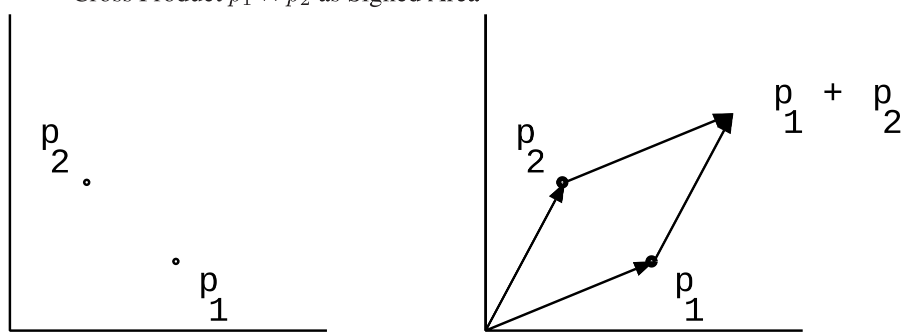
19-0: **Cross Products**

- Given any two points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$
  - Cross Product: $p_1 \times p_2 = x_1y_2 - x_2y_1$

$$\begin{aligned} p_1 \times p_2 &= x_1y_2 - x_2y_1 \\ &= -1 * (x_2y_1 - x_1y_2) \\ &= -p_2 \times p_1 \end{aligned}$$

19-1: **Cross Products**

- Cross Product $p_1 \times p_2$ as Signed Area

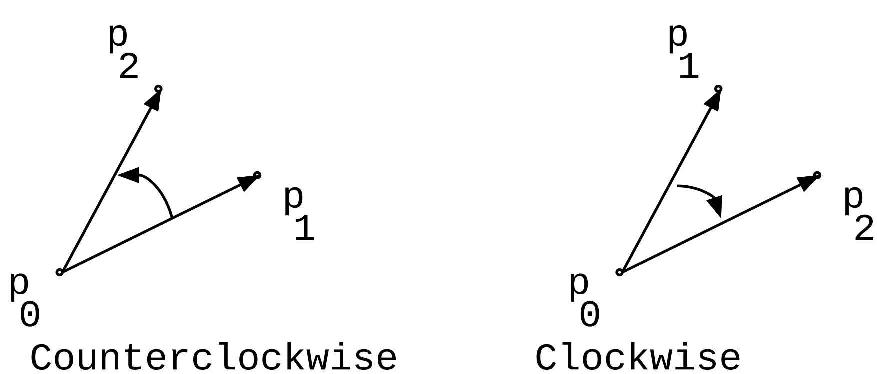


- Area is positive if p_1 is "below" p_2
- Area is negative if p_1 is "above" p_2

19-2: **Cross Products**

- Given two vectors that share an origin:
  - $\overrightarrow{p_0p_1}$ and $\overrightarrow{p_0p_2}$
- Is $\overrightarrow{p_0p_2}$ clockwise or counterclockwise relative to $\overrightarrow{p_0p_2}$?

19-3: **Cross Products**

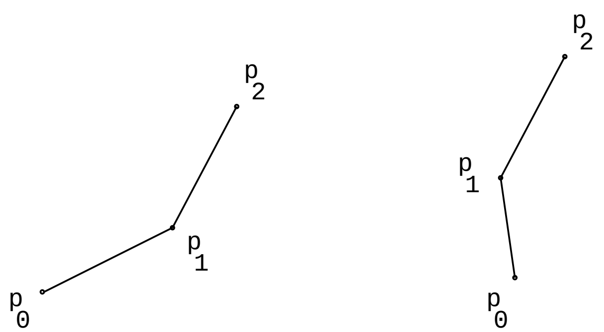


19-4: **Cross Products**

- Given two vectors that share an origin:
  - $\overrightarrow{p_0 p_1}$ and $\overrightarrow{p_0 p_2}$
- Is $\overrightarrow{p_0 p_2}$ clockwise or counterclockwise relative to $\overrightarrow{p_0 p_2}$?
  - $(p_1 - p_0) \times (p_2 - p_0)$ is positive, $\overrightarrow{p_0 p_2}$ is counterclockwise from $\overrightarrow{p_0 p_1}$

19-5: **Cross Products**

- Given two line segments $\overline{p_0 p_1}$ and $\overline{p_1 p_2}$, which direction does angle $\angle p_0 p_1 p_2$ turn?



Left Turn Right Turn

19-6: **Cross Products**

- Given two line segments $\overline{p_0 p_1}$ and $\overline{p_1 p_2}$, which direction does angle $\angle p_0 p_1 p_2$ turn?
  - $(p_2 - p_0) \times (p_1 - p_0)$ is positive, left turn
  - $(p_2 - p_0) \times (p_1 - p_0)$ is negative, right turn
  - $(p_2 - p_0) \times (p_1 - p_0)$ is zero, no turn (colinear)

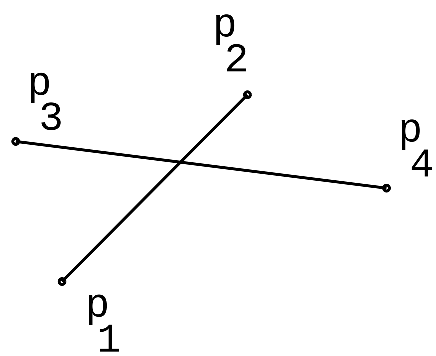
19-7: **Line Segment Intersection**

- Given two line segments $\overline{p_1 p_2}$ and $\overline{p_3 p_4}$, do they intersect?
  - How could we determine this?

19-8: **Line Segment Intersection**

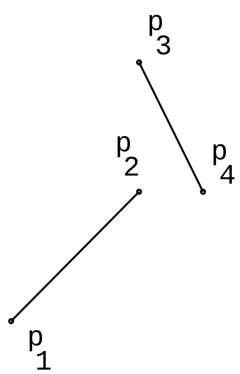
- Given two line segments $\overline{p_1 p_2}$ and $\overline{p_3 p_4}$, do they intersect?
  - Each segment straddles the line containing the other
  - An endpoint of one segment lies on the other segment

19-9: **Line Segment Intersection**



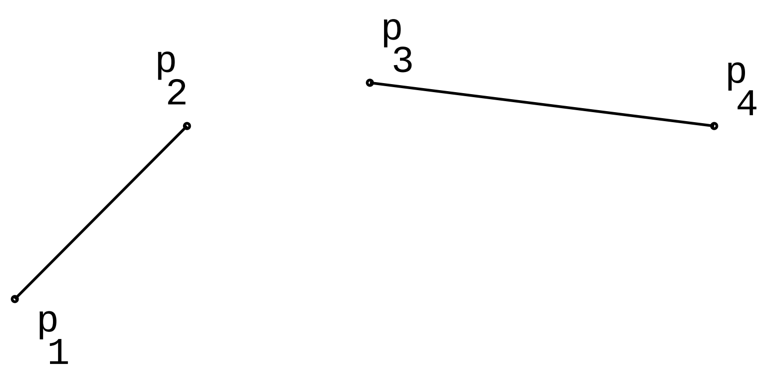
- p_3 and p_4 straddle line defined by p_1 and p_2
- p_1 and p_2 straddle line defined by p_3 and p_4

19-10: **Line Segment Intersection**



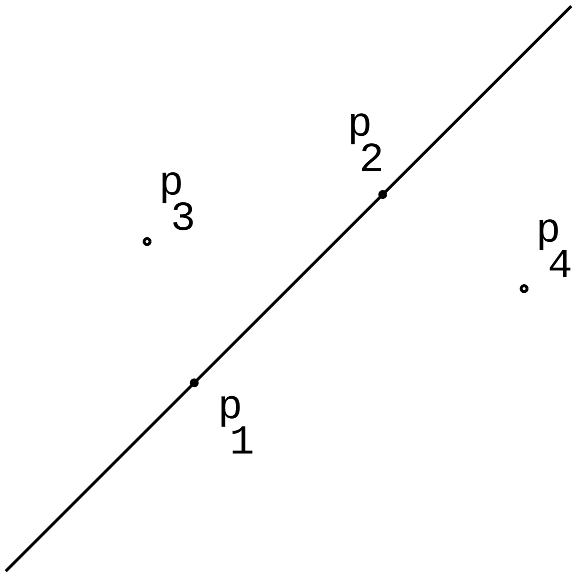
- p_3 and p_4 straddle line defined by p_1 and p_2
- p_1 and p_2 do not straddle line defined by p_3 and p_4

19-11: **Line Segment Intersection**

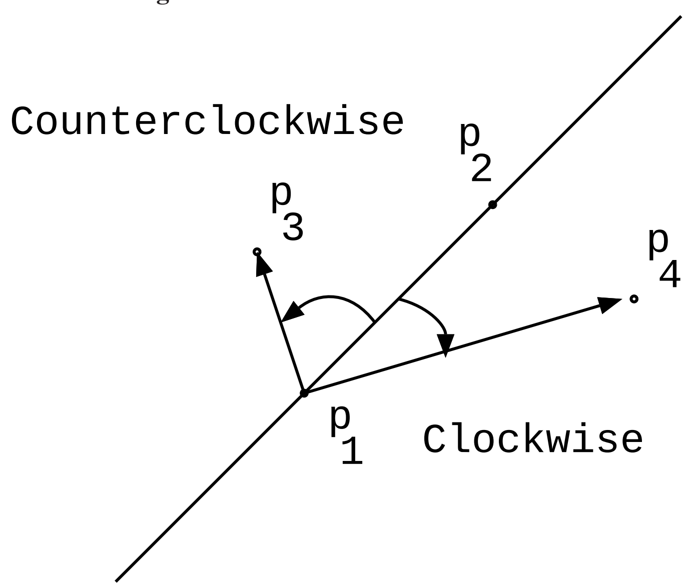


- p_3 and p_4 do not straddle line defined by p_1 and p_2
- p_1 and p_2 do not straddle line defined by p_3 and p_4

