-Relabel Algorithms**



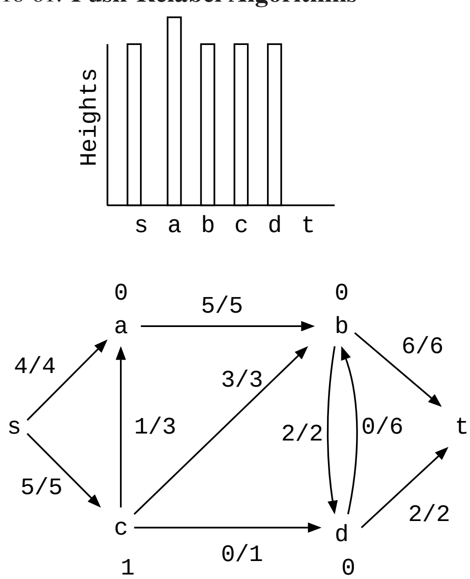
18-79: **Push-Relabel Algorithms**



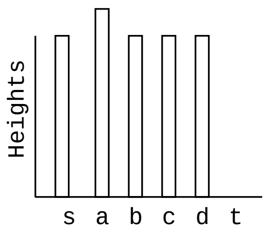
18-80: **Push-Relabel Algorithms**

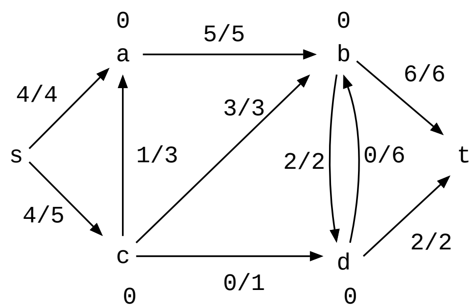


18-81: **Push-Relabel Algorithms**



18-82: **Push-Relabel Algorithms**





18-83: **Push-Relabel Algorithms**

Push(u, v)
  Applies when:
    u is overflowing
    $c_f(u, v) > 0$
    $h[u] = h[v] + 1$
  Action:
    Push $\min$(overflow[u],$c_f(u, v)$) to v

18-84: **Push-Relabel Algorithms**

Relabel(u)
  Applies when:
    u is overflowing
    For all v such that $c_f(u, v) > 0$
      $h[v] \geq h[u]$
  Action:
    $h[u] \leftarrow h[u] + 1$

18-85: **Push-Relabel Algorithms**

Push-Relabel(G)
  Initialize-Preflow(G, s)
  while there exists an applicable push/relabel
    implement push/relabel

18-86: **Push-Relabel Algorithms**

Push-Relabel(G)
  Initialize-Preflow(G, s)

while there exists an applicable push/relabel
    implement push/relabel

- Pick the operations (push/relabel) arbitrarily, time is $O(|V|^2E)$
    - (We won't prove this result, though the proof is in the book)
- Can do better with relabel-to-front
    - Specific ordering for doing push-relabel
    - Time $O(|V|^3)$, also not proven here, proof in text