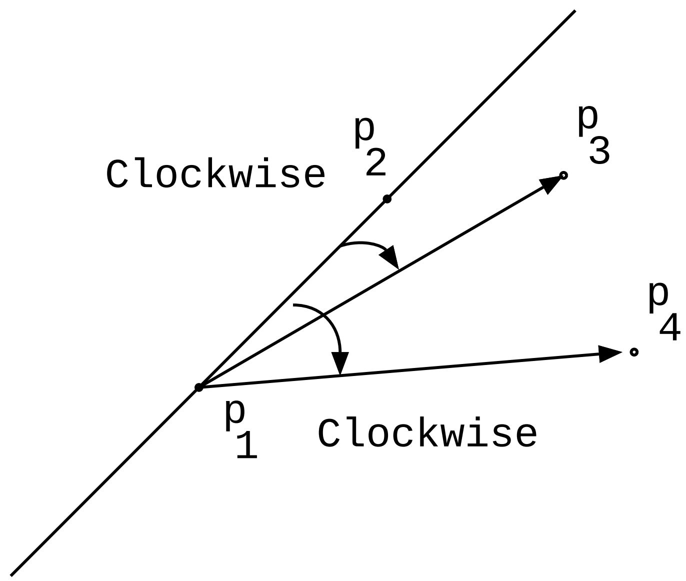
19-12: **Line Segment Intersection**



19-13: **Line Segment Intersection**



19-14: **Line Segment Intersection**



19-15: **Line Segment Intersection**

- p_3 and p_4 straddle line define by p_1 and p_2 if:
  - $\overrightarrow{p_1p_3}$ is counterclockwise of $\overrightarrow{p_1p_2}$ and $\overrightarrow{p_1p_4}$ is clockwise of $\overrightarrow{p_1p_2}$
    - $(p_2 - p_1) \times (p_3 - p_1) > 0$ and $(p_2 - p_1) \times (p_4 - p_1) < 0$
  - $\overrightarrow{p_1p_3}$ is clockwise of $\overrightarrow{p_1p_2}$ and $\overrightarrow{p_1p_4}$ is counterclockwise of $\overrightarrow{p_1p_2}$
    - $(p_2 - p_1) \times (p_3 - p_1) < 0$ and $(p_2 - p_1) \times (p_4 - p_1) > 0$

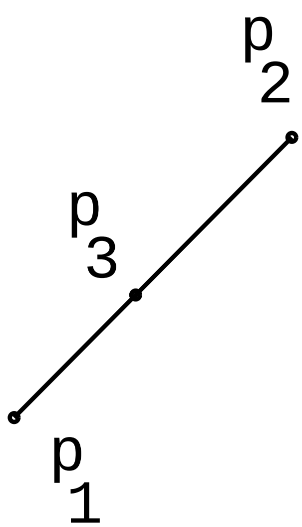
19-16: **Line Segment Intersection**

- How can we determine if p_3 is on the segment $\overline{p_1p_2}$?

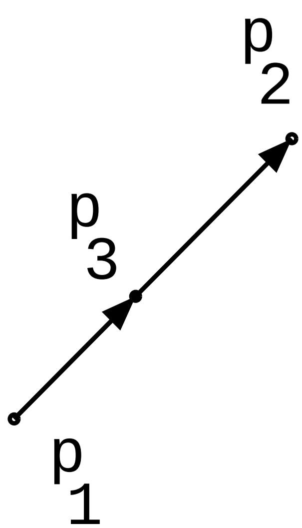
19-17: **Line Segment Intersection**

- How can we determine if p_3 is on the segment $\overline{p_1p_2}$?
  - p_3 is on the line defined by p_1 and p_2
  - p_3 is in the proper range along that line

19-18: **Line Segment Intersection**



19-19: **Line Segment Intersection**



19-20: **Line Segment Intersection**

- How can we determine if p_3 is on the segment $\overline{p_1 p_2}$?
  - p_3 is on the line defined by p_1 and p_2
    - $(p_2 - p_1) \times (p_3 - p_1) = 0$
  - p_3 is in the proper range along that line
    - $p_{3_x} \geq p_{1_x} \&\& p_{3_x} \leq p_{2_x}$ or $p_{3_x} \leq p_{1_x} \&\& p_{3_x} \geq p_{2_x}$
    - $p_{3_y} \geq p_{1_y} \&\& p_{3_y} \leq p_{2_y}$ or $p_{3_y} \leq p_{1_y} \&\& p_{3_y} \geq p_{2_y}$

19-21: **Line Segment Intersection**

- Given a set of n line segments, do any of them intersect?
  - What is a brute force method for solving this problem?
  - How long does it take (if there are n total line segments)

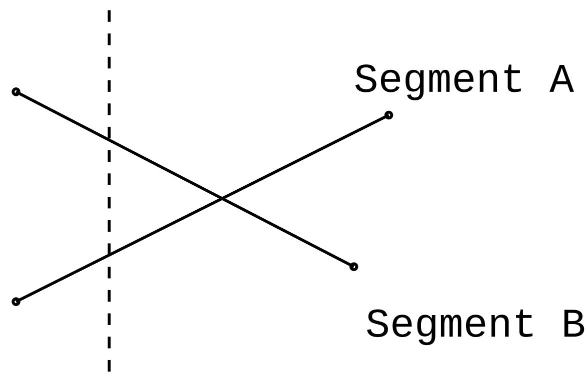
19-22: **Line Segment Intersection**

- Given a set of n line segments, do any of them intersect?
  - What is a brute force method for solving this problem?
    - Check each pair of line segments, see if they intersect using the previous technique
  - How long does it take (if there are n total line segments)
    - Each of the n segments needs to be comparted to $n - 1$ other segments, for a total time of $O(n^2)$
- We can do better!

19-23: **Line Segment Intersection**

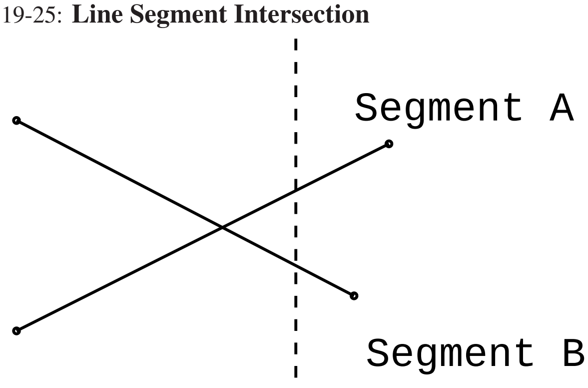
- Basic idea:
  - Assume that there are no vertical line segments
  - Sweep a vertical line across the segments
  - Segment A is above segment B at the line, and as we move the line to the right, Segment B becomes above Segment A, then the segments have crossed

19-24: **Line Segment Intersection**



- Segment B is above Segment A

19-25: **Line Segment Intersection**



- Segment A is above Segment B
- The two segments must have crossed

19-26: **Line Segment Intersection**

- Maintain an ordered list of the segments that intersect with the current sweep line
- Whenever two segments become adjacent on this list, check to see if they intersect
- Only need to check endpoints of segments

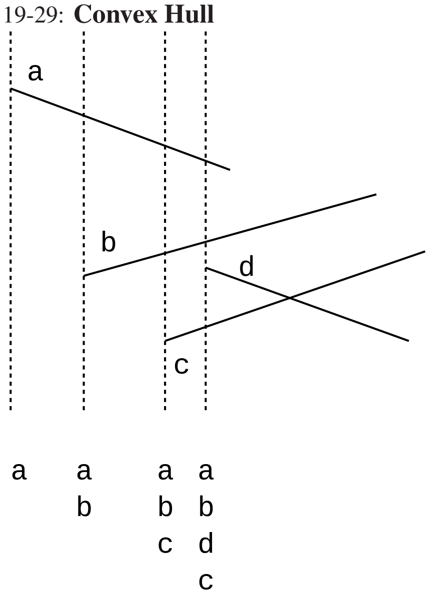
19-27: **Line Segment Intersection**

- Maintian a data structure that lets us:
    - Insert a segment s into T
    - Delete a segment s from T
    - Find the segment above s in T
    - Find the segment below s in T
- Use a red-black tree, using cross products to see if segment 1 is above segment 2 at a certain point

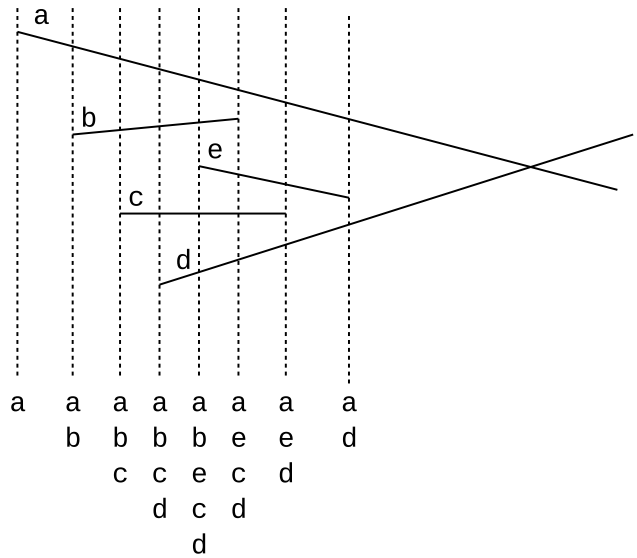
19-28: **Line Segment Intersection**

```
Sort endpoints of segments from left to right
    Break ties:
        Left endpoints before right endpoints
        lowest y-coordinate first
for each point p in endpoint list
    if p is the left endpoint of a segment s
        Insert s into T
        if there is a segment above s in T that intersects s
            or a segment below s in T that intersects s
                return true
    if p is the right endpoint of a segment s
        if there is a segment above s and below s in T
            and these segments intersect
                return true
        Delete s from T
return false
```

19-29: **Convex Hull**



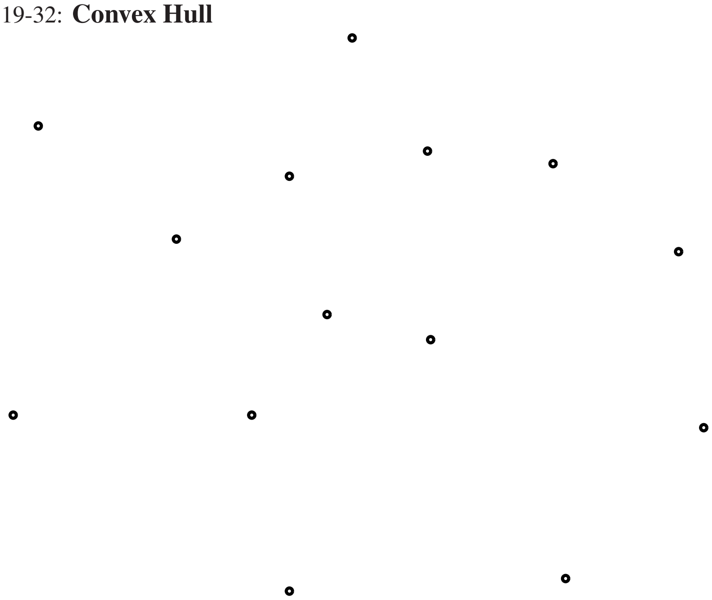
19-30: **Convex Hull**



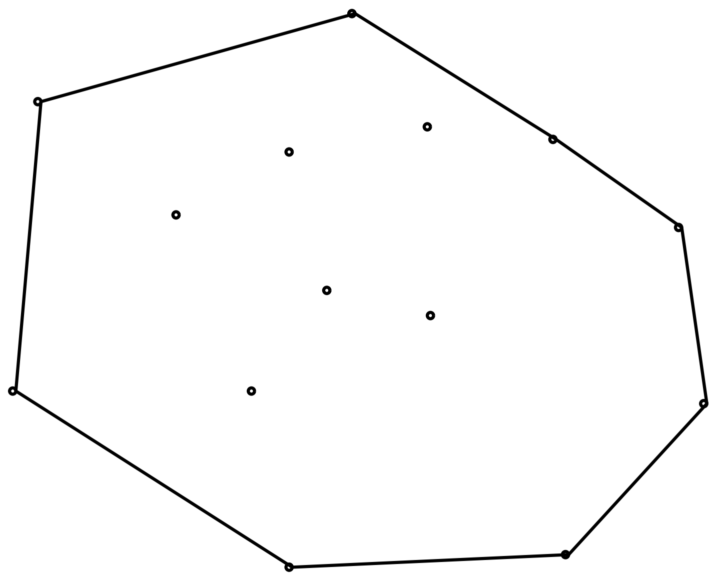
19-31: **Convex Hull**

- Given a set of points, what is the smallest convex polygon that contains all points
- Alternately, if all of the points were nails in a board, and we placed a rubber band around all of them, what shape would it form?

19-32: **Convex Hull**



19-33: **Convex Hull**



19-34: **Convex Hull**

- Several computational geometry problems have finding the convex hull as a subproblem
    - Like many graph algorithms have finding a topological sort as a subproblem
- For instance: Finding the two furthest points
    - Must lie on the convex hull

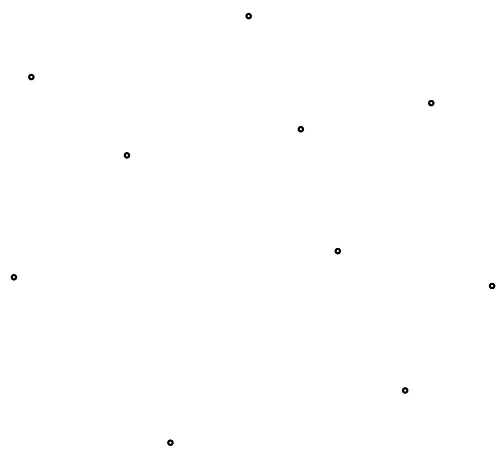
19-35: **Convex Hull**

- Graham's Scan Algorithm
    - Go through all the points in order
    - Push points onto a stack
    - Pop off points that don't form part of the convex hull
    - When we're done, stack contains the points in the convex hull

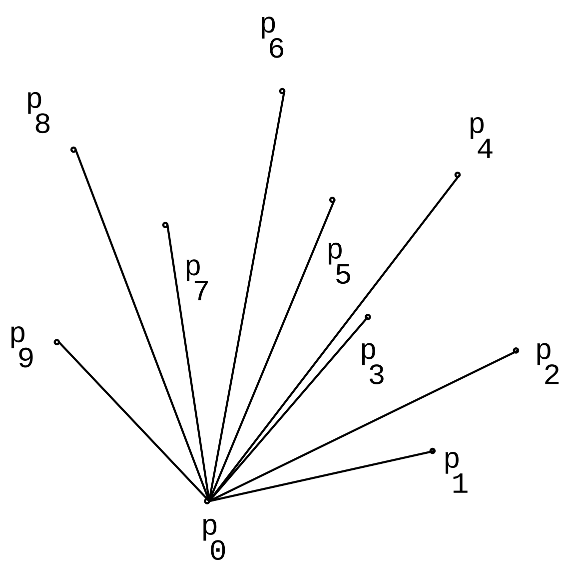
19-36: **Convex Hull**

Gram-Scan
  Let p_0 be the point with the minimum y-coordinate
  Sort the points by increasing polar angle around p_0
  Push p_0, p_1, and p_2 on the stack S
  for $i \leftarrow 3$ to n do
    while angle formed by top two points on S
    doesn't turn left do
      Pop
    Push(p_i)
  return S

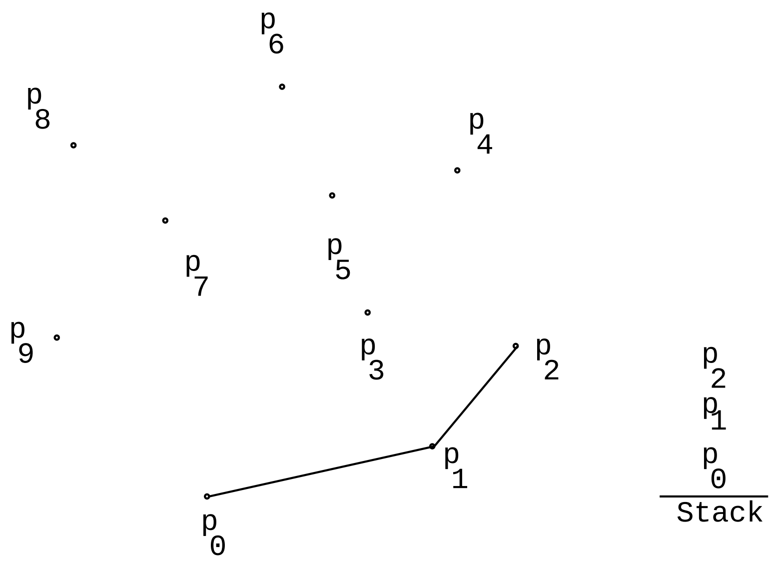
19-37: **Graham's Scan**



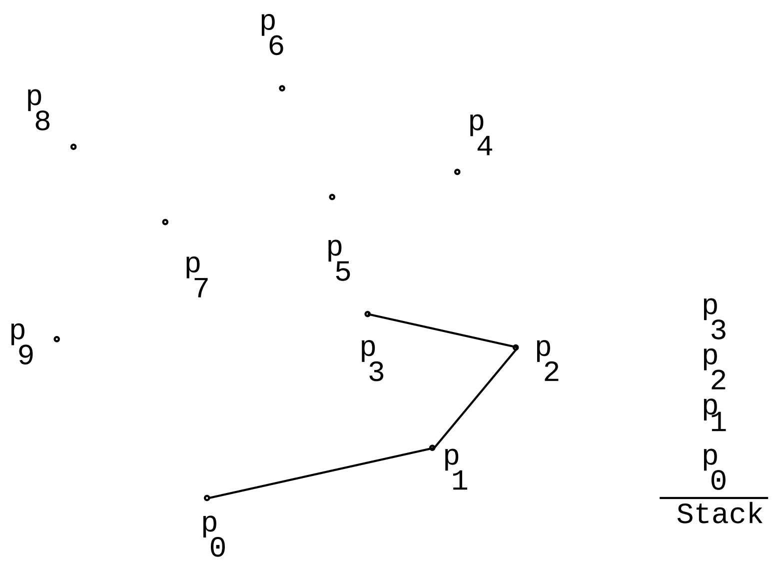
19-38: **Graham's Scan**



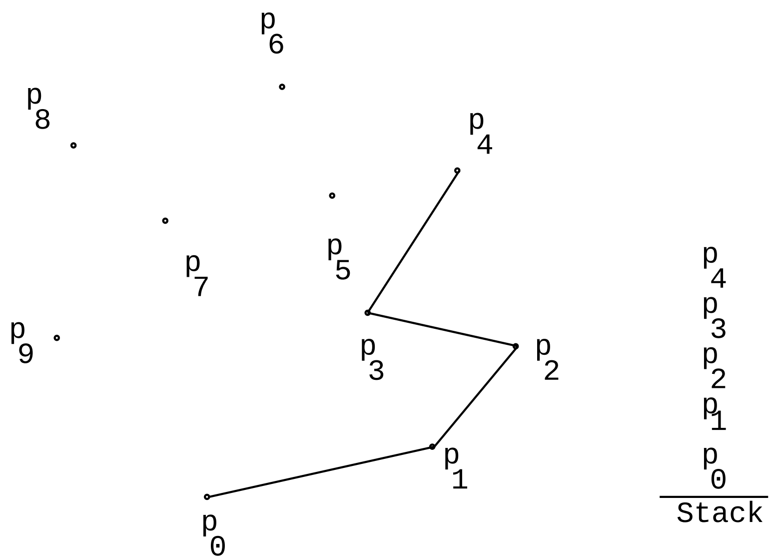
19-39: **Graham's Scan**



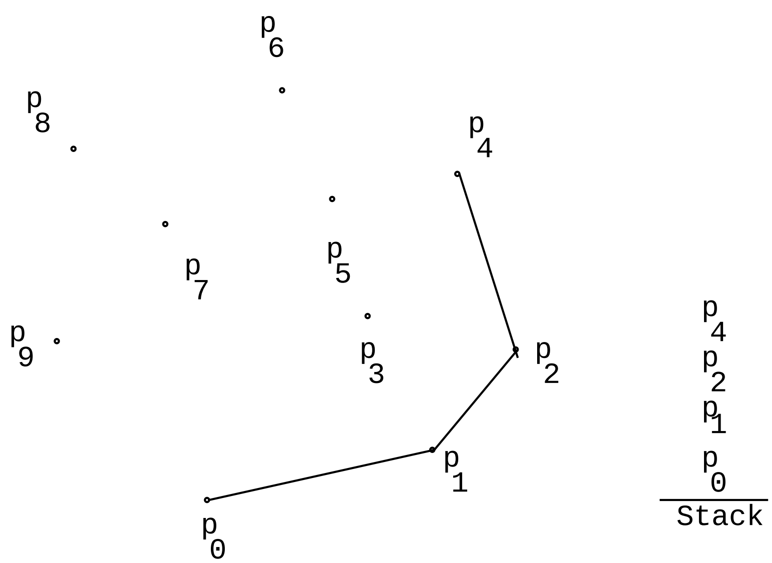
19-40: **Graham's Scan**



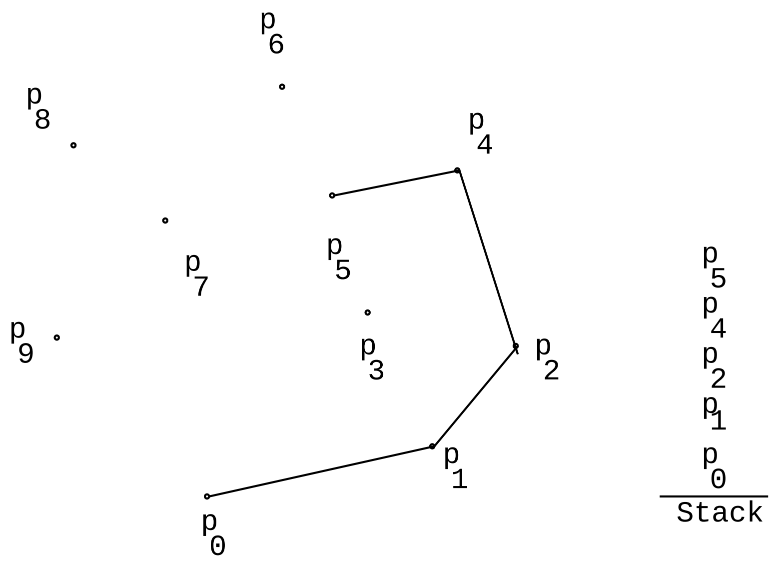
19-41: **Graham's Scan**



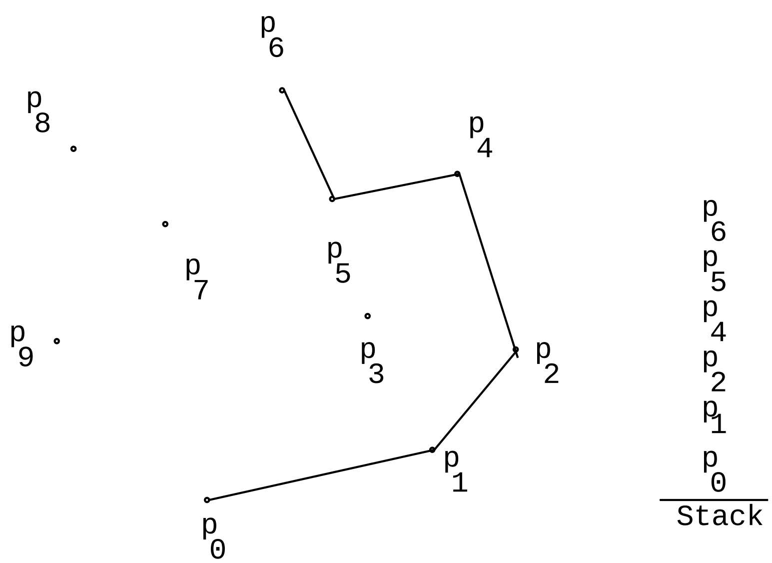
19-42: **Graham's Scan**



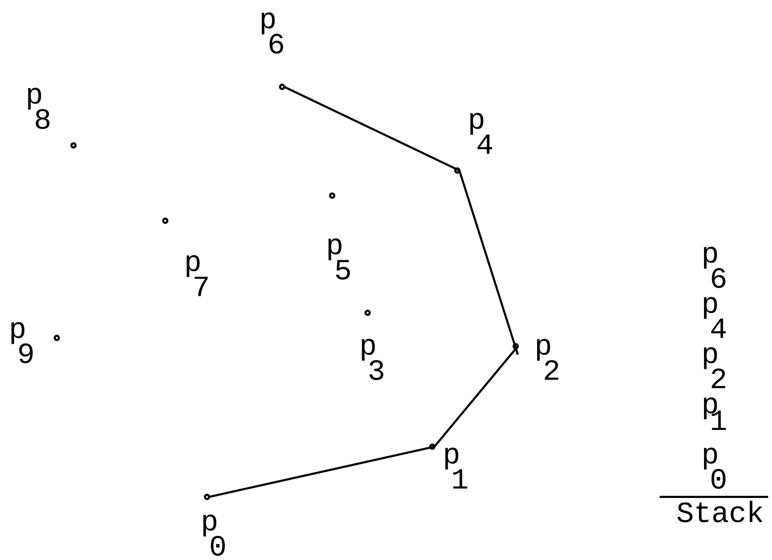
19-43: **Graham's Scan**



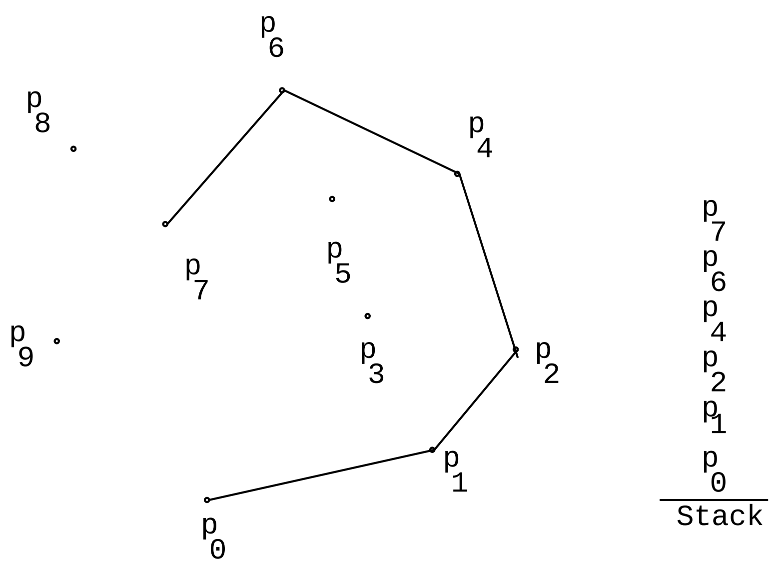
19-44: **Graham's Scan**



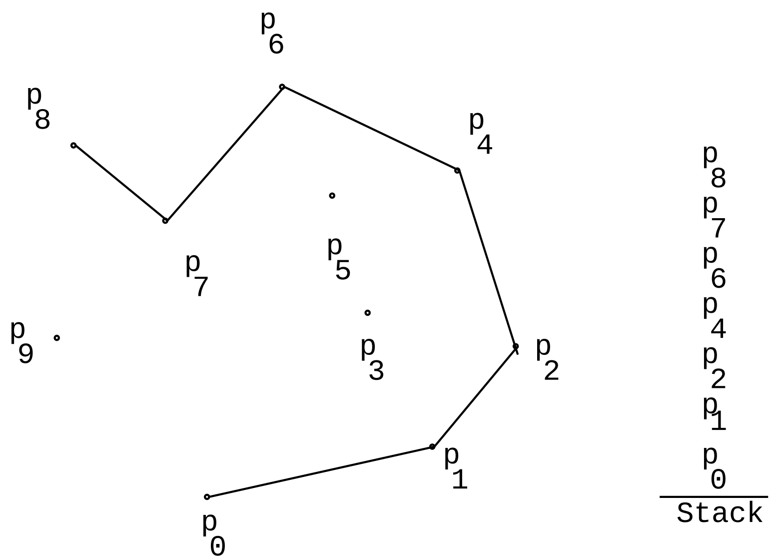
19-45: **Graham's Scan**



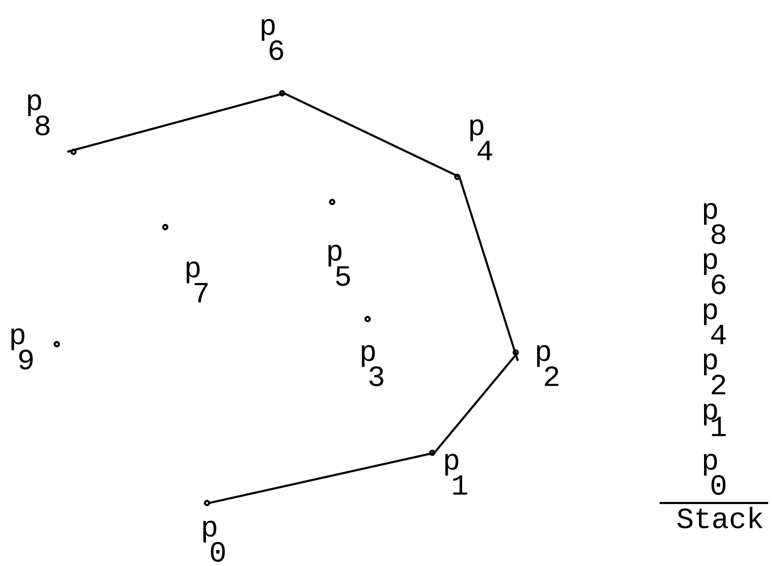
19-46: **Graham's Scan**



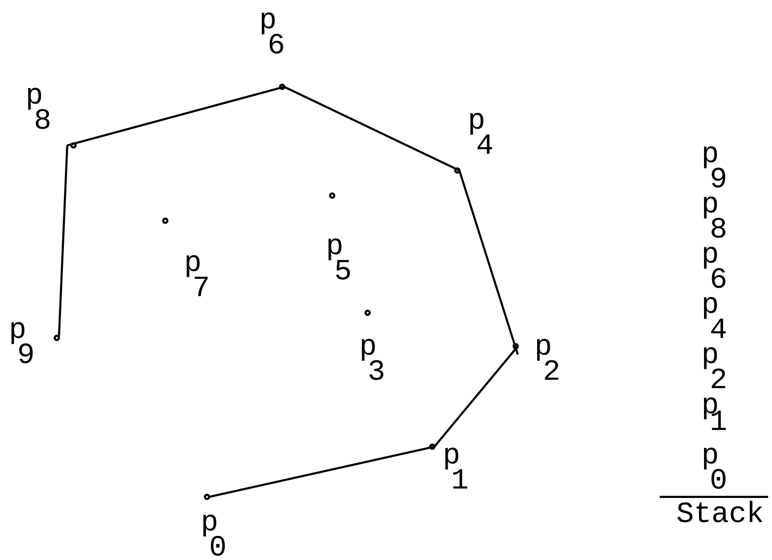
19-47: **Graham's Scan**



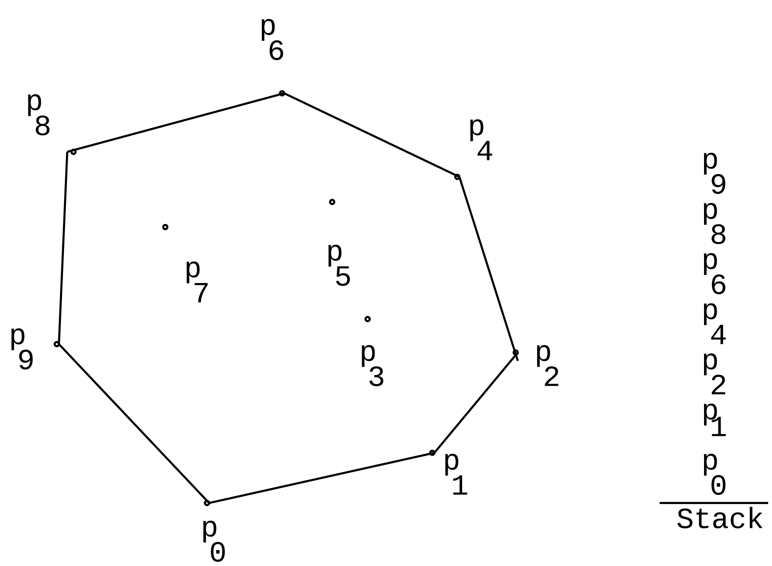
19-48: **Graham's Scan**



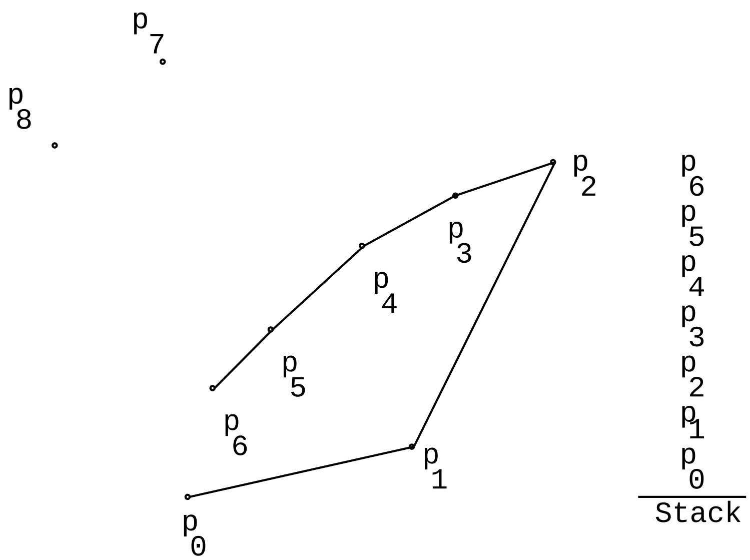
19-49: **Graham's Scan**



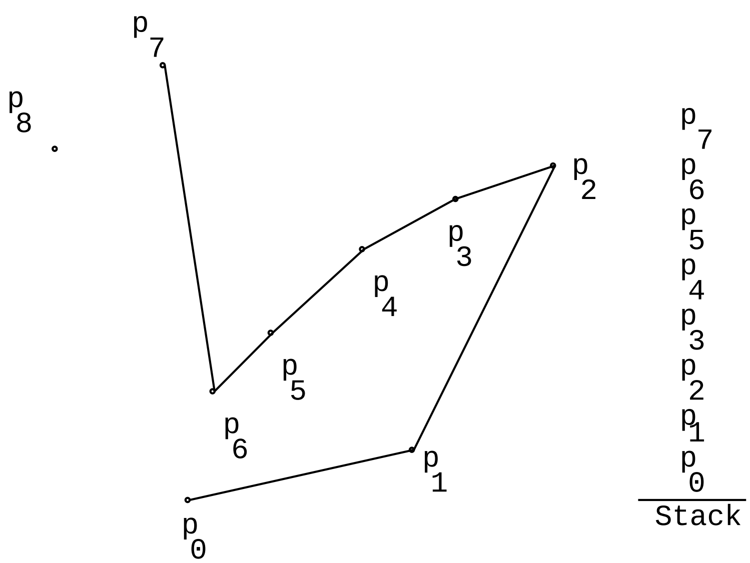
19-50: **Graham's Scan**



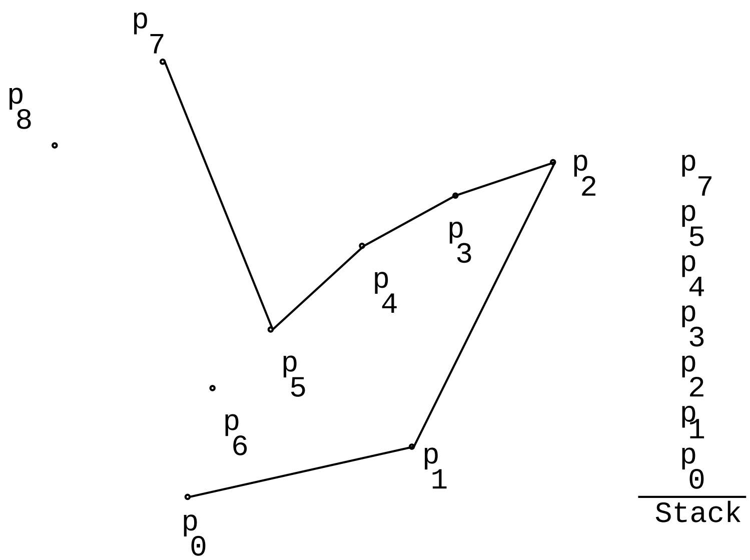
19-51: **Graham's Scan**



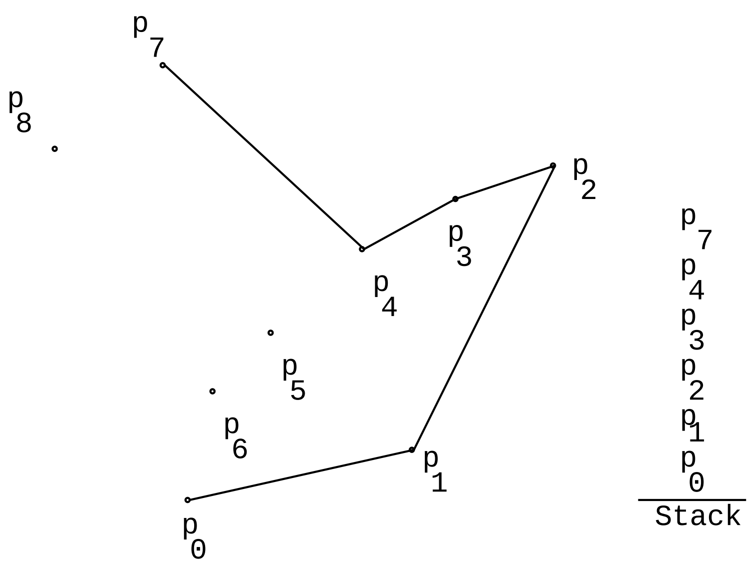
19-52: **Graham's Scan**



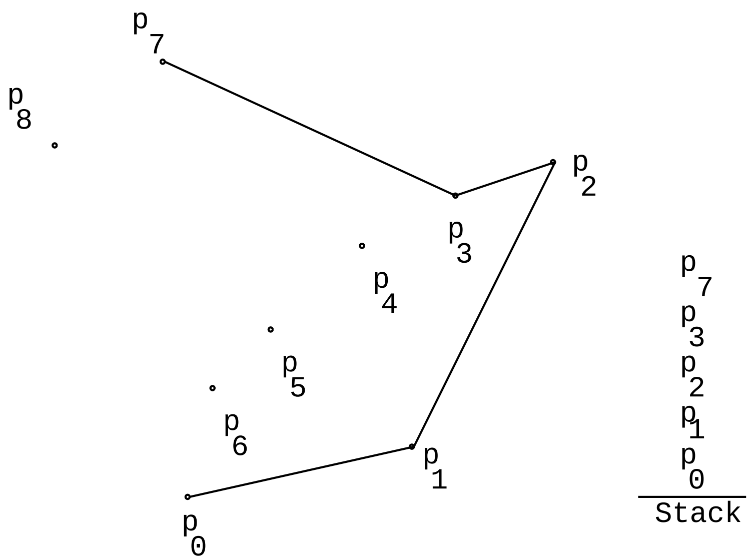
19-53: **Graham's Scan**



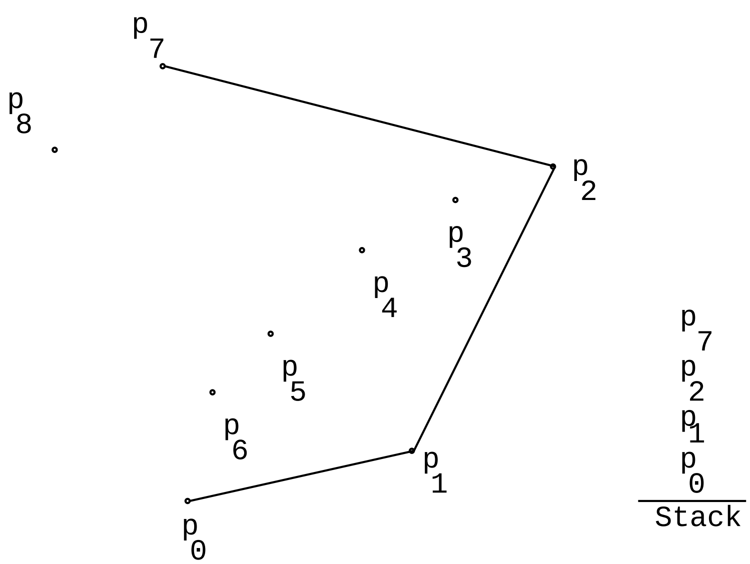
19-54: **Graham's Scan**



19-55: **Graham's Scan**



19-56: **Graham's Scan**



19-57: **Graham's Scan**

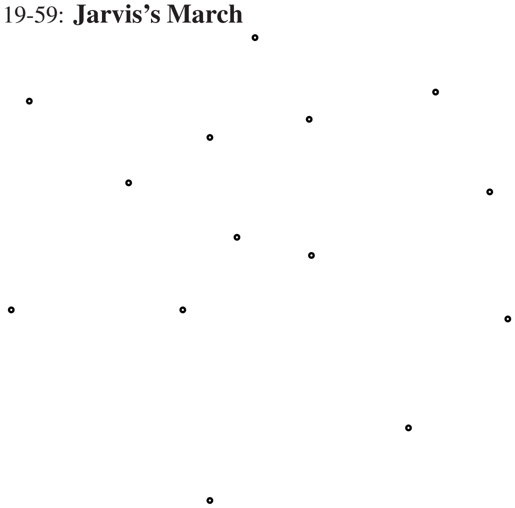
- Time required:
  - $O(n \lg n)$ to sort points by polar degree
    - Note that you don't need to calculate the polar degree, just determine if one vector is clockwise or counterclockwise of another – can be done with a single cross product
  - Each element is added to the stack once, and removed at most once (each taking constant time) for a total time of $O(n)$
  - Total: $O(n \lg n)$

19-58: **Convex Hull**

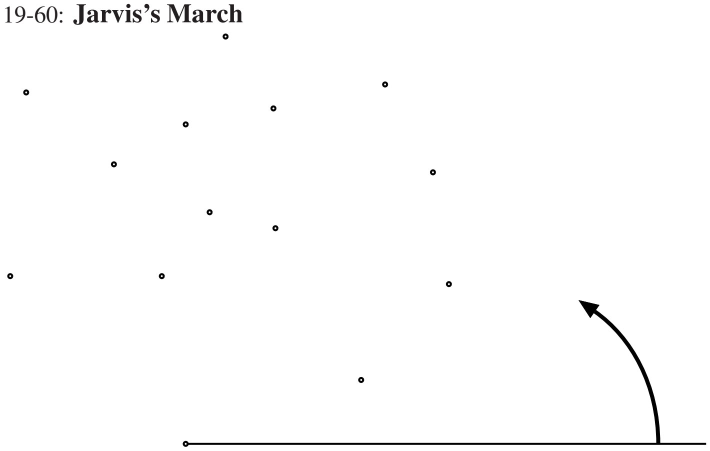
- Different Convex Hull algorithm

- Idea:
  - Attach a string to the lowest point
  - Rotate string counterclockwise, unti it hits a point – this point is in the Convex Hull
  - Keep going until the highest point is reached
  - Continue around back to initial point

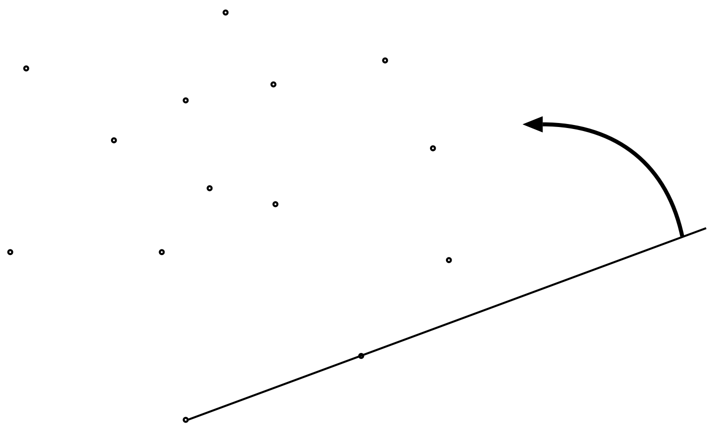
19-59: **Jarvis's March**



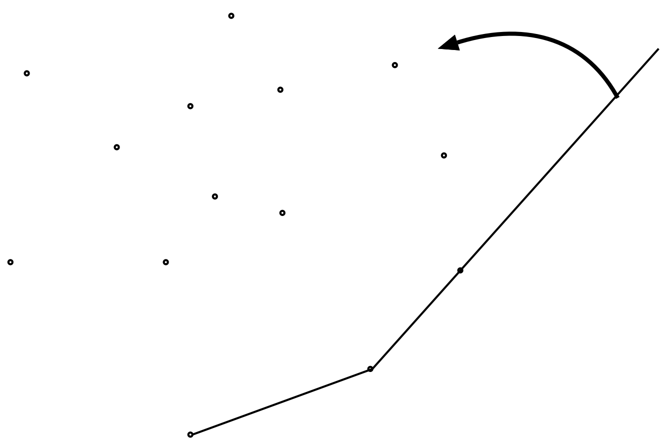
19-60: **Jarvis's March**



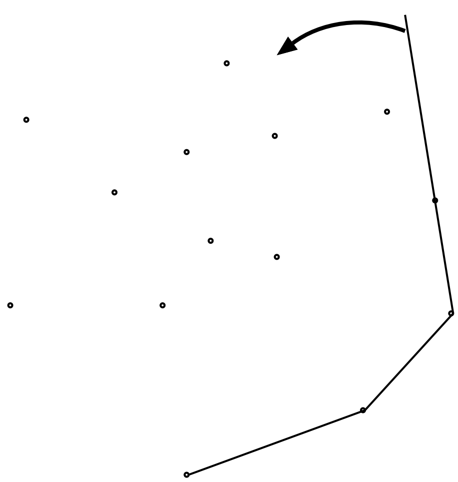
19-61: **Jarvis's March**



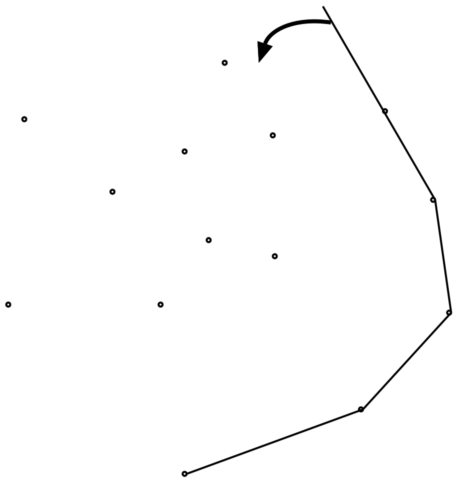
19-62: **Jarvis's March**



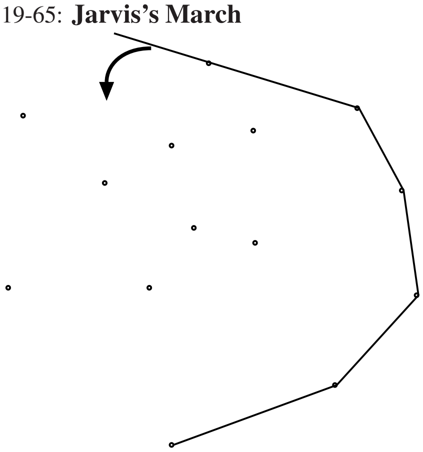
19-63: **Jarvis's March**



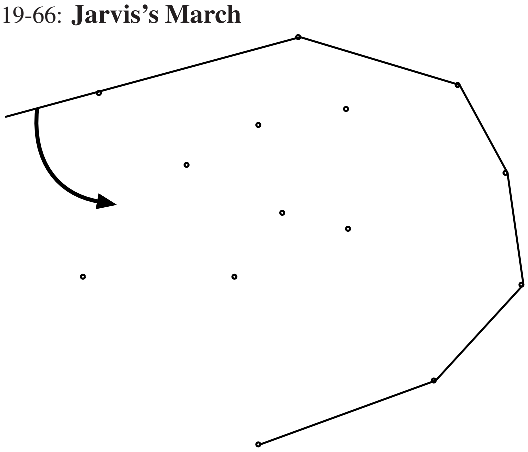
19-64: **Jarvis's March**



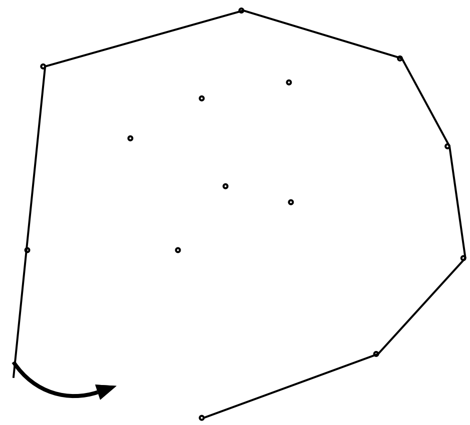
19-65: **Jarvis's March**



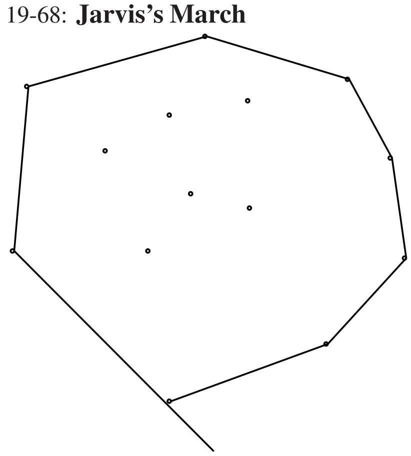
19-66: **Jarvis's March**



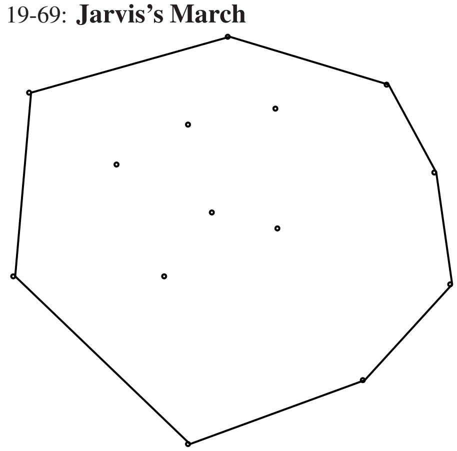
19-67: **Jarvis's March**



19-68: **Jarvis's March**



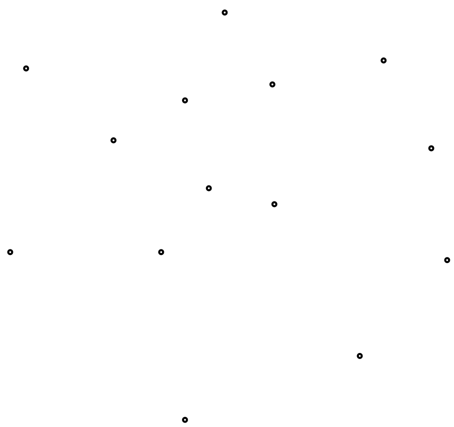
19-69: **Jarvis's March**



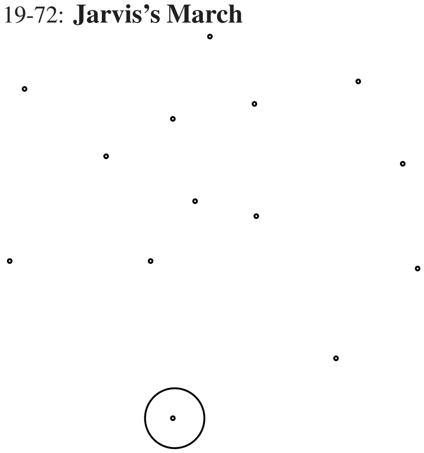
19-70: **Jarvis's March**

- How do we determine which point wraps next?
  - When we're going from lowest to highest point, the smallest polar angle between previous point and the next point
  - When going from highest point back to lowest point, smallest polar angle (from negative)

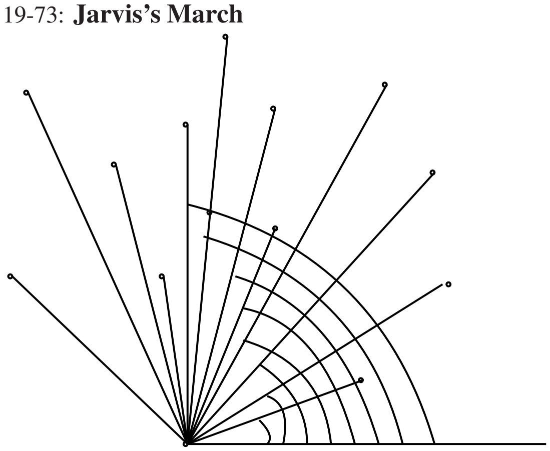
19-71: **Jarvis's March**



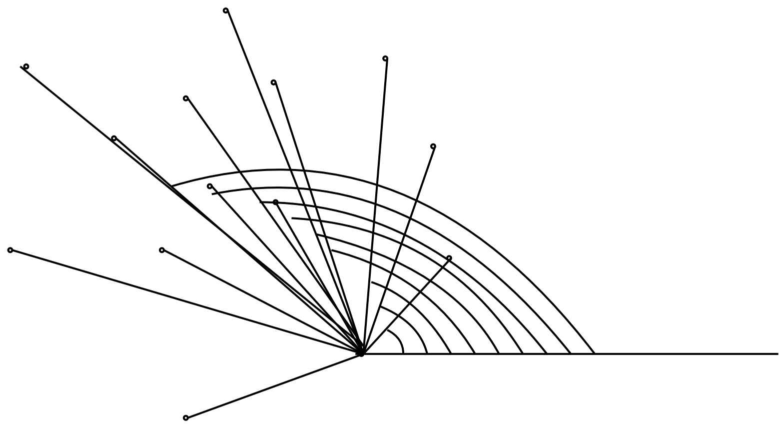
19-72: **Jarvis's March**



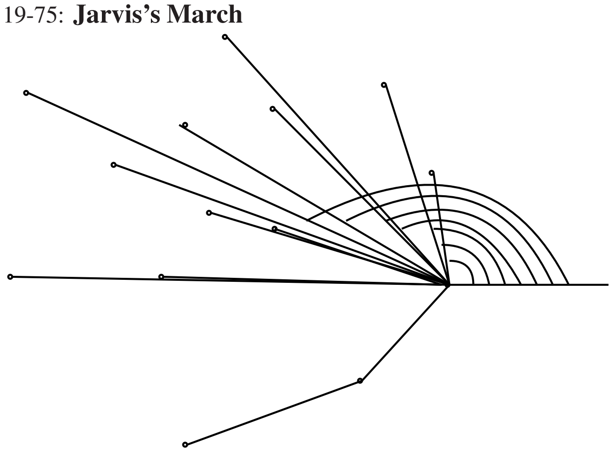
19-73: **Jarvis's March**



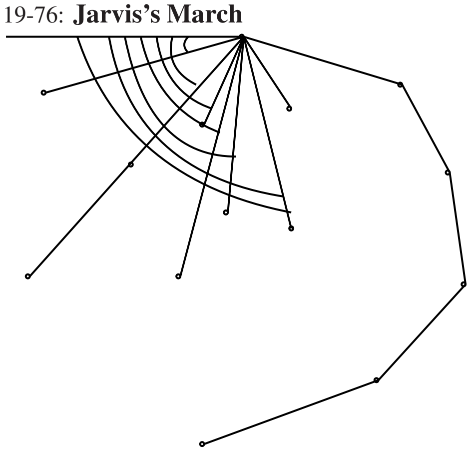
19-74: **Jarvis's March**



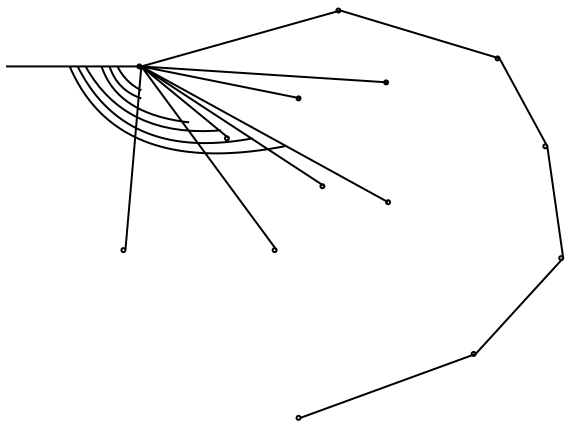
19-75: **Jarvis's March**



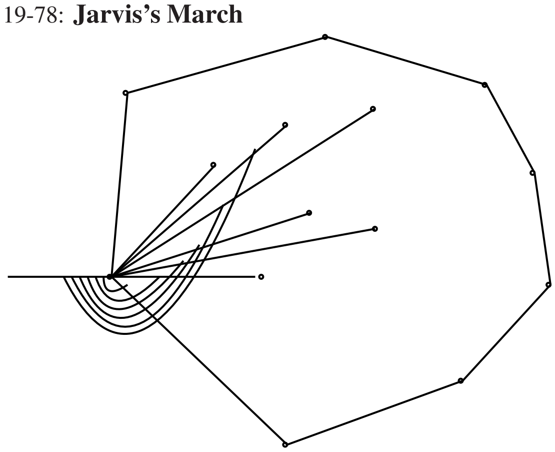
19-76: **Jarvis's March**



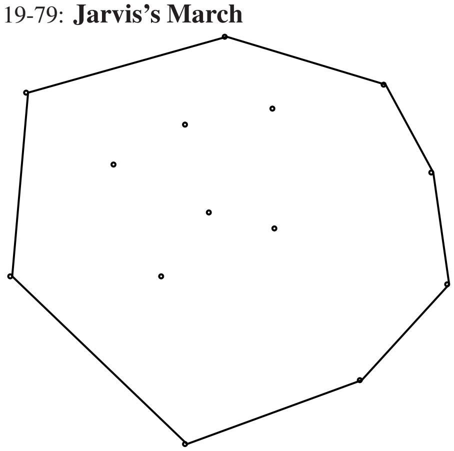
19-77: **Jarvis's March**



19-78: **Jarvis's March**



19-79: **Jarvis's March**



19-80: **Jarvis's March**

- We don't need to actually compute polar angles
    - We just need to compare them, which can be done with a cross product
- From point p_k, comparing angles from p_i and p_j (going up)
    - Is $\overline{p_k p_i}$ clockwise of $\overline{p_k p_j}$?
    - (is $(p_i - p_k) \times (p_j - p_k)$ positive)?

19-81: **Jarvis's March**

- Time for Jarvis's march:
  - For each vertex in the convex hull, we need to look at up to n other vertices to find the next vertex in the convex hull.
  - Total time: $O(nh)$, where h is the number of vertices in the convex hull
  - Is this better or worst than Graham's Scan

19-82: **Closest Pair of Points**

- We have a large number of points $p_1, \ldots, p_n$
- Want to determine which pair of points p_i, p_j is closest together
- How long would a brute force solution take?
- Can you think of another way?

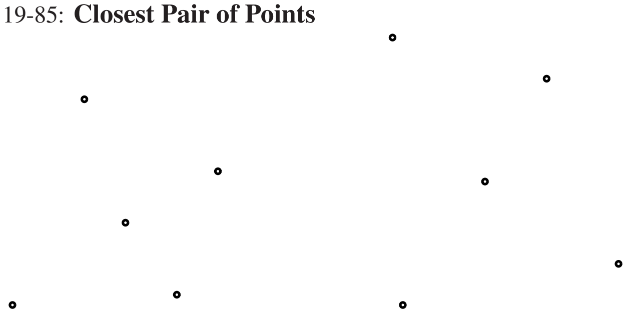
19-83: **Closest Pair of Points**

- Divide & Conquer
  - Divide the list points in half (by a vertical line)
  - Recursively determine the closest pair in each half
  - . . . and then what?

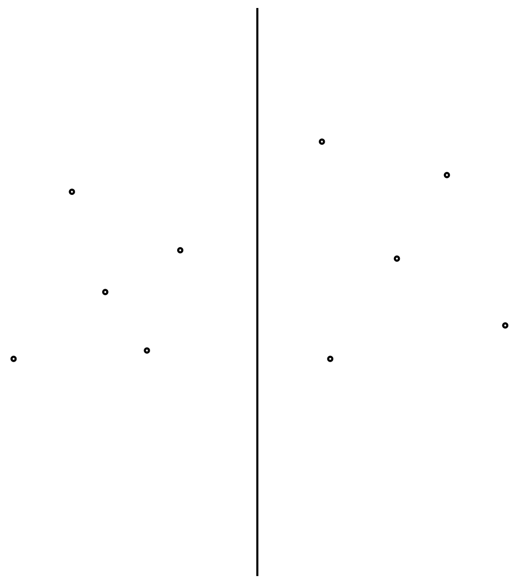
19-84: **Closest Pair of Points**

- Divide & Conquer
  - Divide the list points in half (by a vertical line)
  - Recursively determine the closest pair in each half
  - Smallest distance between points is the minimum of:
    - Smallest distance in left half of points
    - Smallest distance in right half of points
    - Smallest distance that crosses from left to right

19-85: **Closest Pair of Points**



19-86: **Closest Pair of Points**



19-87: **Closest Pair of Points**

- To find smallest distance that crosses from left to right:
  - If we compare all $\frac{n}{2}$ elements in the left sublist with all $\frac{n}{2}$ elements in the right sublist, how much time would that take?

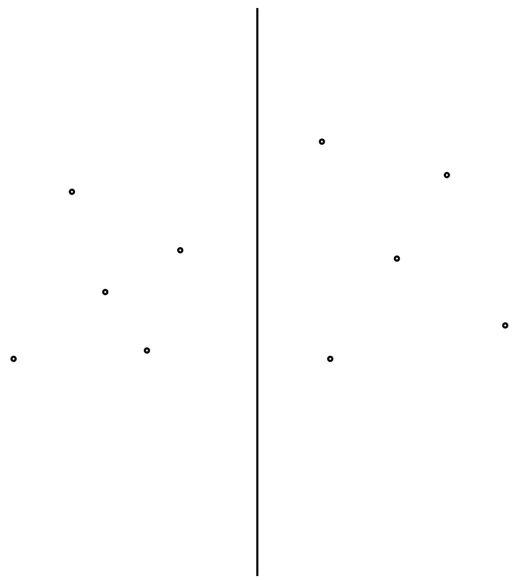
19-88: **Closest Pair of Points**

- To find smallest distance that crosses from left to right:
  - If we compare all $\frac{n}{2}$ elements in the left sublist with all $\frac{n}{2}$ elements in the right sublist, how much time would that take?
  - $\Theta(n^2)$, no better than brute force solution!

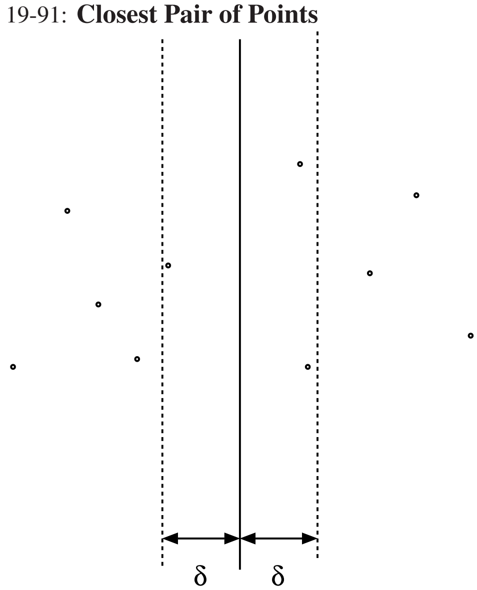
19-89: **Closest Pair of Points**

- To find smallest distance that crosses from left to right:
  - Let δ be the smallest distance in the two sublists
  - Examine only the points that are within δ of the centerline

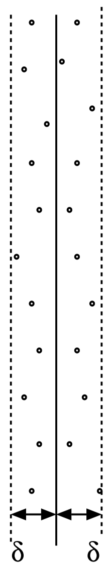
19-90: **Closest Pair of Points**



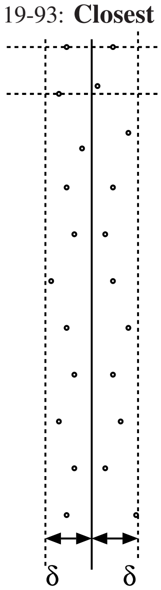
19-91: **Closest Pair of Points**



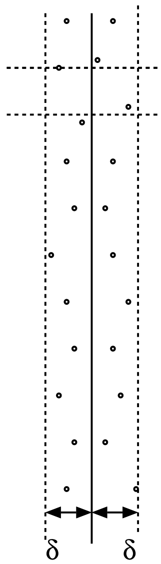
19-92: **Closest Pair of Points**



19-93: **Closest Pair of Points**

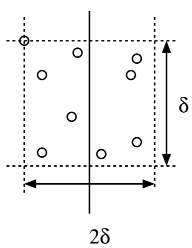


19-94: **Closest Pair of Points**



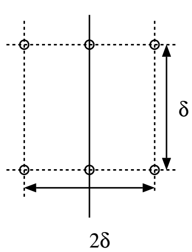
19-95: **Closest Pair of Points**

- How many points can be in the $\delta \times 2\delta$ rectangle?



19-96: **Closest Pair of Points**

- How many points can be in the $\delta \times 2\delta$ rectangle?



19-97: **Closest Pair of Points**

- Create two lists of the points:
  - One sorted by x-coordiate
  - One sorted by y-coordinate
- Call Find-Smallest using these two lists
  - Find-Smallest(XList,YList)

19-98: **Closest Pair of Points**

FindSmallest(L_x, L_y)
  if $|L_x| \leq 3$
    do brute force search on 3 points
  Split list L_x in half
    Put first 1/2 in L_{XL}
    Put second 1/2 in L_{XR}
  Split list L_Y in half
  For each point p in L_y:
    If $p \in L_{XL}$, put p in L_{YL}
    If $p \in L_{XR}$, put p in L_{YR}
  $\delta \leftarrow$ FindSmallest(L_{XL}, L_{YL})
  $\delta \leftarrow$ Min(δ, FindSmallest(L_{XR}, L_{YR}))
  $\delta \leftarrow$ FindSmallestAcross(L_{YR}, L_{YR}, δ)
  return δ

19-99: **Closest Pair of Points**

- Time:
  - Sorting: $O(n \lg n)$ using mergesort
  - Recursive call:

$$\begin{aligned} T(1) &= T(2) = T(3) = c_1 \\ T(n) &= 2T(n/2) + c_2 * n \end{aligned}$$

    - $\Theta(n \lg n)$ by the Master Method
  - Total time: $O(n \lg n)$